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Strongly Semi-Continuous Functions and δ-Stratifiable Spaces

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Abstract. In [6], a sufficient condition for a space to be δ-stratifiable was presented and it was asked whether the condition is necessary. Jin et al [4] gave a negative answer to the question by showing that a space with the condition is zero-dimensional. In this paper, we show that a space with the condition is precisely an almost discrete space. Moreover, we introduce the notions of strongly lower (upper) semi-continuous functions, with which the characterizations of δ-stratifiable spaces are presented.

1. Introduction and preliminaries

Throughout, a space always means a topological space. For a space X, denote by τ and τ^c the topology of X and the family of all closed subsets of X respectively. For a subset A of a space X, we write $\overline{A}$ ($intA$) for the closure (interior) of A in X. Also, we use χ_A to denote the characteristic function of A. The set of all positive integers (real numbers) is denoted by $\mathbb{N}$ ($\mathbb{R}$).

A real-valued function f on a space X is called lower (upper) semi-continuous [2] if for any real number r, the set $\{f > r\} = \{x \in X : f(x) > r\}$ ($\{f < r\} = \{x \in X : f(x) < r\}$) is open. We write $L(X)$ ($U(X)$) for the set of all lower (upper) semi-continuous functions on X and $L^+(X) = \{f \in L(X) : f \geq 0\}$, $U^+(X) = \{f \in U(X) : f \geq 0\}$. $C(X)$ is the set of all continuous functions on X and $C^+(X) = \{f \in C(X) : f \geq 0\}$.

A subset D of a space X is called a regular G_δ-set [7] if there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X such that $D = \bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} \overline{U_n}$. The collection of all regular G_δ-sets of X is denoted by $RG(X)$.

A space X is called δ-normal [7] if every pair of disjoint closed sets, one of which is a regular G_δ-set can be separated by open sets. X is called weakly δ-normal [5] if every pair of disjoint regular G_δ-sets can be separated by open sets. In [3], the notions of lmδn spaces and m$\delta\delta$n spaces were introduced as the monotone versions of δ-normal spaces and weakly δ-normal spaces respectively. In the same paper, the notions of δ-stratifiable spaces and δ-semi-stratifiable were also introduced as the generalizations of stratifiable spaces and semi-stratifiable spaces respectively.

For a space X, consider the following condition.

($*$) There is an order-preserving map $\phi : U^+(X) \to L^+(X)$ such that $\phi(h) \leq h$ for each $h \in U^+(X)$ and $0 < \phi(h)(x) < h(x)$ whenever $h(x) > 0$.

In [6], it was shown that if X satisfies ($*$), then X is δ-stratifiable. It was also asked whether the converse is true. That is, does a δ-stratifiable space satisfy ($*$)? Jin et al [4] showed that if X satisfies ($*$), then X is

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zero-dimensional. Since $\mathbb{R}$ is δ-stratifiable but not zero-dimensional, the answer to the above question is negative. Now, the following two questions arise naturally.

Question 1.1. *What space does condition* $(*)$ *characterize?*

Question 1.2. *How to characterize δ-stratifiable spaces with real-valued functions?*

In this paper, we shall answer Question 1.1 by showing that a space satisfying $(*)$ is precisely an almost discrete space. To answer Question 1.2, we introduce the notions of strongly lower (upper) semi-continuous functions, with which the characterizations of δ-stratifiable spaces are obtained.

Definition 1.3. A space X is called almost discrete if every open subset of X is closed. That is, $\tau = \tau^c$.

Notice that a space is discrete if and only if it is T_0 and almost discrete. Indeed, necessity is clear. Suppose that X is T_0 and almost discrete. Let $x \in X$ and $y \notin \{x\}$. Then $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. If $x \notin \overline{\{y\}}$ then $\overline{\{y\}} \cap \{x\} = \emptyset$ and $\overline{\{y\}}$ is an open neighborhood of y. If $y \notin \overline{\{x\}}$ then $X \setminus \overline{\{x\}}$ is an open neighborhood of y and $(X \setminus \overline{\{x\}}) \cap \{x\} = \emptyset$. This implies that $\{x\}$ is closed and thus open. Therefore, X is discrete.

Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then (X, τ) is almost discrete but not discrete.

Definition 1.4. A real-valued function f on a space X is called strongly lower (upper) semi-continuous if there exist two sequences $\{\alpha_n(f) \in L(X) : n \in \mathbb{N}\}$ and $\{\beta_n(f) \in U(X) : n \in \mathbb{N}\}$ of functions such that $\alpha_n(f) \leq \beta_n(f)$ for each $n \in \mathbb{N}$ and $f = \sup_n \alpha_n(f) = \sup_n \beta_n(f)$ ($f = \inf_n \alpha_n(f) = \inf_n \beta_n(f)$).

$\{\alpha_n(f) \in L(X) : n \in \mathbb{N}\}$ and $\{\beta_n(f) \in U(X) : n \in \mathbb{N}\}$ in the above definition will be called accompanying functions for f.

