Chen-Ricci Inequalities in Slant Submersions for Complex Space Forms

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Abstract. The goal of the present paper is to analyze sharp type inequalities including the scalar and Ricci curvatures of slant submersions in complex space forms.

1. Introduction

The intrinsic and extrinsic invariants that determined an inequality including the square mean curvature and the Ricci curvature of a submanifold in a real space form $\mathbb{R}^n(c)$ devised by B.-Y. Chen in 1999 (see [5]). Also, for arbitrary submanifolds in an arbitrary Riemannian manifold, a generalization of this inequality was demonstrated in 2005 by B.-Y. Chen (see [6]). Subsequently, this inequality has been comprehensively examined for different ambient spaces by some authors who are achieved some results (see [3, 4, 6, 14, 16, 20, 21, 24, 29, 35]).

Given a $C^\infty$–submersion $\varphi$ from a (semi)-Riemannian manifold $(H_M, g_M)$ onto a (semi)-Riemannian manifold $(H_N, g_N)$, according to the circumstances on the map $\varphi : (H_M, g_M) \to (H_N, g_N)$, we get the following: a slant submersion ([11, 28]), an almost Hermitian submersion ([22]), a (pseudo)-Riemannian submersion ([11, 7, 8, 12, 15, 18]), a quaternionic submersion ([13]), an anti-invariant submersion ([27]), ([19]), a Clairaut Submersion ([10]), conformal anti-invariant submersion([2]), a semi-invariant submersion ([17]), etc. As far as we know, Riemannian submersions were presented by B. O’Neill ([15]) and A. Gray ([8]) in 1960s, independently. Especially, by utilizing the notion of almost Hermitian submersions, B. Watson ([22]) presented some differential geometric features among fibers, base manifolds, and total manifolds. Subsequently, many results occur on this topic.

Watson in ([22]) studied Riemannian submersions between almost Hermitian manifolds under the name of holomorphic submersions. One of the most important consequences of this idea is that vertical and horizontal distributions are invariant under almost complex structure. He indicated that if the total manifold is a Kaehler manifold, then the base manifold is a Kaehler manifold. Slant submersions from almost Hermitian manifolds to Riemannian manifolds was introduced by Şahin in 2011 ([26]). In this article he showed that the geometry of slant submersions is quite different from holomorphic submersions.

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Let $H_M$ be an almost Hermitian manifold with complex structure $J$ of type $(1,1)$ and $H_M$ a Riemannian manifold isometrically immersed in $H_M$. We can say that submanifolds of a Kaehler manifold are determined by the tangent bundle of the submanifold under the action of the complex structure of the ambient manifold. $T^H M$ indicates the tangent space to $H_M$ at the point $x$ such that if $J(T_x H_M) \subset T_x H_M$, for every $x \in H_M$, then a submanifold $H_M$ is called holomorphic (complex). $T^H M$ indicates the normal space to $H_M$ at the any point $x$ such that if $J(T_x H_M) \subset T^H M$ for every $x \in H_M$, then $H_M$ is called totally real. Slant submanifolds which a generalization of holomorphic and totally real submanifolds were introduced by Chen in ([25]).

If for any $X \in T_x H_M$ and $x \in H_M$, the angle between $T^H M$ and $JX$ is a constant $\theta(x) \in \left[0, \frac{\pi}{2}\right]$, then the submanifold $H_M$ is called slant ([25]) i.e., it does not depend on the choice of $X \in T_x H_M$ and $x \in H_M$. We can say that invariant and totally real immersions are slant immersions with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor totally real is called a proper slant immersion.

In ([30]), the authors mainly studied Chen-Ricci inequality for Riemannian submersions from real space form. They also studied Chen-Ricci inequality for Riemannian maps from real space form and complex space forms by considering various cases such as anti-invariant, semi-invariant, Lagrangian etc. In this paper we consider Chen-Ricci inequality for slant Riemannian submersions from complex space forms. The rest of the paper is organized as follows: After giving some basic definitions and formulas in the second part, we study several inequalities including the Ricci and the scalar curvatures on vertical $(ker\varphi_q)$ and horizontal $(ker\varphi_x)$ distributions of slant submersions in complex space forms and then, we acquire Chen-Ricci inequalities on $(ker\varphi_q)$ and $(ker\varphi_x)$ of slant submersions in complex space forms.

