Inequalities of Generalized Euclidean Berezin Number

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Abstract. In this paper, we present several Berezin number inequalities involving extensions of Euclidean Berezin number for \( n \) operators. Among other inequalities for \( (T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{H}) \) we show that

\[
\text{ber}_p(T_1, \ldots, T_n) \leq \frac{1}{2^p} \text{ber}\left(\sum_{i=1}^{n} ||T_i||^p\right),
\]

where \( p > 1 \).

1. Introduction

Let \( \mathcal{B}(\mathcal{H}) \) denote the \( C^* \)-algebra of all bounded linear operators on a complex Hilbert space \( \mathcal{H} \) with an inner product \( \langle \cdot, \cdot \rangle \) and the corresponding norm \( ||\cdot|| \). An operator \( A \in \mathcal{B}(\mathcal{H}) \) is called positive if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \), and then we write \( A \geq 0 \).

A functional Hilbert space \( \mathcal{H} = \mathcal{H}(\Omega) \) is a Hilbert space of complex valued functions on a nonempty set \( \Omega \), which has the property that point evaluations are continuous i.e. for each \( \lambda \in \Omega \) the map \( f \mapsto f(\lambda) \) is a continuous linear functional on \( \mathcal{H} \). The Riesz representation theorem ensure that for each \( \lambda \in \Omega \) there is a unique element \( k_{\lambda} \in \mathcal{H} \) such that \( f(\lambda) = \langle f, k_{\lambda} \rangle \) for all \( f \in \mathcal{H} \). The collection \( \{k_{\lambda} : \lambda \in \Omega\} \) is called the reproducing kernel of \( \mathcal{H} \). If \( \{e_n\} \) is an orthonormal basis for a functional Hilbert space \( \mathcal{H} \), then the reproducing kernel of \( \mathcal{H} \) is given by \( k_{\lambda}(z) = \sum_n e_n(\lambda)e_n(z) \); (see [13, Problem 37]). For \( \lambda \in \Omega \), let \( \tilde{k}_{\lambda} = \frac{k_{\lambda}}{||k_{\lambda}||} \) be the normalized reproducing kernel of \( \mathcal{H} \). For a bounded linear operator \( A \) on \( \mathcal{H} \), the function \( \tilde{A} \) defined on \( \Omega \) by \( \tilde{A}(\lambda) = \langle \tilde{A}k_{\lambda}, k_{\lambda} \rangle \) is the Berezin symbol of \( A \), which firstly have been introduced by Berezin [5, 6]. The Berezin set and the Berezin number of the operator \( A \) are defined by

\[
\text{Ber}(A) := \{|\tilde{A}(\lambda) : \lambda \in \Omega\} \quad \text{and} \quad \text{ber}(A) := \sup\{|\tilde{A}(\lambda) : \lambda \in \Omega|, \]

respectively, (see [15]). The numerical radius of \( A \in \mathcal{B}(\mathcal{H}) \) is defined by \( w(A) := \sup\{|\langle Ax, x \rangle : x \in \mathcal{H}, ||x|| = 1\} \). It is clear that

\[
\text{ber}(A) \leq w(A) \leq ||A||
\]
for all $A \in \mathcal{B}(\mathcal{H})$. Moreover, the Berezin number of operators $A, B$ satisfy the following properties:

(i) $\text{ber}(\alpha A) = |\alpha| \text{ber}(A)$ for all $\alpha \in \mathbb{C}$;

(ii) $\text{ber}(A + B) \leq \text{ber}(A) + \text{ber}(B)$.