The collection of all strongly lower (upper) semi-continuous functions on a space X is denoted by $SL(X)$ ($SU(X)$) and $SL^+(X) = \{f \in SL(X) : f \geq 0\}$, $SU^+(X) = \{f \in SU(X) : f \geq 0\}$.

Notice that $SL(X) \subset L(X)$, $SU(X) \subset U(X)$ and $C(X) = SL(X) \cap SU(X)$.

It is known that a space X is perfectly normal if and only if for each $F \in \tau^c$, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X such that $F = \bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} \overline{U_n}$. That is, $\tau^c \subset RG(X)$. Hence, by Lemma 2.8 in Section 2, X is perfectly normal if and only if for each $F \in \tau^c$, $\chi_F \in SU(X)$. Therefore, for a space X which is not perfectly normal, there must exist an $F \in \tau^c$ such that $\chi_F \notin SU(X)$ while $\chi_F \in U(X)$. Dually, there exists a $U \in \tau$ such that $\chi_U \notin SL(X)$ while $\chi_U \in L(X)$.

Definition 1.5. ([3]) A space X is called δ-stratifiable (δ-semi-stratifiable) if there exists a map $\sigma : \mathbb{N} \times RG(X) \to \tau$ such that

(1) $D = \bigcap_{n\in\mathbb{N}} \sigma(n, D) = \bigcap_{n\in\mathbb{N}} \overline{\sigma(n, D)}$ ($D = \bigcap_{n\in\mathbb{N}} \sigma(n, D)$) for each $D \in RG(X)$.

(2) If $D, E \in RG(X)$ and $D \subset E$ then $\sigma(n, D) \subset \sigma(n, E)$ for each $n \in \mathbb{N}$.

Notice that, without loss of generality, we may assume that σ is decreasing with respect to n.

2. Some basic lemmas

In this section, we list some properties of strongly lower (upper) semi-continuous functions and some basic lemmas for later use.

First notice that if $f \in SL(X)$ ($f \in SU(X)$) then $rf \in SL(X)$ ($rf \in SU(X)$) whenever $r > 0$, and $rf \in SU(X)$ ($rf \in SL(X)$) whenever $r < 0$.

Proposition 2.1. *If $f \in SL(X)$ ($f \in SU(X)$) then there exist two increasing (decreasing) sequences $\{\delta_n(f) \in L(X) : n \in \mathbb{N}\}$ and $\{\eta_n(f) \in U(X) : n \in \mathbb{N}\}$ of functions such that $\delta_n(f) \leq \eta_n(f)$ for each $n \in \mathbb{N}$ and $f = \sup_n \delta_n(f) = \sup_n \eta_n(f)$ ($f = \inf_n \delta_n(f) = \inf_n \eta_n(f)$).*

Proof. Let $f \in SL(X)$ be accompanied by $\{\alpha_n(f) \in L(X) : n \in \mathbb{N}\}$ and $\{\beta_n(f) \in U(X) : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $\delta_n(f) = \max\{\alpha_i(f) : i \leq n\}$ and $\eta_n(f) = \max\{\beta_i(f) : i \leq n\}$. Then $\{\delta_n(f) \in L(X) : n \in \mathbb{N}\}$, $\{\eta_n(f) \in U(X) : n \in \mathbb{N}\}$ are increasing, $\delta_n(f) \leq \eta_n(f)$ for each $n \in \mathbb{N}$ and $\sup_n \delta_n(f) = \sup_n \alpha_n(f)$, $\sup_n \eta_n(f) = \sup_n \beta_n(f)$.

For $f \in SU(X)$ applying the fact that if $f \in SU(X)$ then $-f \in SL(X)$. □

Proposition 2.2. *If $f, g \in SL(X)$ ($f, g \in SU(X)$) then $f + g \in SL(X)$ ($f + g \in SU(X)$).*

Proof. Let $f, g \in SL(X)$ be accompanied by $\{\alpha_n(f) \in L(X) : n \in \mathbb{N}\}$, $\{\beta_n(f) \in U(X) : n \in \mathbb{N}\}$ and $\{\alpha_n(g) \in L(X) : n \in \mathbb{N}\}$, $\{\beta_n(g) \in U(X) : n \in \mathbb{N}\}$ respectively. By Proposition 2.1, we may assume that the four accompanying sequences are increasing. Hence, $f = \lim_n \alpha_n(f) = \lim_n \beta_n(f)$ and $g = \lim_n \alpha_n(g) = \lim_n \beta_n(g)$. It follows that $f + g = \lim_n(\alpha_n(f) + \alpha_n(g)) = \lim_n(\beta_n(f) + \beta_n(g))$. This in turn implies that $f + g = \sup_n(\alpha_n(f) + \alpha_n(g)) = \sup_n(\beta_n(f) + \beta_n(g))$, in which $\alpha_n(f) + \alpha_n(g) \in L(X)$, $\beta_n(f) + \beta_n(g) \in U(X)$ and $\alpha_n(f) + \alpha_n(g) \leq \beta_n(f) + \beta_n(g)$ for each $n \in \mathbb{N}$.