2. Preliminaries

Let $(H_M, g_M)$ be an almost Hermitian manifold. This implies ([23]) that $H_M$ admits a tensor field $J$ of type $(1,1)$ on $H_M$ such that, $\forall X_1, X_2 \in \chi(H_M)$, we have

$$J^2 = -I, \quad g_M(JX_1, X_2) = -g_M(X_1, JX_2).$$

An almost Hermitian manifold $H_M$ is called Kaehler manifold if

$$(\nabla^LC_X)_X = 0, \quad \forall X_1, X_2 \in \chi(H_M),$$

here $\nabla^L$ is the Levi-Civita connection on $H_M$. If $\{X_1, [JX_1]\}$ spans a plane section, the sectional curvature $F_{H_M}(X_1) = K_{H_M}(X_1 \wedge [JX_1])$ of span$\{X_1, [JX_1]\}$ is called a sectional curvature. The Riemannian-Christoffel curvature tensor of a complex space form ([23]) $H_M(a)$ of constant sectional curvature $a$ satisfies

$$R_{H_M}(X_1, X_2, X_3, X_4) = \alpha \frac{a}{4}(g_M(X_1, X_4)g_M(X_2, X_3) - g_M(X_1, X_3)g_M(X_2, X_4) + g_M(X_1, X_3)g_M(X_2, X_4) - g_M(X_1, X_4)g_M(X_2, X_3) + 2g_M(X_1, JX_2)g_M(X_3, X_4))$$

for all $X_1, X_2, X_3, X_4 \in \chi(H_M)$.

Let $(H_M, g_M)$ and $(H_N, g_N)$ be Riemannian manifolds. A Riemannian submersion is a smooth map $\varphi : H_M \rightarrow H_N$ which is onto and satisfies the following conditions:

(i) $\varphi_q : T_x H_M \rightarrow T_{\varphi(x)} H_N$ is onto for all $q \in H_M$;
(ii) The fibres $\varphi^{-1}_x, x \in H_M$, are Riemannian submanifolds of $H_M$;
(iii) $\varphi_q$ preserves the length of the horizontal vectors.

The vectors tangent to the fibres are called vertical and those normal to the fibres are called horizontal. The tangent bundle of $H_M$ splits as the Whitney sum of two distributions, the vertical one $ker\varphi_q$ and the orthogonal complementary distribution $(ker\varphi_q)^\perp$ called horizontal, and we denote by $h$ and $v$ the horizontal and vertical projections, respectively. A horizontal vector field $X_1$ on $H_M$ is called as basic if $X_1$ is $\varphi$-related.
to a vector field \(X_\theta\) on \(H_N([15])\). A Riemannian submersion \(\varphi : H_M \to H_N\) specifies two \((1, 2)\) tensor fields \(\mathcal{T}\) and \(\mathcal{A}\) on \(H_M\), by the formulae([15]):

\[
\mathcal{T}(E, G) = \mathcal{T}_E G = h^E \mathcal{L}_E vG + v^E \mathcal{L}_E hG
\]

(4)

and

\[
\mathcal{A}(E, G) = \mathcal{A}_E G = v^E \mathcal{L}_E hG + h^E \mathcal{L}_E vG
\]

(5)

for all \(E, G \in \chi(H_M)\).

**Lemma 2.1.** ([15]) Let \(\varphi : (H_M, g_M) \to (H_N, g_N)\) be a Riemannian submersion. Then, we have:

\[
\mathcal{A}_X X_2 = -\mathcal{A}_X X_1, \quad X_1, X_2 \in \chi((\ker \varphi)^\bot);
\]

(6)

\[
\mathcal{T}_U U_2 = \mathcal{T}_U U_1, \quad U_1, U_2 \in \chi((\ker \varphi));
\]

(7)

\[
g_M(\mathcal{T}_U U_2, X_3) = -g_M(\mathcal{T}_U U_3, X_2), \quad U_1 \in \chi((\ker \varphi)), \quad X_2, X_3 \in \chi(H_M);
\]

(8)

\[
g_M(\mathcal{A}_X X_2, X_3) = -g_M(\mathcal{A}_X X_3, X_2), \quad X_1 \in \chi((\ker \varphi)^\bot), \quad X_2, X_3 \in \chi(H_M).
\]

(9)

Let \(R^{H_M}, R^{H_N}, R^{\ker \varphi}\) and \(R^{(\ker \varphi)^\bot}\) stand for the Riemannian curvature tensors of Riemannian manifolds \(H_M, H_N\), the vertical distribution \(\ker \varphi\), and the horizontal distribution \((\ker \varphi)^\bot\), respectively.