Let $A_i \in \mathcal{B}(\mathcal{H})$ ($1 \leq i \leq n$). The generalized Euclidean Berezin number of $A_1, ..., A_n$ is defined in [1] as follows:

$$\text{ber}_p(A_1, ..., A_n) := \sup_{\lambda \in \Omega} \left( \sum_{i=1}^{n} \left| \langle A\hat{k}_i, \hat{k}_i \rangle \right|^p \right)^{\frac{1}{p}}, \quad (p \geq 1).$$

In the case $p = 2$, we have the Euclidean Berezin number and denote by

$$\text{ber}_e(A_1, ..., A_n) := \sup_{\lambda \in \Omega} \left( \sum_{i=1}^{n} \left| \langle A\hat{k}_i, \hat{k}_i \rangle \right|^2 \right)^{\frac{1}{2}}.$$

For $p = 1$, we have $\text{ber}_1(A_1, ..., A_n)$, such that if $A_1 = A_2 = \ldots = A_n = A$, then $\text{ber}_1(A_1, ..., A_n) = n\text{ber}(A)$.

The generalized Euclidean Berezin number $\text{ber}_p(p \geq 1)$ has the following properties:

(i) $\text{ber}_p(\alpha A_1, ..., \alpha A_n) = |\alpha| \text{ber}_p(A_1, ..., A_n)$ for all $\alpha \in \mathbb{C}$;

(ii) $\text{ber}_p(A_1 + B_1, ..., A_n + B_n) \leq \text{ber}_p(A_1, ..., A_n) + \text{ber}_p(B_1, ..., B_n)$;

(iii) $\text{ber}_p(X^* A_1 X, ..., X^* A_n X) \leq \|X\| \text{ber}_p(A_1, ..., A_n)$;

(iv) $\text{ber}_p(A_1 A_2, ..., A_n) = \text{ber}_p(A_1^*, A_2^*, ..., A_n^*)$,

where $A_i, B_j \in \mathcal{B}(\mathcal{H})$, $X \in \mathcal{B}(\mathcal{H}(\Omega))$ ($1 \leq i \leq n$).

The proof of the properties (i) – (iv) immediately comes from definition of the generalized Berezin number.

Namely, the Berezin symbol have been investigated in detail for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it is widely applied in the various questions of analysis and uniquely determines the operator (i.e., for all $\lambda \in \Omega, \hat{A}(\lambda) = \hat{B}(\lambda)$ implies $A = B$). For further information about Berezin symbol we refer the reader to [1, 9, 12, 16, 19–21] and references therein.

In this paper, we want to examine the properties of the generalized Euclidean Berezin number. Moreover, we obtain some generalization of the Euclidean Berezin number inequalities. For these goals, we will apply some methods from [18].

2. Main results

In this section we would like to check some properties about the generalized Euclidean Berezin number and then we state some inequalities related to this concept.

In the next lemma, we show that the generalized Euclidean Berezin number is weakly unitarily invariant.

**Lemma 2.1.** Suppose $A_i, U_i \in \mathcal{B}(\mathcal{H}(\Omega))$ ($i = 1, \ldots, n$) such that $U_i$’s are unitarily invariant operators. Then

$$\text{ber}_p(U_1^* A_1 U_1, ..., U_n^* A_n U_n) = \text{ber}_p(A_1, ..., A_n).$$

**Proof.** From the definition of the generalized Euclidean Berezin number, we have

$$\text{ber}_p(U_1^* A_1 U_1, ..., U_n^* A_n U_n) = \sup_{\lambda} \left( \sum_{i=1}^{n} \left| \langle U_i^* A_1 U_i \hat{k}_i, \hat{k}_i \rangle \right|^p \right)^{\frac{1}{p}}$$

$$= \sup_{\lambda} \left( \sum_{i=1}^{n} \left| \langle A\hat{k}_i, \hat{k}_i \rangle \right|^p \right)^{\frac{1}{p}}$$

$$= \sup_{\lambda} \left( \sum_{i=1}^{n} \left| \langle \hat{k}_i, \hat{k}_i \rangle \right|^p \right)^{\frac{1}{p}}$$

$$= \text{ber}_p(A_1, ..., A_n).$$
The next result follows from the Jensen’s inequality, which asserts:

$$
\left( \frac{1}{n} \sum_{i=1}^{n} a_i^p \right)^{1/p} \leq \frac{1}{n} \sum_{i=1}^{n} a_i
$$

(2)

for any real positive sequence \((a_i)^n\) and \(p \geq 1\).

