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Carleson Measures Induced by Higher Order Schwarzian Derivatives and Derivatives of Analytic Functions

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Abstract. Let G be a finitely generated Fuchsian group of the second kind without any parabolic element and f be a univalent analytic function in the unit disk $\mathbb{D}$ compatible with G. In this paper, we study the higher order Schwarzian derivatives: $\sigma_{n+1}(f) = \sigma_n'(f) - (n-1)\frac{f''}{f'} \cdot \sigma_n(f)$, $n \geq 3$, where $\sigma_3(f)$ stands for the Schwarzian derivatives of f, and $S_n(f) = (f')^{\frac{n-1}{2}} D^n (f')^{-\frac{n-1}{2}}$, $n \geq 2$. For $p > 0$, we show that if $|\sigma_n(f)(z)|^p (1-|z|^2)^{p(n-1)-1} dxdy$ (resp. $|S_n(f)(z)|^p (1-|z|^2)^{pn-1} dxdy$) satisfies the Carleson condition on the infinite boundary of the Dirichlet fundamental domain $\mathcal{F}$ of G, then $|\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1} dxdy$ (resp. $|S_n(f)(z)|^p (1-|z|^2)^{pn-1} dxdy$) is a Carleson measure in $\mathbb{D}$. Similarly, for $p > 0$ and a bounded analytic function f in the unit disk $\mathbb{D}$ compatible with G, we prove that if $|f'(z)|^p (1-|z|^2)^{p-1} dxdy$ satisfies the Carleson condition on the infinite boundary of the Dirichlet fundamental domain $\mathcal{F}$ of G, then $|f'(z)|^p(1-|z|^2)^{p-1} dxdy$ is a Carleson measure in $\mathbb{D}$.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane and $\mathbb{S} = \{z : |z| = 1\}$ be the unit circle. We denote by $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$ the closed unit disk.

We say that a positive measure λ on a simply connected domain Ω is a Carleson measure if there exists a $C > 0$ (independent of r) such that for all $0 < r <$ diameter$(\partial\Omega)$ and $z \in \partial\Omega$, $\lambda(\Omega \cap B(z, r)) \leq Cr$, where $B(z, r)$ stands for the disk with center z and radius r. The Carleson norm $\|\lambda\|_\Omega$ of λ is then defined by

$$\|\lambda\|_\Omega = \sup\left\{\frac{\lambda(\Omega \cap B(z,r))}{r} : z \in \partial\Omega, 0 < r < \text{diameter}(\Omega)\right\} < \infty.$$

We denote by $CM(\Omega)$ the set of all Carleson measures on Ω. When $\Omega = \mathbb{D}$, $CM(\Omega)$ corresponds to $CM(\mathbb{D})$.

A Fuchsian group $\mathcal{G}$ is of the first kind if its limit set is $\mathbb{S}$, otherwise it is of the second kind. For a Fuchsian group $\mathcal{G}$, it is cocompact if $\mathbb{D}/\mathcal{G}$ is compact and is convex cocompact if $\mathcal{G}$ is finitely generated without parabolic elements. All cocompact groups are the first kind and convex cocompact groups minus

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cocompact groups are the second kind. A finitely generated Fuchsian group $\mathcal{G}$ is called to be of divergence type if

$$\sum_{g\in\mathcal{G}}(1-|g(0)|)=\infty \text{ or } \sum_{g\in\mathcal{G}}\exp(-\rho(0,g(0)))=\infty,$$

where $\rho(\cdot,\cdot)$ is the hyperbolic distance, and to be of convergence type otherwise. All finitely generated Fuchsian groups of the second kind are of convergence type. For more details on Fuchsian group and finitely Fuchsian group, one refer to see [5, 9].

It is well known that the Schwarzian derivative of a locally univalent analytic function in $\mathbb{D}$ plays a key role in the theory of univalent functions and Teichmüller spaces (see [1–3, 13, 14, 17]). Recall that the Schwarzian derivative of a locally univalent analytic function f in $\mathbb{D}$ is defined as

$$S(f)=(N_f)'-\frac{1}{2}(N_f)^2,$$

where $N_f=(\log f')'$ denotes the pre-Schwarzian derivatives of f. If f is univalent in $\mathbb{D}$, then $|S(f)|(1-|z|^2)^2\leq 6$ for all $z\in\mathbb{D}$ (see [13]).

Following [16], Schippers defined the higher order Schwarzian derivatives of a locally univalent analytic function f in $\mathbb{D}$ as

$$\sigma_{n+1}(f)=\sigma_n'(f)-(n-1)N_f\cdot\sigma_n(f), n\geq 3,$$

where $\sigma_3(f)=S(f)$. It's easy to show that $\sigma_n(f\circ T)=\sigma_n(f)\circ T\cdot(T')^{n-1}$ for all Möbius transformations T. In addition, Schippers [16, Theorem 2] proved that

$$|\sigma_n(f)(z)|(1-|z|^2)^{n-1}\leq 6\cdot 4^{n-3}(n-2)!$$

for all univalent analytic functions f in $\mathbb{D}$. Like Schwarzian derivatives of univalent analytic functions in $\mathbb{D}$, higher order Schwarzian derivatives of them also paly an important role in Teichmüller spaces [7, 10, 18, 19].

