Filomat 36:15 (2022), 5289–5296
https://doi.org/10.2298/FIL2215289B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Observations on the Mildly Menger Property and Topological Games

Manoj Bhardwaj[a], **Alexander V. Osipov**[b]

[a]*Department of Mathematics, University of Delhi, New Delhi-110007, India*
[b]*Krasovskii Institute of Mathematics and Mechanics,*
Ural Federal University, Ural State University of Economics, Yekaterinburg, Russia

Abstract. In this paper, we defined two new games - the mildly Menger game and the compact-clopen game. In a zero-dimensional space, the Menger game is equivalent to the mildly Menger game and the compact-open game is equivalent to the compact-clopen game. An example is given for a space on which the mildly Menger game is undetermined. Also we introduced a new game namely $\mathcal{K}$-quasi-component-clopen game and proved that this game is equivalent to the compact-clopen game. Then we proved that if a topological space is a union of countably many quasi-components of compact sets, then TWO has a winning strategy in the mildly Menger game.

1. Introduction

In 1924, Menger [9] (see also [5]) introduced covering property in topological spaces. A space X is said to have *Menger property* if for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\bigcup \mathcal{V}_n$ for some n.

In covering properties, Menger property is one of the most important property. This property is stronger than Lindelöf and weaker than σ- compactness.

Usually, each selection principle $S_{fin}(\mathcal{A}, \mathcal{B})$ can be associated with some topological game $G_{fin}(\mathcal{A}, \mathcal{B})$. So the Menger property $S_{fin}(\mathcal{O}, \mathcal{O})$ is associated with the Menger game $G_{fin}(\mathcal{O}, \mathcal{O})$.

In [5] Hurewicz proved that a topological space X is Menger if and only if ONE does not have a winning strategy in the Menger game on X. Thus, the Menger property can be investigated from the point of view of topological game theory.

In ([14], Corollary 3), R. Telgársky proved that ONE has a winning strategy in the compact-open game if and only if TWO has a winning strategy in the Menger game. Telgársky also observes (Proposition 1, [14]) ONE having a winning strategy in the Menger game implies TWO having a winning strategy in the compact-open game.

Lj.D.R. Kočinac define and study a version of the classical Hurewicz covering property by using clopen covers. He calls this property *mildly Hurewicz*. In [8], game-theoretic and Ramsey-theoretic characteristics of this property are given.

2020 *Mathematics Subject Classification*. Primary 54D20; Secondary 54A20, 91A44
Keywords. Selection principles, compact-clopen game, zero-dimensional space, $\mathcal{K}$-quasi-component-clopen game, Menger space, Menger game, mildly Menger space, mildly Menger game
Received: 16 September 2021; Revised: 29 January 2022; Accepted: 31 January 2022
Communicated by Ljubiša D.R. Kočinac
Email addresses: manojmnj27@gmail.com (Manoj Bhardwaj), OAB@list.ru (Alexander V. Osipov)

In this paper, we define two new games - the mildly Menger game and the compact-clopen game. In a zero-dimensional space, the Menger game is equivalent to the mildly Menger game and the compact-open game is equivalent to the compact-clopen game. Also we introduced a new game namely $\mathcal{K}$-quasi-component-clopen game and proved that this game is equivalent to the compact-clopen game.

2. Preliminaries

Let (X, τ) or X be a topological space. We will denote by $Cl(A)$ and $Int(A)$ the closure of A and the interior of A, for a subset A of X, respectively. If a set is open and closed in a topological space, then it is called *clopen*. Recall that a space X is called *zero-dimensional* if it is nonempty and has a base consisting of clopen sets, i.e., if for every point $x \in X$ and for every neighborhood U of x there exists a clopen subset $C \subseteq X$ such that $x \in C \subseteq U$. It is clear that a nonempty subspace of a zero-dimensional space is again zero-dimensional.

Note that separable zero-dimensional metric spaces are homeomorphic to subsets of the irrational numbers ([4],[E, 6.2.16]). For the terms and symbols that we do not define follow [3].

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a topological space X.

The symbol $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection principle that for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\omega} \mathcal{V}_n$ is an element of $\mathcal{B}$ [11].