**Proposition 2.2.** Let \(A_i \in B(\mathcal{H}(\Omega))(i = 1, \ldots, n)\). Then

(i) \(\text{ber}_p(A_1, \ldots, A_n) \leq \text{ber}_r(A_1, \ldots, A_n) \leq n^{-\frac{1}{2}} \text{ber}_r(A_1, \ldots, A_n)\) for \(p \geq 1\);

(ii) \(\text{ber}_p(A_1, \ldots, A_n) \leq n^{\frac{1}{2}-\frac{1}{p}} \text{ber}_r(A_1, \ldots, A_n)\) for \(q \geq p \geq 1\).

In particular,

\[
\text{ber}_r(A_1, \ldots, A_n) \geq n^{\frac{1}{2}-\frac{1}{r}} \text{ber}_r(A_1, \ldots, A_n).
\]

**Proof.** The part (i) follows from (2) by letting \(a_i = |\langle A_i \hat{k}_i, \hat{k}_i \rangle|\)\((i = 1, \ldots, n)\).

For part (ii), from power inequality, we have

\[
\left( \frac{1}{n} \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \leq \left( \frac{1}{n} \sum_{i=1}^{n} a_i^q \right)^{\frac{1}{q}}
\]

(p ≤ q),

if we take \(a_i = |\langle A_i \hat{k}_i, \hat{k}_i \rangle|\)\((i = 1, \ldots, n)\), then

\[
\left( \frac{1}{n} \sum_{i=1}^{n} |\langle A_i \hat{k}_i, \hat{k}_i \rangle|^p \right)^{\frac{1}{p}} \leq \left( \frac{1}{n} \sum_{i=1}^{n} |\langle A_i \hat{k}_i, \hat{k}_i \rangle|^q \right)^{\frac{1}{q}}
\]

(p ≤ q)

Taking the supremum over \(\lambda \in \Omega\), we deduce the desired result. □

To prove more the Berezin number inequalities, we need several well known lemmas.

The following lemma is a simple consequence of the classical Jensen and Young inequalities(see [14]).

**Lemma 2.3.** Let \(a, b \geq 0\) and \(0 \leq \nu \leq 1\). Then

\[
a^\nu b^{1-\nu} \leq \nu a + (1-\nu)b \leq (\nu a^\nu + (1-\nu)b^\nu)^{\frac{1}{\nu}}
\]

(3)

for \(r \geq 1\).

The following lemma is known as the generalized mixed Schwarz inequality [17].

**Lemma 2.4.** Let \(A \in B(\mathcal{H})\) and \(x, y \in \mathcal{H}\) be any vectors.

(a) If \(0 \leq \nu \leq 1\), then

\[
|\langle Ax, y \rangle|^2 \leq \langle |A|^2 \nu x, x \rangle \langle |A|^2 (1-\nu) y, y \rangle,
\]

where \(|A| = (A^*A)^{\frac{1}{2}}\) is the absolute value of \(A\).

(b) If \(f, g\) are nonnegative continuous functions on \([0, \infty)\) which are satisfying the relation \(f(t)g(t) = t(t \in [0, \infty))\), then

\[
|\langle Ax, y \rangle| \leq \|f(|A|)x\| \|g(|A^*|)y\|
\]

for all \(x, y \in \mathcal{H}\).
The next lemma follows from the spectral theorem for positive operators and the Jensen’s inequality (see [17]).

**Lemma 2.5. (McCarthy inequality).** Let $A \in B(\mathcal{H})$, $A \geq 0$ and $x \in \mathcal{H}$ be a unit vector. Then

(a) $\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle$ for $r \geq 1$;
(b) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for $0 < r \leq 1$.

Now, we are in the position to state our results.