Let G be a finitely generated Fuchsian group of the second kind without any parabolic element. Let $\mathcal{F}$ be the Dirichlet fundamental domain of G centered at $z=0$ and $\mathcal{F}(\infty)$ be the boundary at infinity of $\mathcal{F}$. We say that a locally univalent analytic function f compatible with G if and only if $f\circ A=f$ for any $A\in G$. If μ is bounded measurable on $\mathbb{D}$ satisfying $\|\mu\|_\infty<1$ and $\mu(z)=\mu(g(z))\overline{g'(z)}/g'(z)$ for all $g\in G$, then μ is said to be G-compatible Beltrami coefficients. Recently, Huo [11] considered a Beltrami coefficient μ in $\mathbb{D}$ compatible with G and showed that if $|\mu|^2/(1-|z|^2)dxdy$ satisfies the Carleson condition on $\mathcal{F}(\infty)$, then $|\mu|^2/(1-|z|^2)dxdy$ is a Carleson measure in $\mathbb{D}$. It is noticed that $|N_f(z)|^2(1-|z|^2)dxdy$, $|S(f)(z)|^2(1-|z|^2)^3dxdy$ and $|\mu|^2/(1-|z|^2)dxdy$ play a crucial role in the theory of BMO-Teichmüller space [4, 8, 17], where μ denotes the Beltrami coefficient in $\mathbb{D}$. Due to the importance of higher order Schwarzian derivatives $\sigma_n(f)$ in Teichmüller spaces, by the basic properties of $\sigma_n(f)$ and the result by Huo [11], one may naturally ask whether it is right for the higher order Schwarzian derivative $\sigma_n(f)$ of a G-compatible locally univalent analytic function f or not. In fact, we obtain the following result.

In the following, we always suppose that $p>0$ and χ_A is the characteristic function of the set A.

Theorem 1.1. *Suppose that f is a univalent analytic function in $\mathbb{D}$ compatible with G. If there exists a C (>0) such that, for any $\xi\in\mathcal{F}(\infty)$ (i.e. ξ is in the free edges of $\mathcal{F}$) and $0<r<2$,*

$$\iint_{B(\xi,r)}|\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}\chi_{\mathcal{F}}(z)dxdy\leq Cr,\ n\geq 3,$$

then $|\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy\in CM(\mathbb{D})$.

Apart from $\sigma_n(f)$, there are some other kinds of higher order Schwarzian derivatives of a locally univalent function f in $\mathbb{D}$ (see [6, 12]). In [6], Bertilsson considered a general differential operators of univalent analytic functions f in $\mathbb{D}$ as follows:

$$S_n(f)=(f')^{\frac{n-1}{2}}D^n(f')^{-\frac{n-1}{2}},\ n\geq 1.$$

It is easy to see that $S_2(f) = -\frac{1}{2}S(f)$ and $S_n(f \circ \tau) = (S_n(f)) \circ \tau \cdot (\tau')^n$ for all Möbius transformations τ. Bertilsson [6, Theorem 2] proved that if f is univalent analytic in $\mathbb{D}$, then

$$|S_n(f)(z)|(1-|z|^2)^n \leq (n-1)(n+1)(n+3)\cdots(3n-3),\ n \geq 1.$$

Based on the above properties of $S_n(f)$, we consider and obtain the following result.

Theorem 1.2. *Suppose that f is a univalent analytic function in $\mathbb{D}$ compatible with G. If there exists a C (>0) such that, for any $\xi \in \mathcal{F}(\infty)$ and $0 < r < 2$,*

$$\iint_{B(\xi,r)} |S_n(f)(z)|^p(1-|z|^2)^{pn-1}\chi_{\mathcal{F}}(z)dxdy \leq Cr,\ n \geq 2,$$

then $|S_n(f)(z)|^p(1-|z|^2)^{pn-1}dxdy \in CM(\mathbb{D})$.

For univalent analytic functions f in $\mathbb{D}$, it is failed that if substituting N_f for $S(f)$ or $\sigma_n(f)$ in Theorem 1.1. It is natural to raise that whether we can further find another quantity instead of $\sigma_n(f)$ to get a similar result to Theorem 1.1. In the following theorem, we present an affirmative answer to this question.

