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On Matrix Polynomials $L_n^{(M,\delta,\lambda)}(x)$

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Abstract. The aim of this paper is to introduce matrix polynomials $L_n^{(M,\delta,\lambda)}(x)$ and establish some properties viz, hypergeometric representation, generating matrix relations, integral representations, recurrence relations, summation formulas, series relation, fractional integral and derivative operators.

1. Introduction and preliminaries

The matrix theory plays a pivotal role in every area of mathematics, specifically in orthogonal polynomials and the theory of special functions. The special matrix functions appear in the study of statistics, group representation theory, number theory, Lie theory and in the matrix version of Bessel, Laguerre, Hermite, Gegenbauer, Chebyshev matrix polynomials and Lommel matrix polynomials (see [7–9, 14, 17, 23]). The matrix analogue of the various special functions were studied by many mathematician Bakhet [1], Dwivedi and Sahai [5], J*ó*dar and Cort*é*s [6, 10, 11], Shehata [2, 18–21] and others. They studied orthogonality, differential and integral representations, finite summation formulas and generating functions of unique matrix functions and polynomials. These matrix functions and polynomials play a significant part in resolving a wide range of statistical, probabilistic, engineering, and mathematical science issues.

Let $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be the complex space of all **r**- square complex matrices. For $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$, its spectrum $\sigma(M)$ denotes the set of all eigenvalues of M and $\alpha(M) = \max[\Re(\rho) : \rho \in \sigma(M)]$, $\beta(M) = \min[\Re(\rho) : \rho \in \sigma(M)]$. Any square matrix $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ is a positive stable, if $\Re(\rho) > 0$ for all $\rho \in \sigma(M)$. A matrix norm is a vector norm on $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$. That is, if $\|M\|$ denotes the norm of the matrix M, then the operator norm corresponding to the 2-norm for vectors is

$$\|M\| = \sup_{x\neq 0} \frac{\|Mx\|_2}{\|x\|_2} = \max\{\sqrt{\lambda} : \lambda \in \sigma(M^*M)\},$$

where for any vector x in r^{th} complex plane, $\|x\|_2 = (x^*x)^{1/2}$ is the Euclidean norm of x and M^* denotes the transposed conjugate of M. The identity matrix in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ is symbolized by I. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z and defined in an open set Ω in complex plane, then from the properties of the matrix functional calculus [3], it follows $f(M)g(N) = g(N)f(M)$, where M, N are commuting matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$. The reciprocal gamma function denoted by $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$ is an entire function of the complex variable

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z.
For $M \in \mathbf{C}^{r\times r}$, image of $\Gamma^{-1}(z)$ acting on M and denoted by $\Gamma^{-1}(M)$ which is a well defined matrix.

If $M \in \mathbf{C}^{r\times r}$ and $M+nI$ is invertible matrix for all $n \in \mathbf{Z}^+ \cup \{\mathbf{0}\}$, then the matrix version of the pochhammer symbol $(M)_n$ is defined (Jódar and Cortés [10]) as,

$$(M)_n = \Gamma(M+nI)\Gamma^{-1}(M) = \begin{cases} M(M+I)(M+2I)...(M+(n-1)I) & (n \in \mathbf{N}) \\ I & (n=0). \end{cases} \tag{1}$$

For any matrix $M \in \mathbf{C}^{r\times r}$, one gets the relation due to Jódar and Cortés [10],

$$(1-y)^{-M} = \sum_{n\geq 0} \frac{(M)_n}{n!} y^n, \quad |y| < 1. \tag{2}$$

If M and N are commuting matrices in $\mathbf{C}^{r\times r}$ and for $n \in \mathbf{Z}^+ \cup \{\mathbf{0}\}$, $M+nI, N+nI, M+N+nI$ are invertible, then (Jódar and Cortés [10]),

$$B(M,N) = \Gamma(M)\Gamma(N)\Gamma^{-1}(M+N). \tag{3}$$

Jódar and Cortés [11] defined the Beta matrix function as,

$$B(M,N) = \int_0^1 x^{M-I}(1-x)^{N-I}dx, \tag{4}$$

where M, N are positive stable matrices in $\mathbf{C}^{r\times r}$.

Jódar and Cortés [11] defined the Gamma matrix function as,

$$\Gamma(M) = \int_0^\infty e^{-x}x^{M-I}dx, \quad x^{M-I} = \exp\left((M-I)\ln x\right), \tag{5}$$

where M is a positive stable matrix in $\mathbf{C}^{r\times r}$.

The contour integral representation for the reciprocal Gamma function (see Lebedev [13]):

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_C e^t t^{-z} dt, \tag{6}$$

where C is the path around the origin in the positive direction, beginning at and returning to positive infinity with respect for the branch cut along the positive real axis.

Lemma 1.1. *(See Rainville [15]) Let $A(p,n)$ and $B(p,n)$ be matrices in $\mathbf{C}^{r\times r}$. Then the following series relations are satisfied:*

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} A(p,n) = \sum_{n=0}^{\infty}\sum_{p=0}^{n} A(p,n-p)$$

$$\sum_{n=0}^{\infty}\sum_{p=0}^{n} B(p,n) = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} B(p,n+p).$$

Dwivedi and Sahai [5] defined the generalized hypergeometric matrix function as,

$$\begin{aligned} &{}_pF_q\left[\begin{matrix} M_1, M_2, ..., M_p \\ N_1, N_2, ..., N_q \end{matrix} ;\ x\right] \\ &= \sum_{n\geq 0} \frac{(M_1)_n(M_2)_n...(M_p)_n[(N_1)_n]^{-1}[(N_2)_n]^{-1}...[(N_q)_n]^{-1}x^n}{n!}, \quad |x| < 1, \end{aligned} \tag{7}$$

where $M_i, N_j \in \mathbf{C}^{r\times r}$, $1 \leq i \leq p$, $1 \leq j \leq q$, $p, q \in \mathbf{N}$ and $N_j + kI$ are invertible for all $k \in \mathbf{Z}^+ \cup \{\mathbf{0}\}$.