**Theorem 2.6.** Suppose that $(A_1, A_2, \ldots, A_n) \in B(\mathcal{H})^n$ and $f, g$ are nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, \infty))$. Then

$$ber_p(A_1, \ldots, A_n) \leq ber_p(f'(\langle A_1^r \rangle), \ldots, f'(\langle A_n^r \rangle)) ber_q(g'(\langle A_1^r \rangle), \ldots, g'(\langle A_n^r \rangle)),$$

in which $r \geq 2, p \geq q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** From the Lemmas 2.4(b) and 2.5(a), we have

$$\|\langle Ax, x \rangle\|^r \leq \|f(\langle A_i^r \rangle)x\|^r \|g(\langle A_i^r \rangle)x\|^r$$

$$= (f^2(\langle A_i^r \rangle)x, x) \frac{1}{r} (g^2(\langle A_i^r \rangle)x, x) \frac{1}{r}$$

$$\leq (f^r(\langle A_i^r \rangle)x, g^r(\langle A_i^r \rangle)x, x).$$

By taking the sum over all $i$ from 1 to $n$, and applying the Holder’s inequality, we have

$$\sum_{i=1}^{n} \|\langle Ax, x \rangle\|^r \leq \sum_{i=1}^{n} (f^r(\langle A_i^r \rangle)x, x) (g^r(\langle A_i^r \rangle)x, x)$$

$$\leq \sum_{i=1}^{n} (f^r(\langle A_i^r \rangle)x, x)^p \sum_{i=1}^{n} (g^r(\langle A_i^r \rangle)x, x)^q.$$

Now, by taking the supremum over $\lambda \in \Omega$, we deduce the desired result. □

**Corollary 2.7.** Let $(A_1, A_2, \ldots, A_n) \in B(\mathcal{H})^n$. Then

$$ber_p(A_1, \ldots, A_n) \leq ber_p(|\langle A_1^r \rangle|, |\langle A_n^r \rangle|) ber_q(|\langle A_1^r \rangle|, \ldots, |\langle A_n^r \rangle|),$$

which $p \geq q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** By putting $f(t) = g(t) = t^{\frac{1}{r}}$ and $r = 2$ in (4), we get the desired inequality. □

**Remark 2.8.** By putting $p = q = 2$ in (4) for any $r \geq 2$, we have the following inequality:

$$ber_p(A_1, \ldots, A_n) \leq ber_p(f'(\langle A_1^r \rangle), \ldots, f'(\langle A_n^r \rangle)) ber_q(g'(\langle A_1^r \rangle), \ldots, g'(\langle A_n^r \rangle)).$$

For the next result we applying the following inequality, which is found in [7]:

$$\|DCAx, x\|^2 \leq \langle A^r B^2 Ax, x \rangle (D|C^2|D^* y, y),$$

where $A, B, C, D \in B(\mathcal{H})$ and $x, y \in \mathcal{H}$.

**Theorem 2.9.** Assume that $A_i, B_i, C_i, D_i \in B(\mathcal{H})(i = 1, \ldots, n)$. Then

$$ber_p(D_1 C_1 B_1 A_1, \ldots, D_n C_n B_n A_n) \leq ber_p(A_1^2 B_1^2 A_1, \ldots, A_n^2 B_n^2 A_n) ber_q(D_1 |C_1|^2 D_1^*, \ldots, D_n |C_n|^2 D_n^*)$$

for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
Proof. If $\hat{k}_\lambda$ is the normalized reproducing kernel of $\mathcal{H}(\Omega)$, then by applying (6) with putting $x = y = \hat{k}_\lambda$ and applying the Holder’s inequality, we have