For $f, g \in SU(X)$ applying the fact that $f \in SU(X)$ if and only if $-f \in SL(X)$. □

Proposition 2.3. *Let $f, g \in SL(X)$ and f be accompanied by $\{\alpha_n(f) \in L(X) : n \in \mathbb{N}\}$, $\{\beta_n(f) \in U(X) : n \in \mathbb{N}\}$. If $f \leq g$ then there exist accompanying functions $\{\delta_n(g) \in L(X) : n \in \mathbb{N}\}$, $\{\eta_n(g) \in U(X) : n \in \mathbb{N}\}$ for g such that $\alpha_n(f) \leq \delta_n(g)$ and $\beta_n(f) \leq \eta_n(g)$ for each $n \in \mathbb{N}$.*

Proof. Let g be accompanied by $\{\alpha_n(g) \in L(X) : n \in \mathbb{N}\}$, $\{\beta_n(g) \in U(X) : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $\delta_n(g) = \max\{\alpha_n(f), \alpha_n(g)\}$ and $\eta_n(g) = \max\{\beta_n(f), \beta_n(g)\}$. Then $\delta_n(g) \in L(X)$, $\eta_n(g) \in U(X)$ and $\delta_n(g) \leq \eta_n(g)$, $\alpha_n(f) \leq \delta_n(g)$, $\beta_n(f) \leq \eta_n(g)$. It is clear that $\sup_n \alpha_n(g) \leq \sup_n \delta_n(g)$. Assume that $\sup_n \alpha_n(g)(x) < \sup_n \delta_n(g)(x)$ for some $x \in X$. Then $\sup_n \alpha_n(g)(x) < \delta_m(g)(x) = \max\{\alpha_m(f)(x), \alpha_m(g)(x)\}$ for some $m \in \mathbb{N}$. It follows that $g(x) = \sup_n \alpha_n(g)(x) < \alpha_m(f)(x) \leq \sup_n \alpha_n(f)(x) = f(x)$, a contradiction to $f \leq g$. This implies that $\sup_n \alpha_n(g) = \sup_n \delta_n(g)$. Similarly, $\sup_n \beta_n(g) = \sup_n \eta_n(g)$. Therefore, $g = \sup_n \delta_n(g) = \sup_n \eta_n(g)$. □

Similarly, let $f, g \in SU(X)$ and g be accompanied by $\{\alpha_n(g) \in L(X) : n \in \mathbb{N}\}$, $\{\beta_n(g) \in U(X) : n \in \mathbb{N}\}$. If $f \leq g$ then there exist accompanying functions $\{\delta_n(f) \in L(X) : n \in \mathbb{N}\}$, $\{\eta_n(f) \in U(X) : n \in \mathbb{N}\}$ for f such that $\delta_n(f) \leq \alpha_n(g)$ and $\eta_n(f) \leq \beta_n(g)$ for each $n \in \mathbb{N}$.

Notice that if $f \in SU^+(X)$ then the accompanying functions $\{\alpha_n(f) : n \in \mathbb{N}\}$ and $\{\beta_n(f) : n \in \mathbb{N}\}$ are non-negative. As for $f \in SL^+(X)$, by Proposition 2.3, we may also assume that the accompanying functions are non-negative.

Corollary 2.4. *If $f \in SL^+(X)$ then there exist two sequences $\{\alpha_n(f) \in L^+(X) : n \in \mathbb{N}\}$ and $\{\beta_n(f) \in U^+(X) : n \in \mathbb{N}\}$ of functions such that $\alpha_n(f) \leq \beta_n(f)$ for each $n \in \mathbb{N}$ and $f = \sup_n \alpha_n(f) = \sup_n \beta_n(f)$.*

Proposition 2.5. *If $f, g \in SL(X)$ ($SU(X)$) then $\min\{f, g\}, \max\{f, g\} \in SL(X)$ ($SU(X)$).*

Proof. We shall show that if $f, g \in SL(X)$ then $\min\{f, g\} \in SL(X)$. The others can be shown analogously.

Let $f, g \in SL(X)$ be accompanied by increasing sequences $\{\alpha_n(f) \in L(X) : n \in \mathbb{N}\}$, $\{\beta_n(f) \in U(X) : n \in \mathbb{N}\}$ and $\{\alpha_n(g) \in L(X) : n \in \mathbb{N}\}$, $\{\beta_n(g) \in U(X) : n \in \mathbb{N}\}$ respectively. It is easy to verify that $\min\{\sup_n \alpha_n(f), \sup_n \alpha_n(g)\} = \sup_n \min\{\alpha_n(f), \alpha_n(g)\}$ and $\min\{\sup_n \beta_n(f), \sup_n \beta_n(g)\} = \sup_n \min\{\beta_n(f), \beta_n(g)\}$. Therefore, $\min\{f, g\} = \sup_n \min\{\alpha_n(f), \alpha_n(g)\} = \sup_n \min\{\beta_n(f), \beta_n(g)\}$ which implies that $\min\{f, g\} \in SL(X)$. □