Theorem 1.2. *(Dwivedi and Sahai [5]) Let $M_1, M_2, ..., M_p$ and $N_1, N_2, ..., N_q$ be positive stable matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ and $p \leq q+1$. Then the series (1) is absolutely convergent for $|x| = 1$ if*

$$\beta(N_1) + \beta(N_2) + ... + \beta(N_q) > \alpha(M_1) + \alpha(M_2) + ... + \alpha(M_p). \tag{8}$$

The Riemann-Liouville fractional integral and derivative operator are defined (see Samko et al. [22]) as,

$$I_a^{\mu}(f)(x) = \frac{1}{\mu}\int_a^x (x-t)^{\mu-1} f(t)dt, \quad (\mu \in \mathbf{C},\ \Re(\mu) > 0) \tag{9}$$

and

$$\left(D_a^{\mu} f\right)(x) = \left(\frac{d}{dx}\right)^m \left(I_a^{m-\mu} f\right)(x). \tag{10}$$

Definition 1.3. *(Bakhet et al. [1]) Let $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a positive stable matrix and $\mu \in \mathbf{C}$ with $\Re(\mu) > 0$. Then the Riemann- Liouville fractional integral of order μ is :*

$$I^{\mu}(x^M) = \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1} t^M dt. \tag{11}$$

Lemma 1.4. *(Bakhet et al. [1]) Let $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a positive stable matrix and $\mu \in \mathbf{C}$ with $\Re(\mu) > 0$. Then the Riemann- Liouville fractional integral of order μ is :*

$$I^{\mu}(x^{M-I}) = \Gamma(M)\Gamma^{-1}(M+\mu I)x^{M+(\mu-1)I}. \tag{12}$$

In 1994, Jódar et al. [6] introduced the Laguerre matrix polynomial and defined as,

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k}{k!(n-k)!}(A+I)_n[(A+I)_k]^{-1} x^k, \tag{13}$$

where λ is a complex number with $\Re(\lambda) > 0$ and A is a matrix in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ with $A + nI$ invertible for all $n \geq 1$. Shehata [20] obtained the connections between Legendre with Hermite and Laguerre matrix polynomials. In sequel to study, we give the extension of Laguerre matrix polynomial $L_n^{(A,\lambda)}(x)$ and discuss some properties.

2. Matrix polynomials $L_n^{(M,\delta,\lambda)}(x)$

Definition 2.1. *Let $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition*

$$\Re(\rho) > -1, \quad \forall \rho \in \sigma(M) \tag{14}$$

and $\delta \in \mathbf{Z}^+$, $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$. Then the matrix polynomials $L_n^{(M,\delta,\lambda)}(x)$ define as:

$$L_n^{(M,\delta,\lambda)}(x) = \frac{\Gamma(M+(\delta n+1)I)}{n!}\sum_{p=0}^{n} \frac{(-nI)_p \lambda^p x^p}{p!}\Gamma^{-1}(M+(\delta p+1)I). \tag{15}$$

Remark 2.2. *If we set $\delta = 1$, this reduces to the Laguerre matrix polynomial $L_n^{(M,\lambda)}(x)$ (Jódar et al. [6]).*

2.1. Hypergeometric representation

On using (7), we can represent (15) in terms of the hypergeometric matrix function as,

$$L_n^{(M,\delta,\lambda)}(x) = \frac{(M+I)_{\delta n}}{n!}\, {}_1F_\delta\left[\begin{matrix} -nI \\ \Delta(\delta; M+I) \end{matrix} ;\ \frac{\lambda x}{\delta^\delta}\right], \tag{16}$$

where $\Delta(\delta; M+I)$ is the array of δ parameters:

$$\left(\frac{M+I}{\delta}\right), \left(\frac{M+2I}{\delta}\right), \left(\frac{M+3I}{\delta}\right), \ldots, \left(\frac{M+\delta I}{\delta}\right).$$

Remark 2.3. *If we set $\delta = 1$, this yields the corresponding known result of Shehata [16],*

$$L_n^{M,\lambda}(x) = \frac{(M+I)_n}{n!}\, {}_1F_1\left[\begin{matrix} -nI \\ M+I \end{matrix} ;\ \lambda x\right]. \tag{17}$$

3. Generating matrix functions

In this section, we establish generating matrix relations of matrix polynomials (15).

Theorem 3.1. *Let M and N be commuting matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$, $|\omega| < 1$ and $|\frac{-\omega\lambda x}{(1-\omega)\delta^\delta}| < 1$. Then the following generating matrix relation holds:*

$$\sum_{n=0}^{\infty} (N)_n \left[(M+I)_{\delta n}\right]^{-1} L_n^{(M,\delta,\lambda)}(x)\omega^n = (1-\omega)^{-N}\, {}_1F_\delta\left[\begin{matrix} N \\ \Delta(\delta; M+I) \end{matrix} ;\ \frac{-\omega\lambda x}{(1-\omega)\delta^\delta}\right]. \tag{18}$$

Proof. The left hand side of (18), gives

$$\sum_{n=0}^{\infty} (N)_n \left[(M+I)_{\delta n}\right]^{-1} L_n^{(M,\delta,\lambda)}(x)\omega^n = \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{(-1)^p \lambda^p x^p}{(n-p)!p!} (N)_n \left[(M+I)_{\delta p}\right]^{-1} \omega^n.$$

On applying the Lemma 1.1, we get

$$\begin{aligned} \sum_{n=0}^{\infty} (N)_n \left[(M+I)_{\delta n}\right]^{-1} L_n^{(M,\delta,\lambda)}(x)\omega^n &= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{\lambda^p (-x)^p}{n!p!} (N)_{n+p} \left[(M+I)_{\delta p}\right]^{-1} \omega^{n+p} \\ &= \sum_{p=0}^{\infty} \frac{(-\omega\lambda x)^p}{p!} \left[(M+I)_{\delta p}\right]^{-1} \sum_{n=0}^{\infty} \frac{(N+p)_n \omega^n}{n!} \\ &= (1-\omega)^{-N} \sum_{p=0}^{\infty} \frac{\left(\frac{-\omega\lambda x}{(1-\omega)\delta^\delta}\right)^p (N)_p}{p!} \prod_{m=1}^{\delta} \left[\left(\frac{M+mI}{\delta}\right)_p\right]^{-1}, \end{aligned}$$

on applying (7), this immediately completes the proof of result (18). □

On plugging $\delta = 1$ in (18), this deduces to the following corollary.