$$
sup_{A} \left( \sum_{i=1}^{n} \left| \left( D_{i}C_{i}B_{i}A_{\hat{k}_{\lambda}}A_{\hat{k}_{\lambda}} \right) \right|^{\frac{1}{p}} \right) \leq \sup_{A} \left( \sum_{i=1}^{n} \left( A_{i}^{*}B_{i}^{*}A_{i}\hat{k}_{\lambda},\hat{k}_{\lambda} \right) \right)^{\frac{1}{p}} \left( D_{i}C_{i}B_{i}A_{\hat{k}_{\lambda}}A_{\hat{k}_{\lambda}} \right) \leq \sup_{A} \left( \sum_{i=1}^{n} \left( A_{i}^{*}B_{i}^{*}A_{i}\hat{k}_{\lambda},\hat{k}_{\lambda} \right) \right)^{\frac{1}{p}} \left( D_{i}C_{i}B_{i}A_{\hat{k}_{\lambda}}A_{\hat{k}_{\lambda}} \right) \leq \text{ber}_p(A_{1}^{*}|B_{1}|^{2}A_{1},\ldots,A_{n}^{*}|B_{n}|^{2}A_{n})\text{ber}_q(D_{1}|C_{1}|^{2}D_{1}^{*},\ldots,D_{n}|C_{n}|^{2}D_{n}^{*}).\]

$\square$

Corollary 2.10. Let $B_{i} \in \mathcal{B}(\mathcal{H}(\Omega))(i = 1,\ldots,n)$. Then

$$
\text{ber}_p(B_{1}^{2},\ldots,B_{n}^{2}) \leq \text{ber}_p(|B_{1}|^{2},\ldots,|B_{n}|^{2})\text{ber}_q(|B_{1}|^{2},\ldots,|B_{n}|^{2})
$$

(8)

for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

In particular for $p = q = 2$,

$$
\text{ber}_r(B_{1}^{2},\ldots,B_{n}^{2}) \leq \text{ber}_e(|B_{1}|^{2},\ldots,|B_{n}|^{2}).
$$

(9)

Proof. By putting $C_{i} = B_{i}^{*}, A_{i} = U_{i}$ and $D_{i} = U_{i}^{*}$ in (7), where $U_{i}(i = 1,\ldots,n)$ are unitaries, we have

\[
\text{ber}_r(U_{1}^{*}B_{1}^{2}U_{1},\ldots,U_{n}^{*}B_{n}^{2}U_{1}) \leq \text{ber}_p(U_{1}^{*}|B_{1}|^{2}U_{1},\ldots,U_{1}^{*}|B_{n}|^{2}U_{1})\text{ber}_q(U_{1}^{*}|B_{1}|^{2}U_{1},\ldots,U_{n}^{*}|B_{n}|^{2}U_{1}).
\]

The result deduce from that fact $\text{ber}_p(\cdot)$ is weakly unitarily invariant. $\square$

Furuta [8] proved the following generalization of the Kato’s inequality as follows:

$$
|\langle A|A^{\alpha+\beta-1}x,y \rangle|^{2} \leq |\langle A|A^{\alpha}x,y \rangle| |\langle A|A^{\beta}y,y \rangle|
$$

(10)

for all $A \in \mathcal{B}(\mathcal{H}), x, y \in \mathcal{H}$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \geq 1$.

Applying (10), we have the next result for the Berezin number concept.

 Proposition 2.11. Assume that $A \in \mathcal{B}(\mathcal{H}(\Omega))$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \geq 1$. Then

$$
\text{ber}(A|A|^{\alpha+\beta-1}) \leq \frac{1}{2}(\text{ber}(|A|^{\alpha}) + \text{ber}(|A|^{\beta})).
$$

(11)

Proof. If $\hat{k}_\lambda$ is the normalized reproducing kernel of $\mathcal{H}(\Omega)$, by applying (10) with putting $x = y = \hat{k}_\lambda$, and the arithmetic-geometric mean, we have

\[
|\langle A|A^{\alpha+\beta-1}\hat{k}_{\lambda}^{\alpha},\hat{k}_{\lambda}^{\beta} \rangle| \leq \left( |\langle A^{\alpha}\hat{k}_{\lambda}^{\alpha},\hat{k}_{\lambda}^{\alpha} \rangle| |\langle A^{\beta}\hat{k}_{\lambda}^{\beta},\hat{k}_{\lambda}^{\beta} \rangle| \right)^{\frac{1}{2}} \leq \frac{1}{2} \left( |\langle A^{\alpha}\hat{k}_{\lambda}^{\alpha},\hat{k}_{\lambda}^{\beta} \rangle| + |\langle A^{\beta}\hat{k}_{\lambda}^{\alpha},\hat{k}_{\lambda}^{\beta} \rangle| \right) \leq \frac{1}{2}(\text{ber}(|A|^{\alpha}) + \text{ber}(|A|^{\beta})).
\]