**Lemma 2.2.** ([15]) Let \(\varphi : (H_M, g_M) \to (H_N, g_N)\) be a Riemannian submersion. Then, we have:

\[
R^{H_M}(U_1, U_2, U_3, U_4) = R^{\ker \varphi}(U_1, U_2, U_3, U_4) + g_M(\mathcal{T}_U U_4, \mathcal{T}_U U_3)
- g_M(\mathcal{T}_U U_4, \mathcal{T}_U U_3),
\]

(10)

\[
R^{H_M}(X_1, X_2, X_3, X_4) = R^{(\ker \varphi)^\bot}(X_1, X_2, X_3, X_4) - 2g_M(\mathcal{A}_X X_2, \mathcal{A}_X X_3, X_2, X_3)
+ g_M(\mathcal{A}_X X_3, \mathcal{A}_X X_2) - g_M(\mathcal{A}_X X_3, \mathcal{A}_X X_2),
\]

(11)

\[
R^{H_M}(X_1, U_1, X_2, U_2) = g_M(\mathcal{L}_E \mathcal{T}(U_1, U_2), X_2) + g_M(\mathcal{L}_E \mathcal{A}(X_1, X_2), U_2)
- g_M(\mathcal{T}_U X_1, \mathcal{T}_U X_2) + g_M(\mathcal{A}_X U_2, \mathcal{A}_X U_1),
\]

(12)

for all \(X_1, X_2, X_3, X_4 \in \chi((\ker \varphi)^\bot)\) and \(U_1, U_2, U_3, U_4 \in \chi((\ker \varphi))\).

Further, the \(\mathcal{H}\) mean curvature of every fibre of \(\varphi\) Riemannian submersion is defined

\[
\mathcal{H} = \frac{1}{s} N, \quad N = \sum_{p=1}^{s} \mathcal{T}_{E_p} E_p,
\]

(13)

where \(\{E_1, E_2, ..., E_s\}\) forms an orthonormal basis for the vertical distribution \(\ker \varphi\). Also, \(\varphi\) has totally geodesic fibres if \(\mathcal{T} = 0\) on \(\ker \varphi\).

**Definition 2.3.** ([26]) Let \(\varphi\) be a Riemannian submersion from an almost Hermitian manifold \((H_M, g_M, \mathcal{J})\) onto a Riemannian manifold \((H_N, g_M)\). If the angle \(\theta(X)\) between \(X\) and the space \(\ker \varphi q\) is a constant for any non-zero vector \(X \in \ker \varphi q, q \in H_M\), i.e. it is independent of the point \(q \in H_M\) and choice of the tangent vector \(X\) in \(\ker \varphi q\), then we say that \(\varphi\) is a slant submersion. In this case, the angle \(\theta\) is called the slant angle of the slant submersion.
Example 2.4. Define a map \( \psi : \mathbb{R}^4 \to \mathbb{R}^2 \) by
\[
\psi(x_1, x_2, x_3, x_4) = (x_1 \cos a - x_3 \sin a, x_2 \sin b - x_4 \cos b),
\]
here \( a \) and \( b \) are constant. Then the map \( \psi \) is a slant submersion with the slant angle \( \theta \) with \( \cos \theta = |\sin(a + b)| \), where \( J \) is the canonical complex structure of \( \mathbb{R}^4 \) defined by \( J(a_1, a_2, a_3, a_4) = (-a_2, a_1, -a_4, a_3) \).