Corollary 3.2. *Let M and N be commuting matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ satisfying the spectral condition (14) and $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$, $|\omega| < 1$ and $|\frac{-\omega\lambda x}{(1-\omega)}| < 1$. Then the following generating matrix relation holds:*

$$\sum_{n=0}^{\infty} (N)_n \left[(M+I)_n\right]^{-1} L_n^{(M,\lambda)}(x)\omega^n = (1-\omega)^{-N}\, {}_1F_1\left[\begin{matrix} N \\ M+I \end{matrix} ;\ \frac{-\omega\lambda x}{(1-\omega)}\right]. \tag{19}$$

Theorem 3.3. *Let $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$ and $|\frac{-\omega\lambda x}{\delta^\delta}| < 1$. Then the following generating matrix relation holds:*

$$\sum_{n=0}^{\infty} [(M+I)_{\delta n}]^{-1} L_n^{(M,\delta,\lambda)}(x)\omega^n = e^{\omega}\, {}_0F_\delta\left[\begin{matrix} - \\ \Delta(\delta; M+I) \end{matrix} ; \frac{-\omega\lambda x}{\delta^\delta}\right]. \tag{20}$$

Proof. Consider the left hand side of (20),

$$\sum_{n=0}^{\infty} [(M+I)_{\delta n}]^{-1} L_n^{(M,\delta,\lambda)}(x)\omega^n = \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{(-1)^p\lambda^p x^p}{(n-p)!p!}\left[(M+I)_{\delta p}\right]^{-1}\omega^n,$$

on applying the Lemma 1.1, we get

$$\begin{aligned}\sum_{n=0}^{\infty} [(M+I)_{\delta n}]^{-1} L_n^{(M,\delta,\lambda)}(x)\omega^n &= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{\lambda^p(-x)^p}{n!p!}\left[(M+I)_{\delta p}\right]^{-1}\omega^{n+p} \\ &= \sum_{n=0}^{\infty}\frac{\omega^n}{n!}\sum_{p=0}^{\infty}\frac{(-\omega\lambda x)^p}{p!}\left[(M+I)_{\delta p}\right]^{-1} \\ &= e^{\omega}\sum_{p=0}^{\infty}\frac{\left(\frac{-\omega\lambda x}{\delta^\delta}\right)^p}{p!}\prod_{m=1}^{\delta}\left[\left(\frac{M+mI}{\delta}\right)_p\right]^{-1},\end{aligned}$$

on applying (8), this yields the result (20). □

Remark 3.4. *If we set $\delta = 1$ in (20), yields the corresponding known result of Jódar and Sastre [12],*

$$\sum_{n=0}^{\infty} [(M+I)_n]^{-1} L_n^{(M,\lambda)}(x)\omega^n = e^{\omega}\, {}_0F_1\left[\begin{matrix} - \\ M+I \end{matrix} ; -\omega\lambda x\right]. \tag{21}$$

Corollary 3.5. *Let $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$ and $|\frac{-\omega\lambda x}{\delta^\delta}| < 1$. Then the following generating matrix relation holds:*

$$\sum_{n=0}^{\infty} \Gamma^{-1}(M+(\delta n+1)I) L_n^{(M,\delta,\lambda)}(x)\omega^n = \Gamma^{-1}(M+I)\, e^{\omega}\, {}_0F_\delta\left[\begin{matrix} - \\ \Delta(\delta; M+I) \end{matrix} ; \frac{-\omega\lambda x}{\delta^\delta}\right]. \tag{22}$$

From the Theorem 3.3, one can obtain the following assertion.

Corollary 3.6. *Let $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (14) and $\delta, p \in \mathbf{Z}^+$, $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$. Then the following explicit formula holds:*

$$x^p I = \sum_{n=0}^{p} \frac{\lambda^{-p} p!\, [(M+I)_{\delta n}]^{-1} (M+I)_{\delta p}\, L_n^{(M,\delta,\lambda)}(x)}{(p-n)!}. \tag{23}$$

Remark 3.7. *If we set $\delta = 1$ in (23), yields the corresponding known result of Jódar and Sastre [12],*

$$x^p I = \sum_{n=0}^{p} \frac{\lambda^{-p} p!\, [(M+I)_n]^{-1} (M+I)_p\, L_n^{(M,\lambda)}(x)}{(p-n)!}. \tag{24}$$

Theorem 3.8. *Let M and N be commuting matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$ and $|\omega\delta^\delta| < 1$. Then the following generating matrix relation holds:*

$$\begin{aligned}&\sum_{n=0}^{\infty}\Gamma(N+\delta nI)\Gamma^{-1}(M+(\delta n+1)I)L_n^{(M,\delta,\lambda)}(x)\omega^n\\&=\sum_{p=0}^{\infty}\frac{(-\omega\lambda x)^p(N)_{\delta p}}{p!}\Gamma^{-1}(M+(\delta p+1)I)\Gamma(N)\,{}_{\delta}F_0\left[\begin{matrix}\Delta(\delta;N+\delta pI)\\-\end{matrix}\;;\;\omega\delta^\delta\right].\end{aligned}\tag{25}$$