By taking the supremum over all $\lambda \in \Omega$, we get the desired result. $\square$
Theorem 2.12. Suppose that $A_i, B_i, C_i, D_i \in \mathbb{B}(\mathcal{H}(\Omega))(i = 1, \ldots, n)$. Then

$$\text{ber}_p^\mu(D_1 C_1 B_1 A_1, \ldots, D_n C_n B_n A_n)$$

$$\leq \frac{1}{2} \text{ber} \left[ \sum_{i=1}^n \left( v(A_i^* B_i^2 A_i) + (1 - v)(D_i |C_i|^2 D_i)^p \right) \right],$$

(12)

where $p > 1$ and $v \in [0, 1]$.

Proof. If $\tilde{k}_3$ is the normalized reproducing kernel of $\mathcal{H}(\Omega)$, by applying (6), and putting $x = y = \tilde{k}_3$, we have

$$\sum_{i=1}^n (D_i C_i B_i A_i \tilde{k}_3, \tilde{k}_3)^p$$

$$= \sum_{i=1}^n \left( \langle A_i^* B_i^2 A_i, \tilde{k}_3 \rangle \right)^p (D_i |C_i|^2 D_i \tilde{k}_3, \tilde{k}_3)^p$$

$$\leq \frac{1}{2} \sum_{i=1}^n \left[ (v(A_i^* B_i^2 A_i, \tilde{k}_3) + (1 - v)(D_i |C_i|^2 D_i \tilde{k}_3, \tilde{k}_3))^p \right.$$

$$\left. + (v(A_i^* B_i^2 A_i, \tilde{k}_3) + (1 - v)(D_i |C_i|^2 D_i \tilde{k}_3, \tilde{k}_3))^p \right]$$

(by $\frac{a + b}{2} \leq \frac{a^2 + b^2}{2}$, and Lemma 2.3)

$$= \frac{1}{2} \sum_{i=1}^n \left( v(A_i^* B_i^2 A_i, \tilde{k}_3) + (1 - v)(D_i |C_i|^2 D_i \tilde{k}_3, \tilde{k}_3) \right)^p + \sum_{i=1}^n \left[ \left( v(A_i^* B_i^2 A_i, \tilde{k}_3) + (1 - v)(D_i |C_i|^2 D_i \tilde{k}_3, \tilde{k}_3) \right)^p \right.$$

(by Lemma 2.5(a))

$$= \frac{1}{2} \left( \sum_{i=1}^n v(A_i^* B_i^2 A_i, (1 - v)(D_i |C_i|^2 D_i \tilde{k}_3, \tilde{k}_3) \right)^p + \left( (1 - v)A_i^* B_i^2 A_i + v(D_i |C_i|^2 D_i \tilde{k}_3, \tilde{k}_3) \right)^p$$

$$\leq \frac{1}{2} \text{ber} \left[ \sum_{i=1}^n \left( v(A_i^* B_i^2 A_i, (1 - v)(D_i |C_i|^2 D_i \tilde{k}_3, \tilde{k}_3) \right)^p + \left( (1 - v)A_i^* B_i^2 A_i + v(D_i |C_i|^2 D_i \tilde{k}_3, \tilde{k}_3) \right)^p \right]$$

Now, by taking the supremum over $\lambda \in \Omega$, we deduce statement desired result. 