Theorem 1.3. *Suppose that f is a bounded analytic function in $\mathbb{D}$ compatible with G. If there exists a C (>0) such that, for any $\xi \in \mathcal{F}(\infty)$ and $0 < r < 2$,*

$$\iint_{B(\xi,r)} |f'(z)|^p(1-|z|^2)^{p-1}\chi_{\mathcal{F}}(z)dxdy \leq Cr,$$

then $|f'(z)|^p(1-|z|^2)^{p-1}dxdy \in CM(\mathbb{D})$.

The paper is organized as follows: In Section 2, we give the proofs of Theorems 1.1 and 1.2. In Section 3, we shall prove Theorem 1.3.

2. Carleson measures induced by two kinds of higher order Schwarzian derivatives of univalent analytic functions in $\mathbb{D}$

In this section, we prove Theorem 1.1 and Theorem 1.2. By some lemmas, we first prove Theorem 1.1.

Lemma 2.1. *Suppose that f is a univalent analytic function in $\mathbb{D}$. If $|\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy \in CM(\mathbb{D})$, then there exists C (>0) such that, for any $\xi \in \overline{\mathbb{D}}$ and $0 < r < 2$,*

$$\iint_{B(\xi,r)\cap\mathbb{D}} |\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy \leq Cr,\ n \geq 3,$$

where C depends only on the Carleson norm of $|\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy$ and $\frac{2\pi}{3}[6\cdot 4^{n-3}(n-2)!]^p$.

Proof. Let $r \in (0,2)$. Firstly, if $\xi \in \mathbb{S}$, the result is obvious. Next we suppose that $\xi \in \mathbb{D}$. To complete the proof, we divide it into two cases.

Case 1. If the Euclidean distance $d(\xi,\mathbb{S}) \geq 2r$ (this case only happens when $0 < r < \frac{1}{2}$), by [16, Theorem 2], we have

$$\begin{aligned}\iint_{B(\xi,r)\cap\mathbb{D}} |\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy &= \iint_{B(\xi,r)} |\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy \\ &\leq \iint_{B(\xi,r)} \frac{[6\cdot 4^{n-3}(n-2)!]^p}{1-|z|^2}dxdy \\ &\leq \frac{[6\cdot 4^{n-3}(n-2)!]^p\pi r^2}{1-|1-r|^2} \\ &= \frac{[6\cdot 4^{n-3}(n-2)!]^p\pi r}{2-r} \\ &\leq \frac{2}{3}[6\cdot 4^{n-3}(n-2)!]^p\pi r.\end{aligned}$$

Case 2. $d(\xi,\mathbb{S}) \leq 2r$. Choosing a $\eta \in \mathbb{S}$ such that $d(\eta,\xi) \leq 2r$. It follows that $B(\xi,r) \subset B(\eta,4r)$ and

$$\begin{aligned}\iint_{B(\xi,r)\cap\mathbb{D}} |\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy &\leq \iint_{B(\eta,4r)\cap\mathbb{D}} |\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy \\ &\leq 4C_1 r,\end{aligned}$$

where C_1 is the Carleson norm of the measure $|\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy$.

By setting $C = \max\{\frac{2}{3}[6\cdot 4^{n-3}(n-2)!]^p\pi, 4C_1\}$, we complete the proof. □

Lemma 2.2. *Let G be a convergence-type Fuchsian group and f be a bounded analytic function in $\mathbb{D}$ compatible with G. If there exists $0 < t < 1$ such that the support set of $\sigma_n(f)\chi_{\mathcal{F}}$ is contained in the ball $B(0,t)$, then $|\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy \in CM(\mathbb{D})$, where $n \geq 3$.*

Proof. Suppose that $B(0,t)$ contains the support set of $\sigma_n(f)(z)\chi_{\mathcal{F}}$. A sequence z_j is an interpolating sequence of $\mathbb{D}$, if $\exists\eta > 0$, $\rho(z_j,z_k) \geq \eta$ for $j \neq k$, $\sum(1-|z_j|^2)\delta_{z_i} \in CM(\mathbb{D})$, where δ_z represents the Dirac mass at z. In [11], Huo showed that $\{g(0)\}_{g\in G}$ is an interpolating sequence in $\mathbb{D}$. Then for any $\xi \in \mathbb{S}$ and $0 < r < 2$, by [16, Theorem 2], we have

$$\begin{aligned}M &:= \iint_{\mathbb{D}\cap B(\xi,r)} |\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy \\ &= \sum_{g\in G}\iint_{g(B(0,t))\cap B(\xi,r)} |\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy \\ &= \sum_{g\in G}\iint_{g(B(0,t))} |\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}\chi_{B(\xi,r)}dxdy \\ &\leq \sum_{g\in G}\iint_{g(B(0,t))} \frac{[6\cdot 4^{n-3}(n-2)!]^p}{(1-|z|^2)}\chi_{B(\xi,r)}dxdy.\end{aligned}$$