Proof. The left hand side of (25), gives

$$\begin{aligned}&\sum_{n=0}^{\infty}\Gamma(N+\delta nI)\Gamma^{-1}(M+(\delta n+1)I)L_n^{(M,\delta,\lambda)}(x)\omega^n\\&=\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{(-\lambda x)^p}{(n-p)!p!}\Gamma^{-1}(M+(\delta p+1)I)\Gamma(N+\delta nI)\omega^n,\end{aligned}$$

on applying the Lemma 1.1, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\Gamma(N+\delta nI)\Gamma^{-1}(M+(\delta n+1)I)L_n^{(M,\delta,\lambda)}(x)\omega^n\\&=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-\omega\lambda x)^p}{n!p!}\Gamma^{-1}(M+(\delta p+1)I)\Gamma(N+\delta nI+\delta pI)\omega^n\\&=\sum_{p=0}^{\infty}\frac{(-\omega\lambda x)^p(N)_{\delta p}}{p!}\Gamma^{-1}(M+(\delta p+1)I)\Gamma(N)\sum_{n=0}^{\infty}\frac{(\omega\delta^\delta)^n}{n!}\prod_{m=1}^{\delta}\left(\frac{N+(\delta p+m-1)I}{\delta}\right)_n,\end{aligned}$$

this immediately leads to (25). □

On setting $\delta = 1$ in (25), this yields the following assertion.

Corollary 3.9. *Let M and N be commuting matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ satisfying the spectral condition (14) and $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$ and $|\omega| < 1$. Then the following generating matrix relation holds:*

$$\begin{aligned}&\sum_{n=0}^{\infty}\Gamma(N+nI)\Gamma^{-1}(M+(n+1)I)L_n^{(M,\lambda)}(x)\omega^n\\&=\sum_{p=0}^{\infty}\frac{(-\omega\lambda x)^p(N)_p}{p!}\Gamma^{-1}(M+(p+1)I)\Gamma(N)\,{}_1F_0\left[\begin{matrix}N+pI\\-\end{matrix}\;;\;\omega\right].\end{aligned}\tag{26}$$

On taking $N = M + I$ in (25), this deduces to the following assertion.

Corollary 3.10. *Let $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$ and $|\omega\delta^\delta| < 1$. Then the following generating matrix relation holds:*

$$\sum_{n=0}^{\infty}L_n^{(M,\delta,\lambda)}(x)\omega^n=\sum_{p=0}^{\infty}\frac{(-\omega\lambda x)^p}{p!}\,{}_{\delta}F_0\left[\begin{matrix}\Delta(\delta;M+(\delta p+1)I)\\-\end{matrix}\;;\;\omega\delta^\delta\right].\tag{27}$$

On setting $\delta = 1$ in (27), this deduces to the following result.

Corollary 3.11. *Let $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (14) and $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$ and $|\omega| < 1$. Then the following generating matrix relation holds:*

$$\sum_{n=0}^{\infty} L_n^{(M,\lambda)}(x)\omega^n = \sum_{p=0}^{\infty} \frac{(-\omega\lambda x)^p}{p!} \, {}_1F_0\left[\begin{matrix} M+(p+1)I \\ - \end{matrix} \; ; \; \omega\right]. \tag{28}$$

Theorem 3.12. *Let M and N be commuting matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$, and $|\frac{\omega(-\delta)^\delta}{1+\omega\lambda x}| < 1$. Then the following generating matrix relation holds:*

$$\sum_{n=0}^{\infty} (N)_n L_n^{(M-\delta nI,\delta,\lambda)}(x)\omega^n = (1+\omega\lambda x)^{-N} \, {}_{\delta+1}F_0\left[\begin{matrix} N, \Delta(\delta;-M) \\ - \end{matrix} \; ; \; \frac{\omega(-\delta)^\delta}{1+\omega\lambda x}\right]. \tag{29}$$

Proof. From the left hand side of (29) and using the Lemma 1.1, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} (N)_n L_n^{(M-\delta nI,\delta,\lambda)}(x)\omega^n \\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} (N)_{n+p}(M+(1-\delta n-\delta p)I)_{\delta n+\delta p}\left[(M+(1-\delta n-\delta p)I)_{\delta p}\right]^{-1} \frac{(-\lambda x)^p\omega^{n+p}}{n!p!} \\
&= \sum_{n=0}^{\infty} (N)_n (M+(1-\delta n)I)_{\delta n}(1+\omega\lambda x)^{-(N+nI)}\frac{\omega^n}{n!} \\
&= (1+\omega\lambda x)^{-N} \sum_{n=0}^{\infty} \frac{(N)_n\left(\frac{(-\delta)^\delta\omega}{1+\omega\lambda x}\right)^n}{n!} \prod_{m=1}^{\delta}\left(\frac{(m-1)I-M}{\delta}\right)_n,
\end{aligned}$$

this immediately leads to (29). □

Theorem 3.13. *Let $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$ and $|\omega(-\delta)^\delta| < 1$. Then the following generating matrix relation holds:*

$$\sum_{n=0}^{\infty} L_n^{(M-\delta nI,\delta,\lambda)}(x)\omega^n = e^{-\omega\lambda x} \, {}_\delta F_0\left[\begin{matrix} \Delta(\delta;-M) \\ - \end{matrix} \; ; \; \omega(-\delta)^\delta\right]. \tag{30}$$