Proposition 2.6. *If $f_n \in SL(X)$ ($f_n \in SU(X)$) for each $n \in \mathbb{N}$ and $\sup_n f_n$ ($\inf_n f_n$) exists then $\sup_n f_n \in SL(X)$ ($\inf_n f_n \in SU(X)$).*

Proof. For each $n \in \mathbb{N}$, let $f_n \in SL(X)$ be accompanied by $\{\alpha_m(f_n) \in L(X) : m \in \mathbb{N}\}$, $\{\beta_m(f_n) \in U(X) : m \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $g_n = \max\{\alpha_i(f_j) : i, j \leq n\}$ and $h_n = \max\{\beta_i(f_j) : i, j \leq n\}$. Then $g_n \in L(X)$, $h_n \in U(X)$ and $g_n \leq h_n$. It is easy to verify that $\sup_n \sup_m \alpha_m(f_n) = \sup_n g_n$ and $\sup_n \sup_m \beta_m(f_n) = \sup_n h_n$. Therefore, $\sup_n f_n = \sup_n g_n = \sup_n h_n$ which implies that $\sup_n f_n \in SL(X)$.

For $f_n \in SU(X)$ applying the fact that $f \in SU(X)$ if and only if $-f \in SL(X)$. □

Corollary 2.7. *Let $f_n \in SL^+(X)$ for each $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} f_n$ exists then $\sum_{n=1}^{\infty} f_n \in SL(X)$.*

Proof. For each $n \in \mathbb{N}$, let $g_n = \sum_{i=1}^{n} f_i$. Then $g_n \in SL^+(X)$ and $\sum_{n=1}^{\infty} f_n = \sup_n g_n$. By Proposition 2.6, $\sum_{n=1}^{\infty} f_n \in SL(X)$. □

Lemma 2.8. *Let $D \subset X$. Then $D \in RG(X)$ if and only if $\chi_D \in SU(X)$.*

Proof. Let $D \in RG(X)$. Then there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X such that $D = \bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} \overline{U_n}$. For each $n \in \mathbb{N}$, let $f_n = \chi_{U_n}$ and $g_n = \chi_{\overline{U_n}}$. Then $f_n \in L(X)$, $g_n \in U(X)$ and $f_n \le g_n$. It is clear that $\chi_D = \inf_n f_n = \inf_n g_n$.

Let $\chi_D \in SU(X)$ be accompanied by $\{\alpha_n(\chi_D) \in L(X) : n \in \mathbb{N}\}$ and $\{\beta_n(\chi_D) \in U(X) : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $U_n = \{\alpha_n(\chi_D) > \frac{1}{2}\}$ and $F_n = \{\beta_n(\chi_D) \ge \frac{1}{2}\}$. Then U_n is open, F_n is closed and $U_n \subset F_n$ which implies that $\overline{U_n} \subset F_n$. If $x \in D$ then $\inf_n \alpha_n(\chi_D)(x) = \chi_D(x) = 1$ which implies that $x \in U_n$ for each $n \in \mathbb{N}$. If $x \in \bigcap_{n\in\mathbb{N}} \overline{U_n}$ then $x \in \overline{U_n} \subset F_n$ for each $n \in \mathbb{N}$ and thus $\chi_D(x) = \inf_n \beta_n(\chi_D)(x) \ge \frac{1}{2}$. It follows that $x \in D$. Therefore, $D = \bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} \overline{U_n}$ which implies that $D \in RG(X)$. □

Lemma 2.9. *Let $D \subset X$. Then $D \in RG(X)$ if and only if there exists $f \in SL^+(X)$ such that $D = f^{-1}(0)$.*

Proof. Let $D \in RG(X)$ and $f = 1 - \chi_D$. By Lemma 2.8, $f \in SL^+(X)$. It is clear that $D = f^{-1}(0)$.

Suppose that $D = f^{-1}(0)$ for some $f \in SL^+(X)$. By Proposition 2.1 and Corollary 2.4, there exist two increasing sequences $\{\alpha_n(f) \in L^+(X) : n \in \mathbb{N}\}$ and $\{\beta_n(f) \in U^+(X) : n \in \mathbb{N}\}$ of functions such that $\alpha_n(f) \le \beta_n(f)$ for each $n \in \mathbb{N}$ and $f = \sup_n \alpha_n(f) = \sup_n \beta_n(f)$. Then

$$D = f^{-1}(0) = \bigcap_{m\in\mathbb{N}} f^{-1}([0, \frac{1}{m})) = \bigcap_{m\in\mathbb{N}} (\sup_n \beta_n(f))^{-1}([0, \frac{1}{m})) \subset \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \beta_n(f)^{-1}([0, \frac{1}{m}))$$

$$\subset \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \overline{\beta_n(f)^{-1}([0, \frac{1}{m}))} \subset \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \alpha_n(f)^{-1}([0, \frac{1}{m}]) = \bigcap_{m\in\mathbb{N}} (\sup_n \alpha_n(f))^{-1}([0, \frac{1}{m}])$$

$$= \bigcap_{m\in\mathbb{N}} f^{-1}([0, \frac{1}{m}]) = f^{-1}(0) = D.$$

Therefore,

$$D = \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \beta_n(f)^{-1}([0, \frac{1}{m})) = \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \overline{\beta_n(f)^{-1}([0, \frac{1}{m}))}.$$