It is easy to show that the hyperbolic radius t_ρ of the Euclidean disk $B(0,t)$ equals to $\ln\frac{1+t}{1-t}$. Therefore, for any $g \in G$, we deduce that $g(B(0,t))$ is a hyperbolic disk with center $g(0)$ and hyperbolic radius t_ρ. From [5], we obtain that the Euclidean disk $B(g(0),R_g)$ ($R_g \leq C_3(1-|g(0)|)$) contains $g(B(0,t))$, where C_3 depends on t. Consequently,

$$\begin{aligned} M \leq & \sum_{g(0)\in B(\xi,r)} \frac{[6\cdot 4^{n-3}(n-2)!]^p \pi R_g^2}{1-|1-R_g|^2} \\ = & \sum_{g(0)\in B(\xi,r)} \frac{[6\cdot 4^{n-3}(n-2)!]^p \pi R_g}{2-R_g} \\ \leq & C_4 \sum_{g(0)\in B(\xi,r)} (1-|g(0)|) \leq C_0 r, \end{aligned}$$

where C_0 depends on C_3, C_4, and the Carleson norm of $\sum_{g\in G}(1-|g(0)|)\delta_{g(0)}$. $\square$

Recalling that a Jordan curve γ is a chord-arc curve, for any ξ_1, $\xi_2 \in \gamma$, if there is a $C > 0$ such that length$(\gamma_{\xi_1,\xi_2}) \leq Cd(\xi_1,\xi_2)$, where γ_{ξ_1,ξ_2} is the shorter arc of γ with endpoints ξ_1, ξ_2.

Zinsmeister [20] showed the following result.

Lemma 2.3. *[20] If Ω is a chord-arc domain, then the following conditions are equivalent:*

1. *dv is a Carleson measure for Ω.*
2. *For $0 < p < \infty$ and $f \in H^p(\Omega) = \{f : f$ is analytic on Ω and $\int_{\partial\Omega} |f|^p ds < \infty\}$,*

$$\iint_{\Omega} |f|^p dv \leq C \int_{\partial\Omega} |f|^p ds,$$

where the positive constant C only depends on the Carleson norm of dv.

Proof of Theorem 1.1. Let f be a univalent analytic function compatible with G. The intersection of the closure of $\mathcal{F}$ with $\mathbb{S}$ contains finitely many intervals called to be free edges of $\mathcal{F}$, and we denote them by $I_1, I_2, \cdots I_n$. Suppose that $q_{i,1}, q_{i,2}$ are the endpoints of I_i $(1 \leq i \leq n)$. Noting that $q_{i,1}$ and $q_{i,2}$ do not belong to the limit set. Both sides of $q_{i,j}$ ($j = 1$ or 2) are free sides of Dirichlet fundamental domains with different centers.

According to the statement of the theorem, there is a $C > 0$ such that for any $1 \leq i \leq n$, choosing a ball B_i such that $B_i \cap \mathbb{S}$ containing no limit points of G and $I_i \subset B_i \cap \mathbb{S}$ and for arbitrary $\xi \in I_i$, $r \in (0, 2)$,

$$\iint_{B(\xi,r)\cap\mathbb{D}} |\sigma_n(f)(z)|^p (1-|z|^2)^{p(n-1)-1} \chi_{B_i\cap\mathbb{D}} dxdy \leq Cr.$$

It is noticed that $\mathcal{F}_c := \overline{\mathcal{F}} - \bigcup_{i=1}^n (B_i \cap \mathcal{F})$ is compact, where $\overline{\mathcal{F}}$ stands for the closure of $\mathcal{F}$.

In view of Lemma 2.1, $|\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy$ is a Carleson measure on $B_i \cap \mathcal{F}$. For convenience, we divide $\sigma_n(f)$ into two parts. Let $\sigma_n(f) = \sum_{g\in G} \sigma_n(f)\chi_{g(\mathcal{F}_c)} + \sum_{g\in G} \sigma_n(f)\chi_{g(B)}$, where $B = \bigcup_{i=1}^n (B_i \cap \mathcal{F})$. By Lemma 2.2, we deduce that $|\sum_{g\in G} \sigma_n(f)\chi_{g(\mathcal{F}_c)}|^p(1-|z|^2)^{p(n-1)-1}dxdy$ is a Carleson measure on $\mathbb{D}$. Without loss of generality, we may assume $\sigma_n(f) = \sum_{g\in G} \sigma_n(f)\chi_{g(B)}$.

Next we show that $|\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy$ is also a Carleson measure. Let an arbitrary point $\xi \in \mathbb{S}$ and $0 < r < 2$. To do so, we shall prove

$$\iint_{B(\xi,r)\cap\mathbb{D}} |\sigma_n(f)(z)|^p (1-|z|^2)^{p(n-1)-1} dxdy \leq Lr,$$

where $L > 0$ is independent of ξ and r.