Proof. From the left hand side of (30) and using the Lemma 1.1, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} L_n^{(M-\delta nI,\delta,\lambda)}(x)\omega^n \\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} (M+(1-\delta n-\delta p)I)_{\delta n+\delta p}\left[(M+(1-\delta n-\delta p)I)_{\delta p}\right]^{-1} \frac{(-\lambda x)^p\omega^{n+p}}{n!p!} \\
&= e^{-\omega\lambda x} \sum_{n=0}^{\infty} \frac{((-\delta)^\delta\omega)^n}{n!} \prod_{m=1}^{\delta}\left(\frac{(m-1)I-M}{\delta}\right)_n,
\end{aligned}$$

on applying (7), this yields the result (30).
□

Considering the double series

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m}\Gamma^{-1}(M+(\delta(n+m)+1)I)L_{n+m}^{(M,\delta,\lambda)}(x)\omega^m\sigma^n$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\Gamma^{-1}(M+(\delta n+1)I)L_{n}^{(M,\delta,\lambda)}(x)\omega^m\sigma^{n-m}$$
$$=\sum_{n=0}^{\infty}\Gamma^{-1}(M+(\delta n+1)I)L_{n}^{(M,\delta,\lambda)}(x)(\omega+\sigma)^n,$$

on using the Corollary 3.5, we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m}\Gamma^{-1}(M+(\delta(n+m)+1)I)L_{n+m}^{(M,\delta,\lambda)}(x)\omega^m\sigma^n$$
$$=\Gamma^{-1}(M+I)\,e^{\omega+\sigma}\,{}_0F_\delta\left[\begin{matrix}-\\ \Delta(\delta;M+I)\end{matrix}\;;\;\frac{-(\omega+\sigma)\lambda x}{\delta^\delta}\right]. \tag{31}$$

This represents double series generating matrix relation for matrix polynomials (15).

On taking $\delta=1$ in (31), we get double series generating matrix relation for Laguerre matrix polynomial (13),

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m}\Gamma^{-1}(M+((n+m)+1)I)L_{n+m}^{(M,\lambda)}(x)\omega^m\sigma^n$$
$$=\Gamma^{-1}(M+I)\,e^{\omega+\sigma}\,{}_0F_1\left[\begin{matrix}-\\ M+I\end{matrix}\;;\;-(\omega+\sigma)\lambda x\right]. \tag{32}$$

4. Integral representations

In this segment, we give the integral representations for matrix polynomials (15) in the form of following theorems.

Theorem 4.1. *Let $M\in\mathbf{C}^{r\times r}$ be a matrix satisfying the spectral condition (14) and $\delta\in\mathbf{Z}^+$, $\lambda\in\mathbf{C}$ with $\Re(\lambda)>0$. Then*

$$L_n^{(M,\delta,\lambda)}(x)=\frac{\Gamma(M+(\delta n+1)I)}{n!\,2\pi i}\int_C(u^\delta-\lambda x)^n e^u u^{-M-(\delta n+1)I}du, \tag{33}$$

where C is the path around the origin in the positive direction, beginning at and returning to positive infinity with respect for the branch cut along the positive real axis.

Proof. The right hand side of (33), gives

$$\frac{\Gamma(M+(\delta n+1)I)}{n!\,2\pi i}\int_C(u^\delta-\lambda x)^n e^u u^{-M-(\delta n+1)I}du$$
$$=\frac{\Gamma(M+(\delta n+1)I)}{n!\,2\pi i}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p x^p}{p}\int_C e^u u^{-M-(\delta p+1)I}du. \tag{34}$$

Using the contour integral representation for the reciprocal Gamma function (6), we get the following integral matrix functional

$$\Gamma^{-1}(M+(\delta p+1)I)=\frac{1}{2\pi i}\int_C e^t t^{-M-(\delta p+1)I}dt. \tag{35}$$

On using (34) and (35), one can easily prove the Theorem 4.1. □

On plugging $\delta = 1$ in (33), this yields the following corollary.

Corollary 4.2. *Let $M \in \mathbf{C}^{r\times r}$ be a matrix satisfying the spectral condition (14) and $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$. Then*

$$L_n^{(M,\lambda)}(x) = \frac{\Gamma(M+(n+1)I)}{n!\,2\pi i}\int_C (u-\lambda x)^n e^u u^{-M-(n+1)I}du, \tag{36}$$

where C is the path around the origin in the positive direction, beginning at and returning to positive infinity with respect for the branch cut along the positive real axis.

Theorem 4.3. *Let $M \in \mathbf{C}^{r\times r}$ be a matrix satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$ and $|\frac{\lambda}{\delta^\delta}| < 1$. Then*

$$\int_0^\infty x^M e^{-x} L_n^{(M,\delta,\lambda)}(x)dx = \frac{\Gamma(M+(\delta n+1)I)}{n!}\,{}_2F_\delta\left[\begin{matrix}-nI, M+I\\ \Delta(\delta; M+I)\end{matrix};\ \frac{\lambda}{\delta^\delta}\right]. \tag{37}$$

Proof. From the left hand side of (37), we find

$$\begin{aligned}
&\int_0^\infty x^M e^{-x} L_n^{(M,\delta,\lambda)}(x)dx\\
&= \frac{\Gamma(M+(\delta n+1)I)\Gamma^{-1}(M+I)}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p}{p!\delta^{\delta p}}\prod_{m=1}^{\delta}\left[\left(\frac{M+pI}{\delta}\right)_n\right]^{-1}\int_0^\infty x^{M+pI}e^{-x}dx\\
&= \frac{\Gamma(M+(\delta n+1)I)}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p(M+I)_p}{p!\delta^{\delta p}}\prod_{m=1}^{\delta}\left[\left(\frac{M+pI}{\delta}\right)_n\right]^{-1}\\
&= \frac{\Gamma(M+(\delta n+1)I)}{n!}\,{}_2F_\delta\left[\begin{matrix}-nI, M+I\\ \Delta(\delta; M+I)\end{matrix};\ \frac{\lambda}{\delta^\delta}\right],
\end{aligned}$$

which completes the proof of result (37). □

Theorem 4.4. *Let M and N be commuting matrices in $\mathbf{C}^{r\times r}$ satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$. Then*

$$L_n^{(M+N,\delta,\lambda)}(x) = \Gamma^{-1}(N)\Gamma(M+N+(\delta n+1)I)\Gamma^{-1}(M+(\delta n+1)I)\int_0^1 u^M(1-u)^{N-I}L_n^{(M,\delta,\lambda)}(xu^\delta)du. \tag{38}$$