For each $n, m \in \mathbb{N}$, let $U_{nm} = \beta_n(f)^{-1}([0, \frac{1}{m}))$. Then $U_{nm} \in \tau$, $\{U_{nm} : m \in \mathbb{N}\}$ is decreasing for each $n \in \mathbb{N}$ and $\{U_{nm} : n \in \mathbb{N}\}$ is decreasing for each $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} U_{nm} \subset U_{mm}$ and thus $\bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} U_{nm} \subset \bigcap_{m\in\mathbb{N}} U_{mm}$. Conversely, let $x \in \bigcap_{n\in\mathbb{N}} U_{nn}$. For each $n, m \in \mathbb{N}$, let $k = \max\{n, m\}$. Then $x \in U_{kk} \subset U_{nm}$. This implies that $\bigcap_{n\in\mathbb{N}} U_{nn} \subset \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} U_{nm}$. Hence, $\bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} U_{nm} = \bigcap_{n\in\mathbb{N}} U_{nn}$. Similarly, $\bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \overline{U_{nm}} = \bigcap_{n\in\mathbb{N}} \overline{U_{nn}}$. Therefore, $D = \bigcap_{n\in\mathbb{N}} U_{nn} = \bigcap_{n\in\mathbb{N}} \overline{U_{nn}}$ which implies that $D \in RG(X)$. □

Corollary 2.10. *If $f \in SL(X)$ ($f \in SU(X)$) then for each $r \in \mathbb{R}$, $\{f \le r\} \in RG(X)$ ($\{f \ge r\} \in RG(X)$).*

Proof. Suppose that $f \in SL(X)$. For $r \in \mathbb{R}$, let $h = \max\{f - r, 0\}$. By Proposition 2.5, $h \in SL^+(X)$. It is clear that $\{f \le r\} = h^{-1}(0)$. By Lemma 2.9, $\{f \le r\} \in RG(X)$.

If $f \in SU(X)$ then $-f \in SL(X)$. For each $r \in \mathbb{R}$, $\{f \ge r\} = \{-f \le -r\} \in RG(X)$. □

3. Main results

In this section, we show that a space satisfying (∗) is precisely an almost discrete space and present some characterizations of δ-stratifiable spaces with real-valued functions.

Theorem 3.1. *For a space X, the following are equivalent.*

(a) X is almost discrete.

(b) There is an order-preserving map $\phi : U^+(X) \to C^+(X)$ such that $\phi(h) \leq h$ for each $h \in U^+(X)$, and $0 < \phi(h)(x) < h(x)$ whenever $h(x) > 0$.

(c) X satisfies $(*)$.

(d) There exists an order-reversing map $\varphi : \tau \to L^+(X)$ such that $U = \varphi(U)^{-1}(0)$ for each $U \in \tau$.

(e) There exists a map $\varphi : \tau \to L^+(X)$ such that $U = \varphi(U)^{-1}(0)$ for each $U \in \tau$.

Proof. (a) ⇒ (b) For each $h \in U^+(X)$ and $n \in \mathbb{N}$, let $U_n(h) = \{h \geq \frac{1}{2^{n-2}}\}$. Then $\{U_n(h) : n \in \mathbb{N}\}$ is an increasing sequence of closed and thus open subsets of X. Let

$$\phi(h) = \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{U_n(h)}$$

Then $\phi(h) \in C^+(X)$. If $h_1 \leq h_2$ then $U_n(h_1) \subset U_n(h_2)$ for each $n \in \mathbb{N}$ and thus $\phi(h_1) \leq \phi(h_2)$.

For each $x \in X$, if $h(x) = 0$ then $x \notin U_n(h)$ for each $n \in \mathbb{N}$ and thus $\phi(h)(x) = 0$. If $h(x) > 0$ then $x \in U_m(h)$ for some $m \in \mathbb{N}$. Let $k = \min\{n \in \mathbb{N} : x \in U_n(h)\}$. Then $x \notin U_n(h)$ for each $n < k$ while $x \in U_n(h)$ for each $n \geq k$. Thus

$$\phi(h)(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{U_n(h)}(x) = \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}}$$

Since $x \in U_k(h)$, we have $h(x) \geq \frac{1}{2^{k-2}} > \phi(h)(x)$.

(b) ⇒ (c) is clear.

(c) ⇒ (d) Let ϕ be the map in property $(*)$. For each $U \in \tau$, let $h_U = 1 - \chi_U$. Then $h_U \in U^+(X)$. Let $\varphi(U) = \phi(h_U)$. Then $\varphi(U) \in L^+(X)$. It is clear that $\varphi(U) \geq \varphi(V)$ whenever $U \subset V$. If $x \in U$, then $h_U(x) = 0$ and thus $\varphi(U)(x) = \phi(h_U)(x) = 0$. If $x \notin U$ then $h_U(x) = 1$ and thus $\varphi(U)(x) = \phi(h_U)(x) > 0$. This implies that $U = \varphi(U)^{-1}(0)$.