Let \( \varphi \) be a Riemannian submersion from an almost Hermitian manifold \( H_M \) with the structure \((g_M, J)\) onto a Riemannian manifold \((H_{M'}, g_{M'})\). Then for \( X \in \Gamma(\ker \varphi) \), we can write
\[
JX = \varphi X + \omega X, \tag{14}
\]
here \( \varphi X \in \Gamma(\ker \varphi) \) and \( \omega X \in \Gamma(\ker \varphi)^\perp \). Also for \( Z \in \Gamma(\ker \varphi)^\perp \), we get
\[
JZ = BZ + CZ, \tag{15}
\]
where \( BZ \) and \( CZ \) are vertical and horizontal parts of \( JZ \). Using (1), (14) and (15) we get
\[
C^2 X = -X - \omega BX \tag{16}
\]

Example 2.5. ([26]) Every Hermitian submersion from an almost Hermitian manifold onto an almost Hermitian manifold is a slant submersion with \( \theta = 0 \).

Example 2.6. ([26]) Every anti-invariant Riemannian submersion from an almost Hermitian manifold to a Riemannian manifold is a slant submersion with \( \theta = \frac{\pi}{2} \).

Theorem 2.7. ([26]) Let \( \varphi \) be a Riemannian submersion from an almost Hermitian manifold \((H_{M'}, g_{M'}, J)\) onto a Riemannian manifold \((H_{M''}, g_{M''})\). Then \( \varphi \) is a proper slant submersion if and only if there exists a constant \( \lambda \in [-1, 0] \) such that
\[
\varphi^2 X = \lambda X \tag{17}
\]
for \( X \in \Gamma(\ker \varphi) \). If \( \varphi \) is a proper slant submersion, then \( \lambda = -\cos^2 \theta \).

Using the above theorem, we have
\[
g_{M'}(\varphi X, \varphi Y) = \cos^2 \theta g_{M'}(X, Y) \tag{18}
g_{M'}(\omega X, \omega Y) = \sin^2 \theta g_{M'}(X, Y) \tag{19}
\]

3. Inequalities for slant submersions

Let’s first give the following result:

Since \( \varphi \) is a slant submersion, and using (3) and (10) we get:

Lemma 3.1. \((H_M(a), g_M)\) and \((H_N, g_N)\) denote a complex space form and a Riemannian manifold and let \( \varphi : (H_M(a), g_M) \to (H_N, g_N) \) be a slant submersion. Then, any for \( U_1, U_2, U_3, U_4 \in \chi(\ker \varphi) \) we obtain

\[
\begin{aligned}
R^{\text{ker} \varphi}(U_1, U_2, U_3, U_4) &= \frac{\alpha}{4}[g_M(U_1, U_4)g_M(U_2, U_3) \\
& \quad - g_M(U_1, U_3)g_M(U_2, U_4) \\
& \quad + g_M(\varphi U_1, U_4)g_M(\varphi U_2, U_3) \\
& \quad - g_M(\varphi U_1, U_3)g_M(\varphi U_2, U_4) \\
& \quad + 2g_M(U_1, \varphi U_2)g_M(\varphi U_3, U_4) \\
& \quad - g_M(T_{U_1} U_4, T_{U_2} U_3) + g_M(T_{U_2} U_4, T_{U_1} U_3),
\end{aligned}
\]
Corollary 3.3. Let the equality status of the inequality satisfies if and only if every fibre is totally geodesic. Let Theorem 3.2. case, Corollary 3.5. Let Einstein manifolds, i.e, where $K$ is a bi-sectional curvature of $\ker \varphi$. From here, we get: From (22), the following result is obtained. From here, the following theorem can be written. Now, let’s find the scalar curvature of the fiber from equation (22). Taking $U_1 = E_j$, $j = 1, ..., 2r$ and using (7), then we obtain

$$2\text{scal}^{\ker \varphi} = \frac{\alpha}{4} \left[ 2r(2r - 1 + 3 \cos^2 \theta) \right] - 4r^2 \|H\|^2 + \sum_{i,j=1}^{2r} g_M(\mathcal{T}_{E_i}E_j, \mathcal{T}_{E_i}E_j).$$

(23)

From here, the following theorem can be written.