Proof. Let

$$\begin{aligned}
\Psi &= \int_0^1 u^M(1-u)^{N-I}L_n^{(M,\delta,\lambda)}(xu^\delta)du\\
&= \frac{\Gamma(M+(\delta n+1)I)}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p x^p}{p!}\Gamma^{-1}(M+\delta nI+I)\int_0^1 u^{M+\delta pI}(1-u)^{N-I}du.
\end{aligned} \tag{39}$$

On using equation (3) and (4), we obtain

$$\begin{aligned}
\Psi &= \frac{\Gamma(N)\Gamma(M+(\delta n+1)I))}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p x^p}{p!}\Gamma^{-1}(M+N+(\delta p+1)I)\\
&= \Gamma(N)\Gamma(M+(\delta n+1)I))\Gamma^{-1}(M+N+(\delta n+1)I)L_n^{(M+N,\delta,\lambda)}(x).
\end{aligned} \tag{40}$$

On using (40) in right hand side of (38), this yields the desired result. □

On taking $\delta = 1$ in (38), one gets the following corollary.

Corollary 4.5. *Let M and N be commuting matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ satisfying the spectral condition (14) and $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$. Then*

$$L_n^{(M+N,\lambda)}(x) = \Gamma^{-1}(N)\Gamma(M+N+(n+1)I)\Gamma^{-1}(M+(n+1)I)\int_0^1 u^M(1-u)^{N-I}L_n^{(M,\lambda)}(xu)du. \tag{41}$$

Theorem 4.6. *Let M and N be commuting matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda, \mu \in \mathbf{C}$ with $\Re(\lambda) > 0$, $\Re(\mu) > 0$. Then*

$$\int_0^\mu (\mu-x)^{N-I}x^M L_n^{(M,\delta,\lambda)}(x^\delta)dx = \mu^{M+N}\Gamma(N)\Gamma(M+I+\delta nI)\Gamma^{-1}(M+N+I+\delta nI)L_n^{(M+N,\delta,\lambda)}(\mu^\delta). \tag{42}$$

Proof. Consider the left hand side of (42),

$$\begin{aligned}&\int_0^\mu (\mu-x)^{N-I}x^M L_n^{(M,\delta,\lambda)}(x^\delta)dx\\ &= \frac{\Gamma(M+(\delta n+1)I)}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p}{p!}\Gamma^{-1}(M+(\delta p+1)I)\int_0^\mu (\mu-x)^{N-I}x^{M+\delta p}dx,\end{aligned}$$

on plugging $\mu - x = \mu(1-t)$, we get

$$\begin{aligned}&\int_0^\mu (\mu-x)^{N-I}x^M L_n^{(M,\delta,\lambda)}(x^\delta)dx\\ &= \frac{\mu^{M+N}\Gamma(M+(\delta n+1)I)}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p\mu^{\delta p}}{p!}\Gamma^{-1}(M+(\delta p+1)I)\int_0^1 (1-t)^{N-I}t^{M+\delta pI}dt,\end{aligned}$$

on applying (4), this leads to the proof of the Theorem 4.6. □

On setting $\delta = 1$ in (42), this deduces the following result.

Corollary 4.7. *Let M and N be commuting matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ satisfying the spectral condition (14) and $\lambda, \mu \in \mathbf{C}$ with $\Re(\lambda) > 0$, $\Re(\mu) > 0$. Then*

$$\int_0^\mu (\mu-x)^{N-I}x^M L_n^{(M,\lambda)}(x)dx = \mu^{M+N}\Gamma(N)\Gamma(M+I+nI)\Gamma^{-1}(M+N+(n+1)I)L_n^{(M+N,\lambda)}(\mu). \tag{43}$$

Theorem 4.8. *Let M and N be commuting matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ and M, N and $M-N$ are positive stable matrices. Then*

$$L_n^{(M,\delta,\lambda)}(x^\delta) = x^{-M}\Gamma(M+(\delta n+1)I)\Gamma^{-1}(N+(\delta n+1)I)\Gamma^{-1}(M-N)\int_0^x (x-\mu)^{M-N-I}\mu^N L_n^{(N,\delta,\lambda)}(\mu^\delta)d\mu, \tag{44}$$

where $\lambda, \mu \in \mathbf{C}$, $\delta \in \mathbf{Z}^+$ with $\Re(\lambda) > 0$, $\Re(\mu) > 0$.

Proof. By making substitution $x - \mu = x(1-t)$ in right hand side of (44) and using Beta matrix function (4), one can easily prove the Theorem 4.8. □

On setting $\delta = 1$ in (44), this deduce the following result.

Corollary 4.9. *Let M and N be commuting matrices in $\mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ and M, N and $M-N$ are positive stable matrices. Then*

$$L_n^{(M,\lambda)}(x) = x^{-M}\Gamma(M+(n+1)I)\Gamma^{-1}(N+(n+1)I)\Gamma^{-1}(M-N)\int_0^x (x-\mu)^{M-N-I}\mu^N L_n^{(N,\lambda)}(\mu)d\mu, \tag{45}$$

where $\lambda, \mu \in \mathbf{C}$ with $\Re(\lambda) > 0$, $\Re(\mu) > 0$.

5. Recurrence relations, summation formulas and series relation

In this section, we derive the following differential recurrence relations, finite summation formulas and series relation of the matrix polynomials (15).