(d) ⇒ (e) is clear.

(e) ⇒ (a) Let φ be the map in (e). Then for each $U \in \tau$, $\varphi(U) \in L^+(X)$, so $U = \varphi(U)^{-1}(0) = \{\varphi(U) \leq 0\}$ is a closed set. Therefore, X is almost discrete. □

Theorem 3.2. *For a space X, the following are equivalent.*

(a) X is δ-stratifiable.

(b) There exist two order preserving maps $\Psi : SL^+(X) \to L^+(X)$ and $\Phi : SL^+(X) \to U^+(X)$ such that $\Psi(h) \leq \Phi(h) \leq h$ for each $h \in SL^+(X)$ and $\Psi(h)(x) > 0$ whenever $h(x) > 0$.

(c)There exists an order preserving map $\Phi : SL^+(X) \to U^+(X)$ such that $\Phi(h) \leq h$ for each $h \in SL^+(X)$ and if $h(x) > 0$ then there exists an open neighborhood O_x of x such that $\inf \Phi(h)(O_x) > 0$.

(d) There exists two order reversing maps $\psi : RG(X) \to L^+(X)$ and $\phi : RG(X) \to U^+(X)$ such that $\psi(D) \leq \phi(D)$ and $D = \psi(D)^{-1}(0) = \phi(D)^{-1}(0)$ for each $D \in RG(X)$.

Proof. (a) ⇒ (b) Let σ be the map in Definition 1.5 which is decreasing with respect to n. For each $h \in SL^+(X)$ and $n \in \mathbb{N}$, let $D_n(h) = \{h \leq \frac{1}{2^{n-1}}\}$. By Corollary 2.10, $\{D_n(h)\}_{n\in\mathbb{N}}$ is a decreasing sequence of regular G_δ-sets of X. Let

$$\Psi(h) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{\overline{\sigma(n,D_n(h))}}, \quad \Phi(h) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{\sigma(n,D_n(h))}.$$

Then $\Psi(h) \in L^+(X)$, $\Phi(h) \in U^+(X)$ and $\Psi(h) \leq \Phi(h)$.

Let $x \in X$.

Case 1. $h(x) = 0$. Then $x \in D_n(h) \subset \sigma(n, D_n(h))$ for each $n \in \mathbb{N}$ from which it follows that $\Phi(h)(x) = 0$.

Case 2. $h(x) > 0$. Then $x \notin D_m(h)$ for some $m \in \mathbb{N}$. It follows that $x \notin \overline{\sigma(i, D_m(h))}$ for some $i \in \mathbb{N}$. Let $j = \max\{i, m\}$. Then $x \notin \overline{\sigma(j, D_j(h))}$ and thus $\Psi(h)(x) > 0$. Let $k = \min\{n \in \mathbb{N} : x \notin \sigma(n, D_n(h))\}$. Then $x \in \sigma(n, D_n(h))$ for each $n < k$ while $x \notin \sigma(n, D_n(h))$ for each $n \geq k$. It follows that

$$\Phi(h)(x) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{\sigma(n,D_n(h))}(x) = 1 - \sum_{n=1}^{k-1} \frac{1}{2^n} = \frac{1}{2^{k-1}}.$$

Since $x \notin \sigma(k, D_k(h)) \supset D_k(h)$, we have $h(x) > \frac{1}{2^{k-1}} = \Phi(h)(x)$.

The above argument shows that $\Phi(h) \leq h$.

Now suppose that $h_1 \leq h_2$. Then $D_n(h_2) \subset D_n(h_1)$ and thus $\sigma(n, D_n(h_2)) \subset \sigma(n, D_n(h_1))$ for each $n \in \mathbb{N}$ which implies that $\Phi(h_1) \leq \Phi(h_2)$. Similarly, $\Psi(h_1) \leq \Psi(h_2)$.

(b) $\Rightarrow$ (c) Let Ψ, Φ be the maps in (b). If $h(x) > 0$ then $\Psi(h)(x) > 0$. Choose $r > 0$ such that $\Psi(h)(x) > r$ and let $O_x = \{\Psi(h) > r\}$. Then O_x is an open neighborhood of x. For each $y \in O_x$, $\Phi(h)(y) \geq \Psi(h)(y) > r$ and thus $\inf \Phi(h)(O_x) \geq r > 0$.

(c) $\Rightarrow$ (d) Let Φ be the map in (c). For each $D \in RG(X)$, let $h_D = 1 - \chi_D$. Then $h_D \in SL^+(X)$. Let $\phi(D) = \Phi(h_D)$. It is clear that $\phi(D) \geq \phi(E)$ whenever $D \subset E$.