Theorem 3.2. Let $\varphi : (H_M(\alpha), g_M) \to (H_N, g_N)$ be a slant submersion. Then, we have

$$\text{Ric}^{\ker \varphi}(U_1) \geq \frac{\alpha}{4} \left[ (2r - 1 + 3 \cos^2 \theta)g_M(U_1, U_1) \right] - 2rg_M(\mathcal{T}_{U_1}U_1, \mathcal{H}).$$

the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

we can state the following result:

Corollary 3.3. Let $\varphi : (H_M(\alpha), g_M) \to (H_N, g_N)$ be an anti-invariant Riemannian submersion with $\theta = \frac{\pi}{2}$. In this case,

$$\text{Ric}^{\ker \varphi}(U_1) \geq \frac{\alpha}{4} \left[ (2r - 1)g_M(U_1, U_1) \right] - 2rg_M(\mathcal{T}_{U_1}U_1, \mathcal{H})$$

the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

Since the fibers are minimal in a Kaehler submersion ([7]), the following result can be written.

Corollary 3.4. Let $\varphi : (H_M(\alpha), g_M) \to (H_N, J_N, g_N)$ be a Hermitian submersion with $\theta = 0$. In this case,

$$\text{Ric}^{\ker \varphi}(U_1) \geq \frac{\alpha}{4} \left[ (r + 1)g_M(U_1, U_1) \right].$$

From (22), the following result is obtained.

Corollary 3.5. Let $\varphi : (H_M(\alpha), g_M) \to (H_N, J_N, g_N)$ be a holomorphic Riemannian submersion. Then, the fibers are Einstein manifolds, i.e,

$$\text{Ric}^{\ker \varphi}(U_1) = \frac{\alpha}{2} (r + 1)g_M(U_1, U_1).$$

Now, let’s find the scalar curvature of the fiber from equation (22). Taking $U_1 = E_j$, $j = 1, ..., 2r$ and using (7), then we obtain

$$2\text{scal}^{\ker \varphi} = \frac{\alpha}{4} \left[ 2r(2r - 1 + 3 \cos^2 \theta) \right] - 4r^2 \|H\|^2 + \sum_{i,j=1}^{2r} g_M(\mathcal{T}_{E_i}E_j, \mathcal{T}_{E_i}E_j).$$

(23)
Theorem 3.6. Let $\varphi : (H_M(\alpha), g_M) \rightarrow (H_N, g_N)$ be a slant submersion. Then, we have

$$2\text{scat}_{\text{ker}} \geq \frac{\alpha}{4} \left[2r(2r - 1 + 3 \cos^2 \theta)\right] - 4r^2\|H\|^2.$$ 

the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

From (23), the following results can be written.

Corollary 3.7. Let $\varphi : (H_M(\alpha), g_M) \rightarrow (H_N, g_N)$ be an anti-invariant Riemannian submersion with $\theta = \frac{\pi}{2}$. In this case,

$$2\text{scat}_{\text{ker}} \geq \frac{\alpha}{2} \left[r(2r - 1)\right] - 4r^2\|H\|^2.$$ 

the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

Corollary 3.8. Let $\varphi : (H_M(\alpha), g_M) \rightarrow (H_N, J_N, g_N)$ be a Hermitian submersion with $\theta = 0$. In this case,

$$2\text{scat}_{\text{ker}} \geq ar(r + 1).$$

Since $\varphi$ is a slant submersion, and using (3),(11), (15) We can write the following lemma:

Lemma 3.9. Let $\varphi : (H_M(\alpha), g_M) \rightarrow (H_N, g_N)$ be a slant submersion. Then, for $X_1, X_2, X_3, X_4 \in \chi(\text{ker} \varphi)$ we have

$$R^{(\text{ker} \varphi)}(X_1, X_2, X_3, X_4) = \frac{\alpha}{4}\left[g_M(X_1, X_4)g_M(X_2, X_3) - g_M(X_1, X_3)g_M(X_2, X_4) + g_M(CX_1, X_4)g_M(CX_2, X_3) - g_M(CX_1, X_3)g_M(CX_2, X_4) + 2g_M(X_1, CX_4)g_M(CX_3, X_4)\right] + 2g_M(\mathcal{A}_X X_2, \mathcal{A}_X X_4) - g_M(\mathcal{A}_X X_3, \mathcal{A}_X X_4) + g_M(\mathcal{A}_X X_3, \mathcal{A}_X X_4),$$