We first consider that there is a $g \in G$ such that $g(B(\xi, r) \cap \mathbb{D}) \subset \mathcal{F}$. Using Lemma 2.1, we find that $|\sigma_n(f)(z)|^p(1-|z|^2)^{p(n-1)-1}dxdy$ is a Carleson measure on $g(B(\xi, r) \cap \mathbb{D})$. Noting that g is a Möbius transformation, it follows that $g(B(\xi, r) \cap \mathbb{D})$ is a chord-arc domain. According to Lemma 2.3 and

$$|g'(z)|(1-|z|^2) = 1-|g(z)|^2 \text{ for all } g \in G,$$

a short calculations lead to

$$
\begin{aligned}
&\iint_{B(\xi,r)\cap\mathbb{D}} |\sigma_n(f)(w)|^p(1-|w|^2)^{p(n-1)-1}dudv \\
&= \iint_{g(B(\xi,r)\cap\mathbb{D})} |\sigma_n(f)\circ g^{-1}(z)|^p(1-|g^{-1}(z)|^2)^{p(n-1)-1}|(g^{-1})'(z)|^2dxdy \\
&= \iint_{g(B(\xi,r)\cap\mathbb{D})} |\sigma_n(f)\circ g^{-1}(z)|^p(1-|z|^2)^{p(n-1)-1}|(g^{-1})'(z)|^{p(n-1)+1}dxdy \\
&= \iint_{g(B(\xi,r)\cap\mathbb{D})} |\sigma_n(f\circ g^{-1}(z))|^p(1-|z|^2)^{p(n-1)-1}|(g^{-1})'(z)|dxdy \\
&= \iint_{g(B(\xi,r)\cap\mathbb{D})} |\sigma_n(f(z))|^p(1-|z|^2)^{p(n-1)-1}|(g^{-1})'(z)|dxdy, \\
&\le C_0\int_{\partial g(B(\xi,r)\cap\mathbb{D})} |(g^{-1})'(z)|ds \\
&= C_0\int_{\partial(B(\xi,r)\cap\mathbb{D})} ds \le 2\pi C_0 r,
\end{aligned}
$$

where C_0 depends on the Carleson norm.

For any $1\le i\le n$, by the fact that $B_i\cap\mathbb{S}$ contains no limit points of G and there are finite $g_1,\cdots,g_m\in G$ such that $(B_i\cap\Delta)\subset\bigcup_1^m g_j(\mathcal{F})$, we yield that $|\sigma_n(f(z))|^p(1-|z|^2)^{p(n-1)-1}dxdy$ is a Carleson measure on $B_i\cap\mathbb{D}$. Denote by G^* the set of all $g\in G$ such that $g(B)\cap B(\xi,r)\neq\emptyset$. To complete the proof, we consider three cases as follows.

Case 1: For $g\in G^*$, there exists $1\le i\le n$, $g(B_i\cap\mathcal{F})\subset B(\xi,r)$. Then

$$
\begin{aligned}
&\iint_{g(B_i\cap\mathcal{F})} |\sigma_n(f(w))|^p(1-|w|^2)^{p(n-1)-1}dudv \\
&\le \iint_{g(B_i\cap\mathbb{D})} |\sigma_n(f(w))|^p(1-|w|^2)^{p(n-1)-1}dudv \\
&= \iint_{B_i\cap\mathbb{D}} |\sigma_n(f)\circ g(z)|^p(1-|g(z)|^2)^{p(n-1)-1}|g'(z)|^2dxdy \\
&= \iint_{B_i\cap\mathbb{D}} |\sigma_n(f)\circ g(z)|^p(1-|z|^2)^{p(n-1)-1}|g'(z)|^{p(n-1)+1}dxdy \\
&= \iint_{B_i\cap\mathbb{D}} |\sigma_n(f\circ g(z))|^p(1-|z|^2)^{p(n-1)-1}|g'(z)|dxdy \\
&= \iint_{B_i\cap\mathbb{D}} |\sigma_n(f(z))|^p(1-|z|^p)^{p(n-1)-1}|g'(z)|dxdy \\
&\le C_i\int_{\partial(B_i\cap\mathbb{D})} |g'(z)|ds \qquad \text{(by Lemma 2.3)} \\
&= C_i\int_{\partial g(B_i\cap\mathbb{D})} ds \le \pi C_i\text{length}(g(B_i\cap\partial\mathbb{D})),
\end{aligned}
$$

where C_i depends only on the Carleson norm of $|\sigma_n(f(z))|^p(1-|z|^2)^{p(n-1)-1}dxdy$ on $B_i\cap\mathbb{D}$.