If $x \in D$, then $h_D(x) = 0$ and thus $\phi(D)(x) = \Phi(h_D)(x) = 0$ which implies that $D \subset \phi(D)^{-1}(0)$. If $x \notin D$ then $h_D(x) = 1$. By (c), there exists an open neighborhood O_x of x and $m \in \mathbb{N}$ such that $\inf \phi(D)(O_x) > \frac{1}{m}$. Thus $\phi(D)(x) > 0$. This implies that $\phi(D)^{-1}(0) \subset D$ and so $D = \phi(D)^{-1}(0)$. From $\inf \phi(D)(O_x) > \frac{1}{m}$ it follows that $O_x \subset \phi(D)^{-1}(\frac{1}{m}, \infty)$ and thus $x \in int(\phi(D)^{-1}(\frac{1}{m}, \infty))$. Let $n_x(D) = \min\{n \in \mathbb{N} : x \in int(\phi(D)^{-1}(\frac{1}{n}, \infty))\}$.

For each $x \in X$, let $\psi(D)(x) = 0$ whenever $x \in D$ and $\psi(D)(x) = \frac{1}{n_x(D)}$ whenever $x \notin D$. Then $D = \psi(D)^{-1}(0)$. For each $x \in X$, if $x \in D$ then $\phi(D)(x) = \psi(D)(x) = 0$. If $x \notin D$ then by the definition of $n_x(D)$, $x \in int(\phi(D)^{-1}(\frac{1}{n_x(D)}, \infty))$ which implies that $\phi(D)(x) > \frac{1}{n_x(D)} = \psi(D)(x)$. Therefore, $\psi(D) \leq \phi(D)$.

To show that $\psi(D) \in L(X)$, suppose that $\psi(D)(x) > r$. Then $r < 1$. If $r < 0$ then X is an open neighborhood of x and $\psi(D)(y) \geq 0 > r$ for each $y \in X$. If $r \geq 0$ then $\psi(D)(x) = \frac{1}{n_x(D)}$. Let $O_x = int(\phi(D)^{-1}(\frac{1}{n_x(D)}, \infty))$. Then O_x is an open neighborhood of x. For each $y \in O_x$, $\phi(D)(y) > \frac{1}{n_x(D)}$ and thus $y \notin D$. By the definition of $n_y(D)$, $n_y(D) \leq n_x(D)$ and thus $\psi(D)(y) \geq \psi(D)(x) > r$. This implies that $\psi(D) \in L(X)$.

Suppose that $D \subset E$. If $x \in E$ then $\psi(E)(x) = 0 \leq \psi(D)(x)$. f $x \notin E$ then $x \notin D$. Since $\phi(D) \geq \phi(E)$, we have $x \in int(\phi(E)^{-1}(\frac{1}{n_x(E)}, \infty)) \subset int(\phi(D)^{-1}(\frac{1}{n_x(E)}, \infty))$ and thus $n_x(D) \leq n_x(E)$. Therefore, $\psi(D)(x) \geq \psi(E)(x)$.

(d) $\Rightarrow$ (a) Let ψ, ϕ be the maps in (d). Then for each $D \in RG(X)$,

$$D = \phi(D)^{-1}(0) = \bigcap_{n \in \mathbb{N}} \phi(D)^{-1}([0, \frac{1}{n})) \subset \bigcap_{n \in \mathbb{N}} \overline{\phi(D)^{-1}([0, \frac{1}{n}))}$$

$$\subset \bigcap_{n \in \mathbb{N}} \psi(D)^{-1}([0, \frac{1}{n}]) = \psi(D)^{-1}(0) = D.$$

For each $D \in RG(X)$ and $n \in \mathbb{N}$, let $\sigma(n, D) = \phi(D)^{-1}([0, \frac{1}{n}))$. Then $\sigma(n, D) \in \tau$ and $D = \bigcap_{n \in \mathbb{N}} \sigma(n, D) = \bigcap_{n \in \mathbb{N}} \overline{\sigma(n, D)}$.

If $D \subset E$, then $\phi(E) \leq \phi(D)$ and thus $\sigma(n, D) = \phi(D)^{-1}([0, \frac{1}{n})) \subset \phi(E)^{-1}([0, \frac{1}{n})) = \sigma(n, E)$ for each $n \in \mathbb{N}$. Therefore, X is δ-stratifiable. □

An analogous argument proves the following.

Proposition 3.3. *For a space X, the following are equivalent.*

(a) X is δ-semi-stratifiable.

(b) There exists an order preserving map $\Phi : SL^+(X) \to U^+(X)$ such that $\Phi(h) \leq h$ for each $h \in SL^+(X)$ and $\Phi(h)(x) > 0$ whenever $h(x) > 0$.

(c) There exists an order reversing map $\phi : RG(X) \to U^+(X)$ such that $D = \phi(D)^{-1}(0)$ for each $D \in RG(X)$.

As another applications of strongly semi-continuous functions, we have the following.

Theorem 3.4. *For a space X, the following are equivalent.*

(a) X is perfectly normal.