Corollary 2.13. Let $A_i, B_i, C_i, D_i \in \mathbb{B}(\mathcal{H}(\Omega))(i = 1, \ldots, n)$. Then

$$\text{ber}_p^\mu(D_1 C_1 B_1 A_1, \ldots, D_n C_n B_n A_n) \leq \frac{1}{2} \text{ber} \left[ \sum_{i=1}^n (A_i^* B_i^2 A_i + D_i |C_i|^2 D_i)^p \right]$$

$$\leq \frac{1}{2} \text{ber} \left[ \sum_{i=1}^n (A_i^* B_i^2 A_i)^p + (D_i |C_i|^2 D_i)^p \right],$$

(13)

where $p > 1$. 
Proof. The inequality (13) immediately comes from (12) by putting \( \nu = \frac{1}{2} \), and applying from that fact \((\frac{a+b}{2})^p \leq \frac{a^p+b^p}{2}\). \(\square\)

In the next result, we state an extension of (11).

**Corollary 2.14.** Assume that \((T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{H}(\Omega))^n\). Then

\[
ber_p(T_1, \ldots, T_n) \leq \frac{1}{2^p} ber\left( \sum_{i=1}^n (|T_i|^2 + |T_i|^2) \right),
\]

where \(\alpha, \beta \in [0, 1], p > 1\) such that \(\alpha + \beta \geq 1\).

**Proof.** Let \(T_i = U_i|T_i|(i = 1, \ldots, n)\) be the polar decomposition of \(T_i\). Then by putting \(D_i = U_i, B_i = 1_{\mathcal{H}(\Omega)}, C_i = |T_i|^p\) and \(A_i = |T_i|^q\) for all \(\alpha, \beta \geq 0\) such that \(\alpha + \beta \geq 1\), we have

\[
D_i C_i B_i A_i = |T_i|^p, \quad A_i^* B_i^2 A_i = |T_i|^q, \quad \text{and} \quad D_i|C_i|^2 D_i^* = |T_i|^q.
\]

So by applying (13) we get the statement result. \(\square\)

**Remark 2.15.** By putting \(\alpha = \beta = \frac{1}{2}\) in (14), we have the following inequality:

\[
ber_p(T_1, \ldots, T_n) \leq \frac{1}{2^p} ber\left( \sum_{i=1}^n (|T_i| + |T_i|^p) \right).
\]

**Theorem 2.16.** Suppose that \(A_i, B_i, C_i, D_i \in \mathcal{B}(\mathcal{H}(\Omega)) (i = 1, \ldots, n)\). Then

\[
ber_4(D_i C_i B_i A_i, \ldots, D_i C_i B_n A_n) \leq ber\left\{ \sum_{i=1}^n \frac{1}{2^p} \left( (A_i^* B_i^2 A_i)^{(1-\nu)p} + (A_i^* B_i^2 A_i)^{q(1-\nu)} \right) + \frac{1}{2^p} \left( (D_i|C_i|^2 D_i^*)^{(1-\nu)q} + (D_i|C_i|^2 D_i^*)^{q(1-\nu)} \right) \right\},
\]

where \(p, q > 1\) and \(\nu \in [0, 1]\) such that \(p\nu \geq 2\) and \(\frac{1}{p} + \frac{1}{q} = 1\).

**Proof.** If \(\hat{k}_i\) is the normalized reproducing kernel of \(\mathcal{H}(\Omega)\), by applying (6), and putting \(x = y = \hat{k}_i\), we have

\[
\sum_{i=1}^n \left| (D_i C_i B_i A_i \hat{k}_i, \hat{k}_i) \right|
\]