Case 2: For $g \in G^*$, there exists $1 \le i \le n$, $g(B_i) \cap B(\xi, r) \neq \emptyset$ and $g(I_i) \subset B(\xi, r) \cap \mathbb{S}$. Then

$$\begin{aligned}&\iint_{g(B_i\cap\mathcal{F})\cap B(\xi,r)} |\sigma_n(f(w))|^p(1-|w|^2)^{p(n-1)-1}dudv\\ \le &\iint_{g(B_i\cap\mathbb{D})\cap B(\xi,r)} |\sigma_n(f(w))|^p(1-|w|^2)^{p(n-1)-1}dudv\\ \le &\pi C_i \text{length}(B_i \cap \mathbb{S}).\end{aligned}$$

Case 3: For $g \in G^*$, there exists $1 \le i \le n$, $g(B_i) \cap B(\xi, r) \neq \emptyset$ and $g(I_i) \cap B(\xi, r) \cap \mathbb{S} \neq \emptyset$. Notice that $g(B_i \cap \mathbb{D}) \cap B(\xi, r)$ is a triangle with three circle-arc and the angle corresponding to the side $g(B_i \cap \mathbb{S}) \cap B(\xi, r)$ is larger than some constant, it follows that

$$\text{length}(\partial(g(B_i \cap \mathbb{D}) \cap B(\xi, r))) \le C_i' \text{length}(g(B_i \cap \mathbb{S}) \cap B(\xi, r)),$$

where C_i' only depends on the Carleson norm of $|\sigma_n(f)|^p(1-|z|^2)^{p(n-1)-1}dxdy$ on $B_i \cap \mathbb{D}$ and the angle between ∂B_i and $\mathbb{S}$.

Similar to Case 1, we deduce

$$\iint_{g(B_i\cap\mathcal{F})\cap B(\xi,r)} |\sigma_n(f(w))|^p(1-|w|^2)^{p(n-1)-1}dudv \le \pi C_i' \text{length}(g(B_i \cap \mathbb{S}) \cap B(\xi, r)).$$

The limit points of G do not belong to the arc $B_i \cap \mathbb{S}$ ($1 \le i \le n$). For any g_1 and g_2 belong to G^*, if $g_1(B_i) \cap B(\xi, r) \neq \emptyset$ and $g_2(B_i) \cap B(\xi, r) \neq \emptyset$, we can obtain that $g_1(B_i \cap \mathbb{S})$ and $g_2(B_i \cap \mathbb{S})$ do not overlap. Consequently

$$\begin{aligned}\iint_{B(\xi,r)\cap\mathbb{D}} |\sigma_n(f(w))|^p(1-|w|^2)^{p(n-1)-1}dudv &\le \pi C^* \sum_{g\in G^*} \text{length}(g(B) \cap B(\xi, r) \cap \mathbb{S})\\ &\le \pi C^* \text{length}(B(\xi, r) \cap \mathbb{S}) \le 2\pi^2 C^* r,\end{aligned}$$

where C^* denotes the maximum value of all constants in the above proof and $B = \bigcup_i^n (B_i \cap \mathbb{D})$. The proof is completed. □

Employing the methods in the proofs of Lemma 2.1 and Lemma 2.2, we can easily prove the following lemmas, here we omit their details.

Lemma 2.4. *Suppose that f is a univalent analytic function in $\mathbb{D}$. If $|S_n(f)(z)|^p(1-|z|^2)^{pn-1}dxdy \in CM(\mathbb{D})$, then there exists a C (> 0) such that, for any $\xi \in \overline{\mathbb{D}}$ and $0 < r < 2$,*

$$\iint_{B(\xi,r)\cap\mathbb{D}} |S_n(f)(z)|^p(1-|z|^2)^{p-1}dxdy \le Cr,\ n \ge 2,$$

where the constant C depends only on the Carleson norm of $|S_n(f)(z)|^p(1-|z|^2)^{pn-1}dxdy$ and $[(n-1)(n+1)(n+3) \cdot \cdot \cdot (3n-3)]^p$.

Lemma 2.5. *Let G be a convergence-type Fuchsian group and f be a bounded analytic function in $\mathbb{D}$ compatible with G. If there exists $0 < t < 1$ such that the support set of $S_n(f)\chi_{\mathcal{F}}$ is contained in the ball $B(0, t)$, then $|S_n(f)(z)|^p(1-|z|^2)^{pn-1}dxdy \in CM(\mathbb{D})$, where $n \ge 2$.*

Proof of Theorem 1.2. Similar to the proof of Theorem 1.1, employing the method in the proof of Theorem 1.1, we can prove Theorem 1.2, here we omit its details. □

3. Carleson measures induced by derivatives of bounded analytic functions in $\mathbb{D}$

To prove Theorem 1.3, we need the following lemmas. It is noticed that for a bounded analytic function f in $\mathbb{D}$, by [15, Chapter 4], we know that $\sup_{z\in\mathbb{D}}|f'(z)|(1-|z|^2)<\infty$, $z\in\mathbb{D}$. Combining the methods in the proofs of Lemma 2.1 and Lemma 2.2, we can easily prove the following lemmas, here we omit their details.