(b) There exists a map $\phi : L^+(X) \to C^+(X)$ such that $\phi(h) \le h$ for each $h \in L^+(X)$ and $\phi(h)(x) > 0$ whenever $h(x) > 0$.

(c) There exists a map $\phi : L^+(X) \to SL^+(X)$ such that $\phi(h) \le h$ for each $h \in L^+(X)$ and $\phi(h)(x) > 0$ whenever $h(x) > 0$.

(d) There exists a map $\varphi : \tau^c \to SL^+(X)$ such that $F = \varphi(F)^{-1}(0)$ for each $F \in \tau^c$.

Proof. The equivalence of (a) and (b) has been shown in [8].

(b) ⇒ (c) is clear.

(c) ⇒ (d) Let ϕ be the map in (c). For each $F \in \tau^c$, $1-\chi_F \in L^+(X)$. Let $\varphi(F) = \phi(1-\chi_F)$. Then $\varphi(F) \in SL^+(X)$. A direct argument shows that $F = \varphi(F)^{-1}(0)$.

(d) ⇒ (a) Let φ be the map in (d). Then for each $F \in \tau^c$, $\varphi(F) \in SL^+(X)$. By Lemma 2.9, $F = \varphi(F)^{-1}(0) \in RG(X)$. Therefore, X is perfectly normal. □

Since stratifiable spaces are monotone versions of perfectly normal spaces, one may conjecture that X is stratifiable if the map ϕ in Theorem 3.4 (c) is order preserving. Actually, it still characterizes perfectly normal spaces.

Theorem 3.5. *For a space X, the following are equivalent.*

(a) X is perfectly normal.

(b) There exists an order reversing map $\varphi : \tau^c \to SL^+(X)$ such that $F = \varphi(F)^{-1}(0)$ for each $F \in \tau^c$.

(c) There exists order preserving map $\phi : L^+(X) \to SL^+(X)$ such that $\phi(h) \le h$ for each $h \in L^+(X)$ and $\phi(h)(x) > 0$ whenever $h(x) > 0$.

Proof. (a) ⇒ (b) Suppose that X is perfectly normal. Then for each $F \in \tau^c$, $F \in RG(X)$. Let $\varphi(F) = 1 - \chi_F$. By Lemma 2.8, $\varphi(F) \in SL^+(X)$. It is clear that $F = \varphi(F)^{-1}(0)$ and $\varphi(F) \ge \varphi(G)$ whenever $F \subset G$.

(b) ⇒ (c) Let φ be the map in (b). For each $F \in \tau^c$, let $\psi(F) = \min\{\varphi(F), 1\}$. Then $\psi : \tau^c \to SL^+(X)$ is an order reversing map such that $F = \psi(F)^{-1}(0)$ for each $F \in \tau^c$.

For each $h \in L^+(X)$ and $n \in \mathbb{N}$, let $F_n(h) = \{h \le \frac{1}{2^{n-1}}\}$. Then $F_n(h) \in \tau^c$. Let

$$\phi(h) = \sum_{n=1}^{\infty} \frac{1}{2^n}\psi(F_n(h)).$$

By Corollary 2.7, $\phi(h) \in SL^+(X)$.

For each $x \in X$, if $h(x) = 0$ then $x \in F_n(h)$ for each $n \in \mathbb{N}$ and thus $\psi(F_n(h))(x) = 0$. It follows that $\phi(h)(x) = 0 = h(x)$. If $h(x) > 0$ then $x \notin F_m(h)$ for some $m \in \mathbb{N}$ and thus $\psi(F_m(h))(x) > 0$. It follows that $\phi(h)(x) > 0$. Now, let $k = \min\{n \in \mathbb{N} : x \notin F_n(h)\}$. Then $x \in F_n(h)$ and thus $\psi(F_n(h))(x) = 0$ for each $n < k$. Thus

$$\phi(h)(x) = \sum_{n=k}^{\infty} \frac{1}{2^n}\psi(F_n(h))(x) \le \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}}$$

Since $x \notin F_k(h)$, we have $h(x) > \frac{1}{2^{k-1}} \ge \phi(h)(x)$.

The above argument shows that $\phi(h) \le h$.

Now suppose that $h_1 \le h_2$. Then $F_n(h_2) \subset F_n(h_1)$ and hus $\psi(F_n(h_1)) \le \psi(F_n(h_2))$ for each $n \in \mathbb{N}$ which implies that $\phi(h_1) \le \phi(h_2)$.

(c) ⇒ (a) follows from Theorem 3.4. □

By their definitions (or characterizations), it is clear that if a pace is δ-stratifiable and perfectly normal then it is stratifiable (the converse is, of course, also true). This can also be deduced from Theorem 3.2 (b), Theorem 3.5 (c) and the following characterization of a stratifiable space [9]: a space X is stratifiable if and only if there exist two order preserving maps $\Psi : L^+(X) \to L^+(X)$ and $\Phi : L^+(X) \to U^+(X)$ such that $\Psi(h) \le \Phi(h) \le h$ for each $h \in L^+(X)$ and $\Psi(h)(x) > 0$ whenever $h(x) > 0$.

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