In this paper $\mathcal{A}$ and $\mathcal{B}$ will be collections of the following open covers of a space X:

$\mathcal{O}$: the collection of all open covers of X.

$C_{\mathcal{O}}$: the collection of all clopen covers of X.

Clearly, X has the Menger property if and only if X satisfies $S_{fin}(\mathcal{O}, \mathcal{O})$.

Definition 2.1. A space X is said to have *mildly Menger property* if for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of clopen covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\bigcup \mathcal{V}_n$ for some n, i.e., X satisfies $S_{fin}(C_{\mathcal{O}}, C_{\mathcal{O}})$.

The proof of the following result easily follows from replacing the open sets with sets of a clopen base of the topological space.

Theorem 2.2. *For a zero-dimensional space X, $S_{fin}(C_{\mathcal{O}}, C_{\mathcal{O}})$ is equivalent to $S_{fin}(\mathcal{O}, \mathcal{O})$.*

3. Games related to $S_{fin}(\mathcal{O}, \mathcal{O})$ and $S_{fin}(C_{\mathcal{O}}, C_{\mathcal{O}})$

The *selection game* $G_{fin}(\mathcal{A}, \mathcal{B})$ is an ω-length game played by two players, ONE and TWO. During round n, ONE choose $A_n \in \mathcal{A}$, followed by TWO choosing $B_n \in [A_n]^{<\omega}$. Player TWO wins in the case that $\bigcup\{B_n : n < \omega\} \in \mathcal{B}$, and Player ONE wins otherwise.

We consider the following selection games:

- $G_{fin}(\mathcal{O}, \mathcal{O})$ - the *Menger game*.
- $G_{fin}(C_{\mathcal{O}}, C_{\mathcal{O}})$ - the *mildly Menger game*.

In [5] Hurewicz proves:

Theorem 3.1. *(Hurewicz) A topological space has the Menger property $S_{fin}(\mathcal{O}, \mathcal{O})$ if, and only if, ONE has no winning strategy in the Menger game $G_{fin}(\mathcal{O}, \mathcal{O})$.*

Telgársky proved that a metric space X is σ-compact if, and only if, TWO has a winning strategy in the Menger game.

If a player has a winning strategy, we write *Player* $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$. If player has no winning strategy, we write *Player* $\not\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.

Note that the following chain of implications always holds:

$$\begin{gathered} X \text{ is } \sigma\text{-compact} \\ \Downarrow \\ TWO \uparrow G_{fin}(\mathcal{O},\mathcal{O}) \Leftrightarrow ONE \uparrow \text{the compact-open game} \\ \Downarrow \\ ONE \not\uparrow G_{fin}(\mathcal{O},\mathcal{O}) \\ \Updownarrow \\ X \text{ has the Menger property.} \end{gathered}$$

The *compact-open game* (*compact-clopen game*) on a space X is played according to the following rules:

In each inning $n \in \omega$, ONE picks a compact set $K_n \subseteq X$, and then TWO chooses an open (clopen) set $U_n \subseteq X$ with $K_n \subseteq U_n$. At the end of the play

$$K_0, U_0, K_1, U_1, K_2, U_2, ..., K_n, U_n, ...,$$

the winner is ONE if $X \subseteq \bigcup_{n\in\omega} U_n$, and TWO otherwise.

Let $\mathcal{K}$ denotes the collection of all compact subsets of a space X. We denote the collection of all clopen subsets of a space by τ_c and the collection of all finite subsets of τ_c by $\tau_c^{<\omega}$.

A strategy for ONE in the compact-clopen game on a space X is a function $\varphi : \tau_c^{<\omega} \to \mathcal{K}$.

A strategy for TWO in the compact-clopen game on a space X is a function $\psi : \mathcal{K}^{<\omega} \to \tau_c$ such that, for all $\langle K_0, K_1, ..., K_n\rangle \in \mathcal{K}^{<\omega} \setminus \{\langle\rangle\}$, we have $K_n \subseteq \psi(\langle K_0, ..., K_n\rangle) = U_n$.