Theorem 5.1. *Let $M \in \mathbf{C}^{r\times r}$ be a matrix satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$. Then the following differential recurrence relations hold:*

$$\frac{d}{dx}\left(x^M L_n^{(M,\delta,\lambda)}(x^\delta)\right) = x^{(M-I)}(M+\delta nI)L_n^{(M-I,\delta,\lambda)}(x^\delta) \tag{46}$$

$$\left(\frac{d}{dx}\right)^m \left[x^M L_n^{(M,\delta,\lambda)}(\omega x^\delta)\right] = x^{M-mI}\Gamma(M+(\delta n+1)I)\Gamma^{-1}(M+(\delta n-m+1)I)L_n^{(M-mI,\delta,\lambda)}(wx^\delta) \tag{47}$$

$$ML_n^{(M,\delta,\lambda)}(x) + x\delta\frac{d}{dx}L_n^{(M,\delta,\lambda)}(x) = (M+\delta nI)L_n^{(M-I,\delta,\lambda)}(x). \tag{48}$$

Proof. The left hand side of (46), gives

$$\begin{aligned}
&\frac{d}{dx}\left(x^M L_n^{(M,\delta,\lambda)}(x^\delta)\right)\\
&= x^{(M-I)}\frac{\Gamma(M+(\delta n+1)I)}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p(M+\delta pI)x^{\delta p}}{p!}\Gamma^{-1}(M+(\delta p+1)I)\\
&= x^{(M-I)}(M+\delta nI)\frac{(M)_{\delta n}}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p x^{\delta p}}{p!}\left[(M)_{\delta p}\right]^{-1},
\end{aligned}$$

this immediately leads to the proof of the first relation (46). Similarly, one can easily prove the relation (47) and (48) . □

Theorem 5.2. *Let $M \in \mathbf{C}^{r\times r}$ be a matrix satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda, \mu \in \mathbf{C}$ with $\Re(\mu) > 0$, $\Re(\lambda) > 0$ and k is a non negative integer. Then the following summation formulas hold:*

$$\sum_{m=0}^{k}\frac{\begin{pmatrix}k\\m\end{pmatrix}x^m}{(M+I-kI)_m}D_x^m L_n^{(M,\delta,\lambda)}(x^\delta) = \frac{(M+I)_{\delta n}}{(M+I-kI)_{\delta n}}L_n^{(M-kI,\delta,\lambda)}(x^\delta) \tag{49}$$

$$\sum_{m=0}^{k}\frac{(\mu)^m}{m!}D_x^m L_n^{(M,\delta,\lambda)}(x) = L_n^{(M,\delta,\lambda)}(x+\mu). \tag{50}$$

Proof. From the left hand side of (49), we get

$$\begin{aligned}
&\sum_{m=0}^{k}\frac{\begin{pmatrix}k\\m\end{pmatrix}x^m}{(M+I-kI)_m}D_x^m L_n^{(M,\delta,\lambda)}(x^\delta)\\
&= \frac{(M+I)_{\delta n}}{n!}\sum_{m=0}^{k}\frac{(-kI)_m}{m!(M+I-kI)_m}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p(-\delta p)_m x^{\delta p}}{p!}\left[(M+I)_{\delta p}\right]^{-1}\\
&= \frac{(M+I)_{\delta n}}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p x^{\delta p}}{p!}\left[(M+I-kI)_{\delta p}\right]^{-1},
\end{aligned}$$

this immediately leads to the proof of the first result (49). Similarly, one can prove the result (50). □

Theorem 5.3. *Let $M, P, Q \in \mathbf{C}^{r\times r}$ be matrices satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$ and $|\frac{-\omega\lambda x}{\delta^\delta}| < 1$. Then the following series relation holds:*

$$\sum_{n=0}^{\infty}(P)_n\,[(Q+I)_n]^{-1}\,\Gamma^{-1}(M+(\delta n+1)I)L_n^{(M,\delta,\lambda)}(x)\,{}_1F_1\left[\begin{matrix}P+nI\\Q+(n+1)I\end{matrix}\;;\,-\omega\right]\omega^n$$
$$= {}_1F_{\delta+1}\left[\begin{matrix}P\\Q+I,\Delta(\delta;M+I)\end{matrix}\;;\,\frac{-\omega\lambda x}{\delta^\delta}\right]. \tag{51}$$

Proof. On using the Corollary 3.5, we get

$$\sum_{n=0}^{\infty}\Gamma^{-1}(M+(\delta n+1)I)L_n^{(M,\delta,\lambda)}(x)\omega^n e^{-\omega} = \Gamma^{-1}(M+I)\,{}_0F_\delta\left[\begin{matrix}-\\\Delta(\delta;M+I)\end{matrix}\;;\,\frac{-\omega\lambda x}{\delta^\delta}\right].$$

Now multiplying both the side by ω^{P-I} and afterwards employing the operator D_ω^{P-Q-I}, where $D_\omega = \frac{d}{d\omega}$, we obtain

$$\sum_{n=0}^{\infty}\Gamma^{-1}(M+(\delta n+1)I)L_n^{(M,\delta,\lambda)}(x)\sum_{r=0}^{\infty}\frac{(-\omega)^r}{r!}\Gamma(P+(n+r)I)\Gamma^{-1}(Q+(n+r+1)I)\omega^n$$
$$= \Gamma^{-1}(M+I)\sum_{p=0}^{\infty}\frac{\left(\frac{-\lambda\omega x}{\delta^\delta}\right)^p}{p!}\Gamma(P+pI)\Gamma^{-1}(Q+(p+1)I)\prod_{m=1}^{\delta}\left[\left(\frac{M+mI}{\delta}\right)_p\right]^{-1},$$

this immediately leads to (51). □

Corollary 5.4. *Let $M, P, Q \in \mathbf{C}^{r\times r}$ be matrices satisfying the spectral condition (14) and $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$ and $|-\omega\lambda x| < 1$. Then the following series relation holds:*

$$\sum_{n=0}^{\infty}(P)_n\,[(Q+I)_n]^{-1}\,\Gamma^{-1}(M+(n+1)I)L_n^{(M,\lambda)}(x)\,{}_1F_1\left[\begin{matrix}P+nI\\Q+(n+1)I\end{matrix}\;;\,-\omega\right]\omega^n$$
$$= {}_1F_2\left[\begin{matrix}P\\Q+I,M+I\end{matrix}\;;\,-\omega\lambda x\right]. \tag{52}$$

6. Fractional integral and derivative operators

In this segment, we establish the composition of the Riemann-Liouville fractional integral and derivative operators with matrix polynomials (15).