Here $K^{(\text{ker} \varphi)}$ is a bi-sectional curvature of $(\text{ker} \varphi)^+.$

Since $\mathcal{A} = 0$ for Kaehler submersion (7), we have:

Corollary 3.10. Let $\varphi : (H_M(\alpha), g_M) \rightarrow (H_N, g_N)$ be a Hermitian submersion with $\theta = 0$. In this case, we have

i. $$R^{(\text{ker} \varphi)}(X_1, X_2, X_3, X_4) = \frac{\alpha}{4}\left[g_M(X_1, X_4)g_M(X_2, X_3) - g_M(X_1, X_3)g_M(X_2, X_4) + g_M(CX_1, X_4)g_M(CX_2, X_3) - g_M(CX_1, X_3)g_M(CX_2, X_4) + 2g_M(X_1, CX_4)g_M(CX_3, X_4)\right].$$

ii. $$K^{(\text{ker} \varphi)}(X_1, X_2) = \frac{\alpha}{4}\left[g_M^2(X_1, X_2) - ||X_1||^2 ||X_2||^2 - 3g_M^2(CX_1, X_2)\right].$$
Theorem 3.11. Let \( \phi : (H_M(\alpha), g_M) \rightarrow (H_N, g_N) \) be a slant submersion. Then,

\[
\text{Ric}^{(\ker \phi)^\perp}(X_1) = \frac{\alpha}{4}(2n + 2)g_M(X_1, X_1) + 3g_M(\omega BX_1, X_1)
\]

From (26), the following result can be written.

Corollary 3.12. Let \( \phi \) be a holomorphic Riemannian submersion from a Kaehler manifold \( (H_M(\alpha), g_M) \) onto a Kaehler manifold \( (H_N, J_N, g_N) \) with \( \theta = 0 \). Then,

\[
\text{Ric}^{(\ker \phi)^\perp}(X_1) \leq \frac{\alpha}{4}(2n + 2)g_M(X_1, X_1) + 3g_M(\omega BX_1, X_1)
\]

the equality status of the inequality satisfies if and only if horizontal distribution is integrable.

From (26), the following result can be written.

Theorem 3.13. Let \( \phi : (H_M(\alpha), g_M) \rightarrow (H_N, g_N) \) be a slant submersion. Then,

\[
2\text{scn}^{(\ker \phi)^\perp} = \frac{\alpha}{4} \left[ 2n(2n + 2) + 6n + 3\text{tr}(\omega B) \right] - 3 \sum_{i,j=1}^{2n} g_M(\mathcal{A}_{\ker \omega c_{\theta}}, \mathcal{A}_{\ker \omega c_{\theta}}, \mathcal{A}_{\ker \omega c_{\theta}}, \mathcal{A}_{\ker \omega c_{\theta}}).
\]

Then, we write

\[
2\text{scn}^{(\ker \phi)^\perp} \leq \frac{\alpha}{4} \left[ 2n(2n + 2) + 6n + 3\text{tr}(\omega B) \right]
\]

Thus, we can give:

Corollary 3.14. Let \( \phi \) be a holomorphic Riemannian submersion from a Kaehler manifold \( (H_M(\alpha), g_M) \) onto a Kaehler manifold \( (H_N, J_N, g_N) \) with \( \theta = 0 \). Then,

\[
2\text{scn}^{(\ker \phi)^\perp} = \frac{\alpha}{4} \left[ 2n(2n + 2) + 6n + 3\text{tr}(\omega B) \right]
\]
4. Chen-Ricci inequalities for slant submersions