A strategy $\varphi : \tau_c^{<\omega} \to \mathcal{K}$ for ONE in the compact-clopen game on X is a winning strategy for ONE if, for every sequence $\langle U_n : n \in \omega\rangle$ of clopen subsets of a space X such that $\forall n \in \omega$, $K_n = \varphi(\langle U_0, U_1, ..., U_{n-1}\rangle) \subseteq U_n$, we have $X \subseteq \bigcup_{n\in\omega} U_n$.

A strategy $\psi : \mathcal{K}^{<\omega} \to \tau_c$ for TWO in the compact-clopen game on X is a winning strategy for TWO if, for every sequence $\langle K_n : n \in \omega\rangle$ of compact subsets of a space X, we have $X \subseteq \bigcup_{n\in\omega}(\psi(\langle K_0, K_1, ..., K_n\rangle) = U_n)$.

Recall that two games G and $G^{'}$ are equivalent (isomorphic) if

1. ONE has a winning strategy in G if and only if ONE has a winning strategy in $G^{'}$;
2. TWO has a winning strategy in G if and only if TWO has a winning strategy in $G^{'}$.

The proof of the following result easily follows from replacing the open sets with sets of a clopen base of the topological space.

Theorem 3.2. *For a zero-dimensional space, the following statements hold:*

1. *The game $G_{fin}(\mathcal{C_O},\mathcal{C_O})$ is equivalent to the game $G_{fin}(\mathcal{O},\mathcal{O})$.*
2. *The compact-clopen game is equivalent to the compact-open game.*

Recall that a topological space X is *mildly compact*, if every clopen cover of X contains a finite subcover; and *mildly Lindelöf* if every clopen cover has a countable subcover [13]. A space X is a *σ-mildly compact space*, if $X = \bigcup_{i\in\omega} A_i$ where A_i is a mildly compact space for all $i \in \omega$.

Note that the mildly Menger property is stronger than mildly Lindelöf and weaker than σ–mildly compactness.

The *mildly compact-clopen game* on a space X is played according to the following rules :

In each inning $n \in \omega$, ONE picks a mildly compact set $K_n \subseteq X$, and then TWO chooses a clopen set $U_n \subseteq X$ with $K_n \subseteq U_n$. At the end of the play

$$K_0, U_0, K_1, U_1, K_2, U_2, ..., K_n, U_n, ...,$$

the winner is ONE if $X \subseteq \bigcup_{n\in\omega} U_n$, and TWO otherwise.

Theorem 3.3. *For a topological space X the following statements hold:*

1. *If ONE has a winning strategy in the compact-clopen game, then ONE has a winning strategy in the mildly compact-clopen game on X.*

2. *If ONE has a winning strategy in the mildly compact-clopen game, then TWO has a winning strategy in the game $G_{fin}(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$ on X.*
3. *If X is a subset of the irrational numbers (a zero-dimensional second-countable space) and TWO has a winning strategy in the game $G_{fin}(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$, then X is σ-compact.*
4. *If X is σ-compact, then ONE has a winning strategy in the compact-clopen game.*
5. *If ONE has a winning strategy in the game $G_{fin}(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$, then TWO has a winning strategy in the mildly compact-clopen game on X.*
6. *If TWO has a winning strategy in the mildly compact-clopen game, then TWO has a winning strategy in the compact-clopen game.*
7. *If X is a zero-dimensional space and TWO has a winning strategy in the compact-clopen game, then TWO has a winning strategy in the mildly compact-clopen game.*

The following diagrams could be helpful in order to show the big picture where $C.\,CL(X)$ and $MC.\,CL(X)$ are designations for the compact-clopen game and the mildly compact-clopen game, respectively.