Theorem 6.1. *Let $M \in \mathbf{C}^{r\times r}$ be a matrix satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^+$, $\lambda, \mu \in \mathbf{C}$ with $\Re(\mu) > 0$, $\Re(\lambda) > 0$. Then*

$$\left(I^\mu\left[x^M L_n^{(M,\delta,\lambda)}(t^\delta)\right]\right)(x) = x^{M+\mu I}\Gamma(M+(\delta n+1)I)\Gamma^{-1}(M+(\delta n+\mu+1)I)$$
$$\times L_n^{(M+\mu I,\delta,\lambda)}(x^\delta). \tag{53}$$

Proof. Consider the left hand side of (53),

$$\left(I^\mu\left[x^M L_n^{(M,\delta,\lambda)}(t^\delta)\right]\right)(x)$$
$$= \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}t^M L_n^{(M,\delta,\lambda)}(t^\delta)dt$$
$$= \frac{\Gamma(M+(\delta n+1)I)}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p}{p!}\Gamma^{-1}(M+(\delta p+1)I)\left(I^\mu t^{M+\delta pI}\right)(x)$$
$$= \frac{x^{M+\mu I}\Gamma(M+(\delta n+1)I}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p x^{\delta p}}{p!}\Gamma^{-1}(M+(\delta p+\mu+1)I),$$

this yields the result (53). □

On setting $\delta = 1$ in (53), this deduces to the following corollary.

Corollary 6.2. *Let $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (14) and $\lambda, \mu \in \mathbf{C}$ with $\Re(\mu) > 0$, $\Re(\lambda) > 0$. Then*

$$\left(I^{\mu}\left[x^{M}L_n^{(M,\lambda)}(t)\right]\right)(x) = x^{M+\mu I}\Gamma(M+(n+1)I)\Gamma^{-1}(M+(n+\mu+1)I)L_n^{(M+\mu I,\lambda)}(x). \tag{54}$$

Theorem 6.3. *Let $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (14) and $\delta \in \mathbf{Z}^{+}$, $\lambda, \mu, \omega \in \mathbf{C}$ with $\Re(\mu) > 0$, $\Re(\omega) > 0$, $\Re(\lambda) > 0$. Then*

$$\left(D^{\mu}\left[u^{M}L_n^{(M,\delta,\lambda)}(\omega u^{\delta})\right]\right)(x) = x^{M-\mu I}\Gamma(M+(\delta n+1)I)\Gamma^{-1}(M+(\delta n-\mu+1)I)L_n^{(M-\mu I,\delta,\lambda)}(\omega x^{\delta}). \tag{55}$$

Proof. From the left hand side of (55), we get

$$\left(D^{\mu}\left[u^{M}L_n^{(M,\delta,\lambda)}(\omega u^{\delta})\right]\right)(x) = \left(\frac{d}{dx}\right)^{m}\left(I^{m-\mu}u^{M}L_n^{(M,\delta,\lambda)}(\omega u^{\delta})\right)(x),$$

on using equation (12), we find that

$$\begin{aligned}
&\left(D^{\mu}\left[u^{M}L_n^{(M,\delta,\lambda)}(\omega u^{\delta})\right]\right)(x)\\
&= \left(\frac{d}{dx}\right)^{m}\left[\frac{1}{\Gamma(m-\mu)}\int_0^x (x-u)^{(m-\mu)-1}u^{M}L_n^{(M,\delta,\lambda)}(\omega u^{\delta})du\right]\\
&= \left(\frac{d}{dx}\right)^{m}\left[\frac{\Gamma(M+(\delta n+1)I)}{n!}\sum_{p=0}^{n}\frac{(-nI)_p\lambda^p\omega^p}{p!}\Gamma^{-1}(M+(\delta p+1)I)\left(I^{m-\mu}u^{M+\delta pI}\right)(x)\right],
\end{aligned}$$

from Lemma 1.4 and equation (15), we find that

$$\begin{aligned}
\left(D^{\mu}\left[u^{M}L_n^{(M,\delta,\lambda)}(\omega u^{\delta})\right]\right)(x) &= \Gamma(M+(\delta n+1)I)\Gamma^{-1}(M+(\delta n+m-\mu+1)I)\\
&\quad\times\left(\frac{d}{dx}\right)^{m}\left[x^{M+(m-\mu)I}L_n^{(M+(m-u)I,\delta,\lambda)}(\omega x^{\delta})\right],
\end{aligned}$$

use of (47), this yields (55). □

On setting $\delta = 1$ in (55), this deduces to the following corollary.

Corollary 6.4. *Let $M \in \mathbf{C}^{\mathbf{r}\times\mathbf{r}}$ be a matrix satisfying the spectral condition (14) and $\lambda, \mu, \omega \in \mathbf{C}$ with $\Re(\mu) > 0$, $\Re(\omega) > 0$, $\Re(\lambda) > 0$. Then*

$$\begin{aligned}
\left(D^{\mu}\left[u^{M}L_n^{(M,\lambda)}(\omega u)\right]\right)(x) &= x^{M-\mu I}\Gamma(M+(n+1)I)\Gamma^{-1}(M+(n-\mu+1)I)\\
&\quad\times L_n^{(M-\mu I,\lambda)}(\omega x).
\end{aligned} \tag{56}$$

Concluding Remarks

In this paper, we discussed the extension of Laguerre matrix polynomials and various properties including the hypergeometric representation, generating matrix relations, integral representations, recurrence relations, summation formulas, series relation, fractional integral and derivative operators and several interesting special cases have been obtained. These results can play a significant role in the Combinatorial Problems, Wireless Communications and Signal Processing, Theory of Special Functions, Operator Theory, Matrix Analysis Theory, Mathematical Physics, Classical Analysis, Fractional Calculus and Statistics.

We can find orthogonality, Rodrigues formula, differential equation, q-analogues and properties using Lie theory approach to the matrix polynomials $L_n^{(M,\delta,\lambda)}(x)$.

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