Lemma 3.1. *Suppose that f is a bounded analytic function in $\mathbb{D}$. If $|f'(z)|^p(1-|z|^2)^{p-1}dxdy\in CM(\mathbb{D})$, then there exists a C (>0) such that, for any $\xi\in\overline{\mathbb{D}}$ and $0<r<2$,*

$$\iint_{B(\xi,r)\cap\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p-1}dxdy\le Cr,$$

where the constant C depends only on the Carleson norm of $|f'(z)|^p(1-|z|^2)^{p-1}dxdy$ and $[\sup_{z\in\mathbb{D}}|f'(z)|(1-|z|^2)]^p$.

Lemma 3.2. *Let G be a convergence-type Fuchsian group and f be a bounded analytic function in $\mathbb{D}$ compatible with G. If there exists $0<t<1$ such that the support set of $f'\chi_{\mathcal{F}}$ is contained in $B(0,t)$, then $|f'(z)|^p(1-|z|^2)^{p-1}dxdy\in CM(\mathbb{D})$.*

Proof of Theorem 1.3. Let f be a bounded analytic function compatible with G. The intersection of the closure of $\mathcal{F}$ with $\mathbb{S}$ contains finitely many intervals which are called free edges of $\mathcal{F}$, denoted by $I_1, I_2, \cdots I_n$. Suppose that $q_{i,1}, q_{i,2}$ are the endpoints of I_i, $1\le i\le n$. It is noticed that $q_{i,1}$ and $q_{i,2}$ do not belong to the limit set. Moreover, both sides of $q_{i,j}$ ($j=1$ or 2) are free sides of Dirichlet fundamental domains with different centers.

According to the statement of the theorem, there is a C (>0) such that for any $1\le i\le n$, choosing a ball B_i such that $B_i\cap\mathbb{S}$ containing no limit points of G and $I_i\subset B_i\cap\mathbb{S}$ and for arbitrary $\xi\in I_i$, $r\in(0,2)$,

$$\iint_{B(\xi,r)\cap\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p-1}\chi_{B_i\cap\mathbb{D}}dxdy\le Cr.$$

Furthermore, the set $\mathcal{F}_c:=\overline{\mathcal{F}}-\bigcup_{i=1}^n(B_i\cap\mathcal{F})$ is compact, where $\overline{\mathcal{F}}$ denotes the closure of $\mathcal{F}$.

In view of Lemma 3.1, $|f'(z)|^p(1-|z|^2)^{p-1}dxdy$ is a Carleson measure on $B_i\cap\mathcal{F}$. Suppose that ξ is an arbitrary point on $\mathbb{S}$ and $0<r<2$. We divide f' into two parts f_1', f_2' as follows: $f_1'=\sum_{g\in G}f'\chi_{g(\mathcal{F}_c)}$, $f_2'=\sum_{g\in G}f'\chi_{g(B)}$, where $B=\bigcup_{i=1}^n(B_i\cap\mathcal{F})$. By Lemma 3.2, we know that the measure $|f_1'(z)|^2(1-|z|^2)dxdy$ is a Carleson measure on $\mathbb{D}$.

Next, we show that $|f_2'(z)|^p(1-|z|^2)^{p-1}dxdy$ is also a Carleson measure. We first consider the case that there exists a $g\in G$ such that $g(B(\xi,r)\cap\mathbb{D})\subset\mathcal{F}$. Using Lemma 3.1, we know that $|f_2'(z)|^p(1-|z|^2)^{p-1}dxdy$ is a Carleson measure on the domain $g(B(\xi,r)\cap\mathbb{D})$. Noting that g is a Möbus transformation, it follows that $g(B(\xi,r)\cap\mathbb{D})$ is a chord-arc domain. By Lemma 2.3 and $|(g^{-1})'(z)|(1-|z|^2)=1-|g^{-1}(z)|^2$ for all $g^{-1}\in G$, we have

$$\begin{aligned}
&\iint_{B(\xi,r)\cap\mathbb{D}}|f_2'(w)|^p(1-|w|^2)^{p-1}dudv\\
&=\iint_{g(B(\xi,r)\cap\mathbb{D})}|f_2'(g^{-1}(z))|^p(1-|g^{-1}(z)|^2)^{p-1}|(g^{-1})'(z)|^2dxdy\\
&=\iint_{g(B(\xi,r)\cap\mathbb{D})}|f_2'(g^{-1}(z))|^p(1-|z|^2)^{p-1}|(g^{-1})'(z)|^{p+1}dxdy\\
&=\iint_{g(B(\xi,r)\cap\mathbb{D})}|f_2'(z)|^p(1-|z|^2)^{p-1}|(g^{-1})'(z)|dxdy\\
&\le C_0\int_{\partial g(B(\xi,r)\cap\mathbb{D})}|(g^{-1})'(z)|ds\\
&=C_0\int_{\partial(B(\xi,r)\cap\mathbb{D})}ds\\
&\le 2\pi C_0 r,
\end{aligned}$$

where C_0 depends on the Carleson norm.