$$\begin{array}{ccccc}
X \text{ is } \sigma\text{-compact} & & & & \\
\Downarrow (4) & & & & \\
ONE \uparrow C.\,CL(X) & \underset{(1)}{\Rightarrow} & ONE \uparrow MC.\,CL(X) & \underset{(2)}{\Rightarrow} & TWO \uparrow G_{fin}(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O}) \\
\Downarrow & & \Downarrow & & \Downarrow \\
TWO \not\uparrow C.\,CL(X) & \underset{(6)}{\Rightarrow} & TWO \not\uparrow MC.\,CL(X) & \underset{(5)}{\Rightarrow} & ONE \not\uparrow G_{fin}(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O}).
\end{array}$$

$$TWO \uparrow G_{fin}(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O}) + (X \subseteq \omega^\omega) \underset{(3)}{\Rightarrow} X \text{ is } \sigma\text{-compact}.$$

$$TWO \not\uparrow MC.\,CL(X) + (X \text{ is a zero-dim. space}) \underset{(7)}{\Rightarrow} TWO \not\uparrow C.\,CL(X).$$

Proof. 1. The proof follows from the fact that every compact subset is mildly compact.

2. Consider a winning strategy φ for ONE in the mildly compact-clopen game. To obtain a winning strategy, we use φ for TWO in the game $G_{fin}(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$ on X.

ONE starts $G_{fin}(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$ with his initial move $\mathcal{U}_0$, a cover by clopen sets of X. Then TWO replies with a finite subset $\mathcal{V}_0$ of $\mathcal{U}_0$ such that $K_0 = \varphi(\langle\rangle) \subseteq \bigcup \mathcal{V}_0$.

If ONE plays $\mathcal{U}_n$ in nth inning, then TWO replies with a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $K_n = \varphi(\langle \mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_{n-1}\rangle) \subseteq \bigcup \mathcal{V}_n$.

In the same manner, the sets $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_n, ...$ played by TWO in the play of the game $G_{fin}(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$ are same as played by TWO in the following play of the mildly compact-clopen game on X :

$$\langle K_0 = \varphi(\langle\rangle), \mathcal{V}_0, K_1 = \varphi(\langle \mathcal{V}_0\rangle), \mathcal{V}_1, ..., K_n = \varphi(\langle \mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_{n-1}\rangle), \mathcal{V}_n, ...\rangle.$$

In the above play of the mildly compact-clopen game on X, ONE uses his winning strategy φ, so $\bigcup_{n\in\omega} \bigcup \mathcal{V}_n = X$. This implies that

$$\langle \mathcal{U}_0, \mathcal{V}_0, \mathcal{U}_1, \mathcal{V}_1, ..., \mathcal{U}_n, \mathcal{V}_n, ...\rangle$$

is a play of the $G_{fin}(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$ on X in which TWO has a winning strategy.

3. If TWO has a winning strategy in $G_{fin}(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$, then TWO has a winning strategy in $G_{fin}(\mathcal{O},\mathcal{O})$ by Theorem 3.2. Rest of the proof follows from Theorem 1 [12]

4. The proof is obvious.

5. Consider a winning strategy φ for ONE in $G_{fin}(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$. To obtain a winning strategy, we use φ for TWO in the mildly compact-clopen game on X.

ONE starts the mildly compact-clopen game with his initial move K_0, a mildly compact subset of X. Then TWO replies with $\bigcup \mathcal{V}_0$ containing K_0 such that $\mathcal{V}_0$ is a finite subset of $\mathcal{U}_0 = \varphi(\langle\rangle)$.

If ONE plays K_1 his next move, then TWO replies with $\bigcup \mathcal{V}_1$ containing K_1 such that $\mathcal{V}_1$ is a finite subset of $\mathcal{U}_1 = \varphi(\langle \mathcal{V}_0\rangle)$.

If ONE plays K_2 his next move, then TWO replies with $\bigcup \mathcal{V}_2$ containing K_2 such that $\mathcal{V}_2$ is a finite subset of $\mathcal{U}_2 = \varphi(\langle \mathcal{V}_0, \mathcal{V}_1\rangle)$ and so on.

If ONE plays K_n in nth inning, then TWO replies with $\bigcup \mathcal{V}_n$ containing K_n such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n = \varphi(\langle \mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_{n-1}\rangle)$.