\[
\leq \frac{1}{2^p} \sum_{i=1}^n \left( (A_i^* B_i^2 A_i \hat{k}_i, \hat{k}_i) \right)^{\frac{1}{2}} \left( (D_i|C_i|^2 D_i^* \hat{k}_i, \hat{k}_i) \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} \sum_{i=1}^n \left( (A_i^* B_i^2 A_i \hat{k}_i, \hat{k}_i) \right)^{\frac{1}{2}} \left( (D_i|C_i|^2 D_i^* \hat{k}_i, \hat{k}_i) \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} \left( \sum_{i=1}^n (A_i^* B_i^2 A_i \hat{k}_i, \hat{k}_i)^{\frac{p}{2}} \right)^{\frac{1}{2}} \left( \sum_{i=1}^n (D_i|C_i|^2 D_i^* \hat{k}_i, \hat{k}_i)^{\frac{q}{2}} \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} \left( \sum_{i=1}^n (A_i^* B_i^2 A_i \hat{k}_i, \hat{k}_i)^{p(1-\nu)} \right)^{\frac{1}{2}} \left( \sum_{i=1}^n (D_i|C_i|^2 D_i^* \hat{k}_i, \hat{k}_i)^{q(1-\nu)} \right)^{\frac{1}{2}}
\]

(by the Holder’s inequality)
Now, by taking the supremum over $\lambda \in \Omega$, we state the desired result. \hfill \Box

By putting $p = q = 2$ in the inequality (16), we get the next result.

**Corollary 2.17.** Let $A_i, B_i, C_i, D_i \in \mathcal{B}(H(\Omega))(i = 1, \ldots, n)$. Then

$$
\text{ber}_1(D_1C_1B_1A_1, \ldots, D_nC_nB_nA_n)
\leq 1/4 \text{ber} \left( \sum_{i=1}^{n} \left( (A_i^*|B_i|^2A_i)^{\nu(1-\nu)} + (A_i^*|B_i|^2A_i)^{2\nu} \right) + \left( (D_i|C_i|^2D_i^*)^{\nu(1-\nu)} + (D_i|C_i|^2D_i^*)^{2\nu} \right) \right),
$$

(17)

where $\nu \in [0, 1]$.

**Corollary 2.18.** Assume that $(T_1, \ldots, T_n) \in \mathcal{B}(H(\Omega))^n$. Then

$$
\text{ber}_1(T_1, \ldots, T_n|T_1|^{\nu_1+\beta_1-1}, \ldots, T_n|T^n_{n+\beta_1-1})
\leq \text{ber} \left( \sum_{i=1}^{n} \frac{1}{2p} \left( |T_i|^{2\nu(1-\nu)p} + |T_i|^{2\nu p} \right) + \frac{1}{2q} \left( |T_i|^{2\nu(1-\nu)q} + |T_i|^{2\nu q} \right) \right),
$$

(18)

where $p, q > 1$ and $\alpha, \beta, \nu \in [0, 1]$ such that $\alpha + \beta \geq 1, p\nu \geq 2$ and $1/p + 1/q = 1$.

**Proof.** The proof of (18) is exactly the same as the proof of Corollary 2.14. \hfill \Box

**Remark 2.19.** By putting $p = q = 2$ in (18), we have the following inequality:

$$
\text{ber}_1(T_1, \ldots, T_n|T_1|^{\nu_1+\beta_1-1}, \ldots, T_n|T^n_{n+\beta_1-1})
\leq 1/4 \text{ber} \left( \sum_{i=1}^{n} \left( |T_i|^{4\nu(1-\nu)} + |T_i|^{4\nu} \right) + \left( |T_i|^{4\nu(1-\nu)} + |T_i|^{4\nu} \right) \right),
$$

(19)

In the next result, we have an upper bound for $\text{ber}_1$.

**Corollary 2.20.** Let $(T_1, \ldots, T_n) \in \mathcal{B}(H(\Omega))^n$. Then

$$
\text{ber}_1(T_1, \ldots, T_n) \leq 1/4 \text{ber} \left( \sum_{i=1}^{n} \left( 1_{H(\Omega)} + |T_i|^2 \right) + \left( |T_i|^2 + 1_{H(\Omega)} \right) \right).
$$

(20)

**Proof.** By putting $\alpha = \beta = 1/2$ and $\nu = 1$ in (19), we get the desired inequality. \hfill \Box
References