For any $1 \leq i \leq n$, since $B_i \cap \partial\mathbb{D}$ contains no limit points of G and there are finitely many $g_1, \cdots, g_m$ belonging to G such that $(B_i \cap \Delta) \subset \bigcup_1^m g_j(\mathcal{F})$, we yield that $|f_2'(z)|^p(1-|z|^2)^{p-1}dxdy$ is a Carleson measure on $B_i \cap \mathbb{D}$. Let G^* be the set of all the elements g in G such that $g(B) \cap B(\xi, r) \neq \emptyset$. To complete the proof, we consider three cases as follows.

Case 1: For $g \in G^*$, there exists $1 \leq i \leq n$, $g(B_i \cap \mathcal{F}) \subset B(\xi, r)$. Then

$$\begin{aligned}\iint_{g(B_i\cap\mathcal{F})} |f_2'(w)|^p(1-|w|^2)^{p-1}dudv &\leq \iint_{g(B_i\cap\mathbb{D})} |f_2'(w)|^p(1-|w|^2)^{p-1}dudv \\ &= \iint_{B_i\cap\mathbb{D}} |f_2'(z)|^p(1-|z|^2)^{p-1}|g'(z)|dxdy \\ &\leq C_i \int_{\partial(B_i\cap\mathbb{D})} |g'(z)|ds \qquad \text{(by Lemma 2.3)} \\ &= C_i \int_{\partial g(B_i\cap\mathbb{D})} ds \leq \pi C_i \text{length}(g(B_i \cap \mathbb{S})),\end{aligned}$$

where C_i depends only on the Carleson norm of $|f_2'(z)|^p(1-|z|^2)^{p-1}dxdy$ on $B_i \cap \mathbb{D}$.

The proofs of Case 2 and Case 3 are similar to the corresponding proofs in Theorem 1.1, here we omit the details. The proof is completed. □

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References

[1] L. Ahlfors, Quasiconformal reflections, Acta Math. 109 (1963) 291-301.
[2] L. Ahlfors, Sufficient conditions for quasiconformal extension, Ann. of Math. Stud. 79 (1974) 23-29.
[3] L. Ahlfors, G. Weill, A uniqueness theorem for Beltrami equations, Proc. Amer. Math. Soc. 13 (1962) 975-978.
[4] K. Astala, M. Zinsmeister, Teichmüller spaces and BMOA, Math. Ann. 289 (1991) 613-625.
[5] A. F. Beardon, The geometry of discrete group, Springer-Verlag, 1983.
[6] D, Bertilsson, Coefficient estimates for negative powers of the derivative of univalent functions, Ark. Mat. 36 (1998) 255-273.
[7] G. Buss, Higher Bers maps, Asian J. Math. **16** (2012), 103-140.
[8] G. Cui, M. Zinsmeister, BMO Teichmüller spaces, Illinois J. Math. 48 (2004) 1223-1233.
[9] F. Dal'Bo, Geodesic and horocyclic Trajectories, Springer-Verlag, 2011.
[10] G. M. Hu, Y. T. Liu, Y. Qi, Q. T. Shi, Morrey type Teichmüller space and higher Bers maps, J. Math. Ineq. 14 (2020) 781-804.
[11] S. J. Huo, On Carleson measures induced by Beltrami coefficients being compatible with Fuchsian groups, Ann. Fenn. Math. 46 (2021) 67-77.
[12] S. L. Krushkal, Differential operators and univalent functions, Complex var. Theory Appl. 7 (1986) 107-127.
[13] O. Lehto, Univalent functions and Teichmüller spaces, New York: Springer-Verlag, 1987.
[14] Z. Nehari, The Schwarzian derivative and schlicht functions, Bull. Amer. Math. Soc. 55 (1949) 545-551.
[15] C. Pommerenke, Boundary behaviour of conformal maps, Berlin: Springer-Verlag, 1991.
[16] E. Schippers, Distorion theorems for higher order Schwarzian derivatives of univalent functions, Proc. Amer. Math. Soc. 128 (2000) 3241-3249.
[17] Y. L. Shen, H. Y. Wei, Universal Teichmüller space and BMO, Adv. Math. 234 (2013) 129-148.
[18] S. A. Tang, G. M. Hu, Q. T. Shi, J. J. Jin, Higher Schwarzian derivative and Dirichlet Morrey space, Filomat 33 (2019) 5489-5498.
[19] S. A. Tang, J. J. Jin, Higher Bers maps and BMO-Teichmüller space, J. Math. Anal. Appl. 460 (2018) 63-75.
[20] M. Zinsmeister, Les domaines de Carleson, Michigan Math. J. 36 (1989) 213-220.