In the same manner, the sets $\bigcup \mathcal{V}_0, \bigcup \mathcal{V}_1, \bigcup \mathcal{V}_2, ..., \bigcup \mathcal{V}_n, ...$ played by TWO in the play of mildly compact-clopen game are same as played by TWO in the following play of the $G_{fin}(C_O, C_O)$ on X :

$$\langle \mathcal{U}_0 = \varphi(\langle\rangle), \mathcal{V}_0, \mathcal{U}_1 = \varphi(\langle \mathcal{V}_0\rangle), \mathcal{V}_1, ..., \mathcal{U}_n = \varphi(\langle \mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_{n-1}\rangle), \mathcal{V}_n, ...\rangle.$$

In the above play of $G_{fin}(C_O, C_O)$ on X, ONE uses his winning strategy φ, so $\bigcup_{n\in\omega} \bigcup \mathcal{V}_n \neq X$. This implies that

$$\langle K_0, \bigcup \mathcal{V}_0, K_1, \bigcup \mathcal{V}_1, ..., K_n, \bigcup \mathcal{V}_n, ...\rangle$$

is a play of the mildly compact-clopen game on X in which TWO has a winning strategy.

6. The proof is obvious.

7. The proof follows from the fact that in a zero-dimensional space, every mildly compact space is compact. □

Corollary 3.4. *For a zero-dimensional separable metric space (X, d), the following statements are equivalent:*

1. *X is σ-compact;*
2. *TWO has a winning strategy in the game $G_{fin}(\mathcal{O}, \mathcal{O})$;*
3. *TWO has a winning strategy in the game $G_{fin}(C_O, C_O)$;*
4. *ONE has a winning strategy in the mildly compact-clopen game;*
5. *ONE has a winning strategy in the compact-clopen game;*
6. *ONE has a winning strategy in the compact-open game.*

4. $\mathcal{K}$-quasi-component-clopen game

Now we consider a new game, namely $\mathcal{K}$-quasi-component-clopen game.

A subset F of a space X is called a *quasi-component of a compact subset* K of X if $F = \bigcap\{U : U$ is clopen in $X, K \subseteq U\}$.

The *$\mathcal{K}$-quasi-component-clopen game* $Q_{\mathcal{K}}C(X)$ on a space X is played according to the following rules :

In each inning $n \in \omega$, ONE picks a quasi-component A_n of a compact subset K_n of X, and then TWO chooses a clopen set $U_n \subseteq X$ with $A_n \subseteq U_n$. At the end of the play

$$A_0, U_0, A_1, U_1, A_2, U_2, ..., A_n, U_n, ...,$$

the winner is ONE if $X \subseteq \bigcup_{n\in\omega} U_n$, and TWO otherwise.

We denote the collection of all quasi-components of compact subsets of a space by $Q_{\mathcal{K}}$ and the collection of all finite subsets of $Q_{\mathcal{K}}$ by $Q_{\mathcal{K}}^{<\omega}$.

A strategy for ONE in the game $Q_{\mathcal{K}}C(X)$ on a space X is a function $\varphi : \tau_c^{<\omega} \to Q_{\mathcal{K}}$.

A strategy for TWO in the game $Q_{\mathcal{K}}C(X)$ on a space X is a function $\psi : Q_{\mathcal{K}}^{<\omega} \to \tau_c$ such that, for all $\langle A_0, A_1, ..., A_n\rangle \in Q_{\mathcal{K}}^{<\omega} \setminus \{\langle\rangle\}$, we have $A_n \subseteq \psi(\langle A_0, ..., A_n\rangle) = U_n$.

A strategy $\varphi : \tau_c^{<\omega} \to Q_{\mathcal{K}}$ for ONE in the game $Q_{\mathcal{K}}C(X)$ on X is a winning strategy for ONE if, for every sequence $\langle U_n : n \in \omega\rangle$ of clopen subsets of a space X such that $\forall n \in \omega$, $A_n = \varphi(\langle U_0, U_1, ..., U_{n-1}\rangle) \subseteq U_n$, we have $X \subseteq \bigcup_{n\in\omega} U_n$. If ONE has a winning strategy in the game $Q_{\mathcal{K}}C(X)$ on X, we write $ONE\uparrow Q_{\mathcal{K}}C(X)$.

A strategy $\psi : Q_{\mathcal{K}}^{<\omega} \to \tau_c$ for TWO in the game $Q_{\mathcal{K}}C(X)$ on X is a winning strategy for TWO if, for every sequence $\langle A_n : n \in \omega\rangle$ of quasi-components of compact subsets of a space X, we have $X \subseteq \bigcup_{n\in\omega}(\psi(\langle A_0, A_1, ..., A_n\rangle) = U_n)$. If TWO has a winning strategy in the game $Q_{\mathcal{K}}C(X)$ on X, we write $TWO\uparrow Q_{\mathcal{K}}C(X)$.

Proposition 4.1. *The compact-clopen game is equivalent to the $\mathcal{K}$-quasi-component-clopen game.*

Proof. Let $\varphi : \tau_c^{<\omega} \to \mathcal{K}$ be a winning strategy for ONE in the compact-clopen game on a space X. Then the function $\psi : \tau_c^{<\omega} \to Q_{\mathcal{K}}$ such that $\psi(\langle U_0, U_1, ..., U_{n-1}\rangle) = Q[\varphi(\langle U_0, U_1, ..., U_{n-1}\rangle)]$ ($Q[K]$ is a quasi-component of $K \in \mathcal{K}$) for every sequence $\langle U_n : n \in \omega\rangle$ of clopen subsets of a space X and $n \in \omega$, is a winning strategy for ONE in the $\mathcal{K}$-quasi-component-clopen game. This follows from the fact that $K_n = \varphi(\langle U_0, U_1, ..., U_{n-1}\rangle) \in Q[K_n] \subseteq U_n$.

Let $\varphi : \tau_c^{<\omega} \to Q_{\mathcal{K}}$ be a winning strategy for ONE in the $\mathcal{K}$-quasi-component-clopen game on a space X. Then the function $\psi : \tau_c^{<\omega} \to \mathcal{K}$ such that $\psi(\langle U_0, U_1, ..., U_{n-1}\rangle) \in \varphi(\langle U_0, U_1, ..., U_{n-1}\rangle)$ for every sequence $\langle U_n : n \in \omega\rangle$ of clopen subsets of a space X and $n \in \omega$, is a winning strategy for ONE in the compact-clopen game. This follows from the fact that if W is a clopen set of X and $K \subseteq W$ then $Q[K] \subseteq W$.

Let $\psi : \mathcal{K}^{<\omega} \to \tau_c$ be a winning strategy for TWO in the compact-clopen game on X. Then the function $\rho : Q_{\mathcal{K}}^{<\omega} \to \tau_c$ such that $\rho(\langle A_0, A_1, ..., A_n\rangle) = \psi(\langle K_0, K_1, ..., K_n\rangle)$ for every sequence $\langle A_n : n \in \omega\rangle$ of quasi-components of compact subsets K_n of a space X and some $K_0, ..., K_n$ that $A_i = Q[K_i]$ for each $i = 0, ..., n$, is a winning strategy for TWO in the $\mathcal{K}$-quasi-component-clopen game.

Let $\psi : Q_{\mathcal{K}}^{<\omega} \to \tau_c$ be a winning strategy for TWO in the $\mathcal{K}$-quasi-component-clopen game on X. Then the function $\rho : \mathcal{K}^{<\omega} \to \tau_c$ such that $\rho(\langle K_0, K_1, ..., K_n\rangle) = \psi(\langle A_0, A_1, ..., A_n\rangle)$ for every sequence $\langle K_n : n \in \omega\rangle$ of points of a space X where $A_i = Q[K_i]$ for each $i = 0, ..., n$, is a winning strategy for TWO in the compact-clopen game. □

Proposition 4.2. *Suppose that X is a union of countably many quasi-components of compact sets. Then TWO has a winning strategy in the game $G_{fin}(\mathcal{C_O}, \mathcal{C_O})$.*

Proof. Let $X = \bigcup_{i\in\omega} Q_{K_i}C(X)$, where $Q_{K_i}C(X)$ is a quasi-component of compact set K_i for each i.

Let ONE starts $G_{fin}(\mathcal{C_O}, \mathcal{C_O})$ with his initial move $\mathcal{U}_0$, a cover by clopen sets of X. Then TWO replies with a finite subset $\mathcal{V}_0$ of $\mathcal{U}_0$ such that $K_1 \subseteq \bigcup \mathcal{V}_0$. Then $Q_{K_1}C(X) \subseteq \bigcup \mathcal{V}_0$.

If ONE plays $\mathcal{U}_1$ his next move, then TWO replies with a finite subset $\mathcal{V}_1$ of $\mathcal{U}_1$ such that $K_2 \subseteq \bigcup \mathcal{V}_1$. Then $Q_{K_2}C(X) \subseteq \bigcup \mathcal{V}_1$.

If ONE plays $\mathcal{U}_2$ his next move, then TWO replies with a finite subset $\mathcal{V}_2$ of $\mathcal{U}_2$ such that $K_3 \subseteq \bigcup \mathcal{V}_2$. Then $Q_{K_3}C(X) \subseteq \bigcup \mathcal{V}_2$ and so on.

If ONE plays $\mathcal{U}_n$ in nth inning, then TWO replies with a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $K_{n+1} \subseteq \bigcup \mathcal{V}_n$. Then $Q_{K_{n+1}}C(X) \subseteq \bigcup \mathcal{V}_n$.

In the same manner, we get a play of the game $G_{fin}(\mathcal{C_O}, \mathcal{C_O})$:

$$\langle \mathcal{U}_0, \mathcal{V}_0, \mathcal{U}_1, \mathcal{V}_1, ..., \mathcal{U}_n, \mathcal{V}_n, ...\rangle.$$

Since $X = \bigcup_{i\in\omega} Q_{K_i}C(X)$, $X = \bigcup_{n\in\omega} \bigcup \mathcal{V}_n$. This completes the proof. □

The following chain of implications always holds. Note that the top equivalence (∗) follows from Telgarsky's equivalence together with Theorem 3.2.

$$\begin{array}{c} X \text{ is a union of countably many quasi-components of compact sets} \\ \Downarrow \\ TWO \uparrow G_{fin}(\mathcal{C_O}, \mathcal{C_O}) \underset{(*)}{\Leftrightarrow} ONE \uparrow \text{the compact-clopen game} \\ \Downarrow \\ ONE \not\uparrow G_{fin}(\mathcal{C_O}, \mathcal{C_O}) \\ \Updownarrow \\ X \text{ has mildly Menger property.} \end{array}$$

Example 4.3. Let $Z = X \times Y$ where X is the one-point compactification of uncountable discrete space D and Y is a connected non-σ-compact space.

Then Z is a quasi-component of compact set $X \times \{y\}$ for some $y \in Y$, but Z is not σ-compact and Z does not consist of countable number of quasi-components.

If possible suppose Z is σ-compact. Then $Z = \cup_{i\in\omega} X_i \times Y_i$, where X_i is compact subset of X and Y_i is a compact subset of Y for each i. This means that $Y = \cup_{i\in\omega} Y_i$ is σ-compact, a contradiction.

For each point (x, y) of $X \times Y$, the quasi-component of (x, y) is $\{x\} \times Y$. Then Z has uncountable number of quasi-components. Since each $\{(x, y)\}$ is compact so it also have uncountable number of quasi-components of compact sets.

Remark 4.4. quasi-component of compact subset B of a zero-dimensional space X is equal to B. It follows that, if X is countable union of quasi-components of compact subsets of X then X is σ-compact.

From Theorem 3.3, we have the following remark.

Remark 4.5. For a zero-dimensional second countable space X, TWO has a winning strategy in the game $G_{fin}(\mathcal{C_O}, \mathcal{C_O})$ if and only if X is a σ-mildly compact space.

5. Determinacy and $G_{fin}(\mathcal{C_O}, \mathcal{C_O})$ game

A game G played between two players ONE and TWO is determined if either ONE has a winning strategy in game G or TWO has a winning strategy in game G. Otherwise G is undetermined.

It can be observed that the game $G_{fin}(\mathcal{C_O}, \mathcal{C_O})$ is determined for every σ–mildly compact space. But in a non σ-mildly compact, mildly Menger and a zero-dimensional metric space, none of the players ONE and TWO have a winning strategy. Since a zero-dimensional metric mildly Menger space is second countable, $G_{fin}(\mathcal{C_O}, \mathcal{C_O})$ is undetermined for a non σ-mildly compact, mildly Menger and a zero-dimensional metric space. Thus every non σ-mildly compact zero-dimensional mildly Menger metric space is undetermined.

Recall that an uncountable set L of reals is a Luzin set if for each meager set M, $L \cap M$ is countable. The Continuum Hypothesis implies the existence of a Luzin set.

Sierpiński showed that Lusin sets of real numbers have the Menger property. Since Lusin sets are not σ-compact they are spaces where neither player has a winning strategy.

Now from Corollary 2 in [12], a Luzin set is an example of a space for which the game $G_{fin}(\mathcal{C_O}, \mathcal{C_O})$ is undetermined.

Then we present several questions, the answers to which will be a natural continuation of the research within the framework of the topic of this paper.

Question 5.1. *Assume that X, Y satisfy $G_{fin}(\mathcal{C_O}, \mathcal{C_O})$ ($S_{fin}(\mathcal{C_O}, \mathcal{C_O})$). Does it follows that $X \times Y$ satisfies $G_{fin}(\mathcal{C_O}, \mathcal{C_O})$ ($S_{fin}(\mathcal{C_O}, \mathcal{C_O})$)?*

Question 5.2. *Is $S_{fin}(\mathcal{C_O}, \mathcal{C_O})$ ($G_{fin}(\mathcal{C_O}, \mathcal{C_O})$ preserved by finite powers?*

Question 5.3. *Are $S_{fin}(\mathcal{C_O}, \mathcal{C_O})$ and $G_{fin}(\mathcal{C_O}, \mathcal{C_O})$ hereditary for subsets representable as a countable union of clopen sets?*

Acknowledgements

The authors would like to thank the referee for careful reading and valuable comments.

The research funding from the Ministry of Science and Higher Education of the Russian Federation (Ural Federal University Program of Development within the Priority-2030 Program) is gratefully acknowledged.

References

[1] L.F. Aurichi, R.R. Dias, A minicourse on topological games, Topology Appl. 258 (2019) 305–335.
[2] E. Borel, Sur la classification des ensembles de mesure nulle, Bull. Soc. Math. France 47 (1919) 97–125.
[3] R. Engelking, General Topology, Revised and completed edition, Heldermann Verlag Berlin (1989).
[4] K.P.Hart, Jun-iti Nagata, J.E.Vaughan (eds.), Encyclopedia of General Topology, Elsevier Science, (2003).
[5] W. Hurewicz, Über eine verallgemeinerung des Borelschen Theorems, Math. Z. 24 (1925) 401–421.
[6] W. Hurewicz, Über Folgen stetiger Funktionen, Fund. Math. 9 (1927) 193–204.
[7] W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki, The combinatorics of open covers (II), Topology Appl. 73 (1996) 241–266.
[8] Lj.D.R. Kočinac, On mildly Hurewicz spaces, Int. Math. Forum. 11 (2016) 573–582.

[9] K. Menger, Einige Überdeckungssätze der punktmengenlehre, Sitzungsberischte Abt. 2a, Mathematik, Astronomie, Physik, Meteorologie und Mechanik (Wiener Akademie, Wien) 133 (1924) 421–444.
[10] A.W. Miller, D.H. Fremlin, Some properties of Hurewicz, Menger and Rothberger, Fund. Math. 129 (1988) 17–33.
[11] M. Scheepers, Combinatorics of open covers (I) : Ramsey theory, Topology Appl. 69 (1996) 31–62.
[12] M. Scheepers, A direct proof of a theorem of Telgársky, Proc. Amer. Math. Soc. 123 (1995) 3483–3485.
[13] R. Staum, The algebra of bounded continuous functions into a nonarchimedean field, Pacific J. Math.,50 (1974) 169–185.
[14] R. Telgársky, On games of Topsoe, Math. Scand. 54 (1984) 170–176.