Let \((H_M(\alpha), g_M)\) be a complex space form, \((H_N, g_N)\) a Riemannian manifold and \(\varphi : H_M(\alpha) \to H_N\) be a slant submersion. For every node \(p \in H_M\), let \(\{E_1, \ldots, E_{2r}\}\) be an orthonormal basis of \(T_pH_M(\alpha)\) such that \(\ker \varphi = \text{span}\{E_1, \ldots, E_{2r}\}\) and \((\ker \varphi)^{\perp} = \text{span}\{\text{csc} \theta \omega_1, \ldots, \text{csc} \theta \omega_{2n}\}\). Let’s state \(T_{ij}^t\) by

\[T_{ij}^t = g_M(T_{E_i}E_j, \text{csc} \theta \omega_t)\]

where \(1 \leq i, j \leq 2r\) and \(1 \leq t \leq 2n\). Similarly, let’s state \(\mathcal{A}_{ij}^t\) by

\[\mathcal{A}_{ij}^t = g_M(\mathcal{A}_{\text{csc} \theta \omega_t}, E_i)\]

in which \(1 \leq i, j \leq 2n\) and \(1 \leq \alpha \leq 2r\) and we employee

\[\delta(N) = \sum_{i=1}^{2n} \sum_{j=1}^{2n} ((\nabla \mathcal{T})_{ij} E_k, \text{csc} \theta \omega_t).\]

Now, from (19), we get

\[2 \text{scal}^{\text{chri}} = \frac{\alpha}{4} \left(2r(2r - 1 + 3 \cos^2 \theta) \right) - 4r^2 \|H\|^2 + \sum_{i,j=1}^{2n} g_M(T_{E_i}E_j, T_{E_i}E_j).\]

Using (7) and (29), we arrive at

\[2 \text{scal}^{\text{chri}} = \frac{\alpha}{4} \left(2r(2r - 1 + 3 \cos^2 \theta) \right) - 4r^2 \|H\|^2 + \sum_{i,j=1}^{2n} \sum_{i,j=1}^{2n} (T_{ij}^t)^2.\] (32)

From (19), we know that

\[\sum_{i=1}^{2n} \sum_{j=1}^{2n} (T_{ij}^t)^2 = \frac{1}{2} 4r^2 \|H\|^2 + \frac{1}{2} \sum_{i=1}^{2n} \left[ T_{i1}^t - T_{i22}^t - \ldots - T_{in}^t \right]^2 \]

\[+ 2 \sum_{i=1}^{2n} \sum_{j=1}^{2n} (T_{ij}^t)^2 - 2 \sum_{i=1}^{2n} \sum_{j=1}^{2n} \left[ T_{ii}^t T_{ij}^t - (T_{ij}^t)^2 \right].\] (33)

If we put (33) in (32), we obtain

\[2 \text{scal}^{\text{chri}} = \frac{\alpha}{4} \left(2r(2r - 1 + 3 \cos^2 \theta) \right) - 2r^2 \|H\|^2 - \frac{1}{2} \sum_{i=1}^{2n} \left[ T_{i1}^t - T_{i22}^t - \ldots - T_{in}^t \right]^2 + 2 \sum_{i=1}^{2n} \sum_{j=1}^{2n} (T_{ij}^t)^2 \]

\[+ 2 \sum_{i=1}^{2n} \sum_{j=1}^{2n} \left[ T_{ii}^t T_{ij}^t - (T_{ij}^t)^2 \right].\] (34)

From here, we have

\[2 \text{scal}^{\text{chri}} \geq \frac{\alpha}{4} \left(2r(2r - 1 + 3 \cos^2 \theta) \right) - 2r^2 \|H\|^2 - 2 \sum_{i=1}^{2n} \sum_{j=1}^{2n} \left[ T_{ii}^t T_{ij}^t - (T_{ij}^t)^2 \right].\] (35)

On the other hand, from (10), taking \(U_1 = U_4 = E_i, U_2 = U_3 = E_j\) and using (29), we have

\[2 \sum_{2s < i < 2r} R^{iu}(E_i, E_i, E_i, E_i) = 2 \sum_{2s < i < 2r} R^{\text{chri}}(E_i, E_i, E_i, E_i) + 2 \sum_{i=1}^{2n} \sum_{j=1}^{2n} \left[ T_{ii}^t T_{ij}^t - (T_{ij}^t)^2 \right].\]
From the last equality, (35) can be written as

\[ 2 \text{scal}^{\text{ker} \psi^*} \geq \alpha \left\{ 2r(2r - 1 + 3 \cos^2 \theta) \right\} - 2r \| \mathcal{H} \|^2 + 2 \sum_{2 \leq i < j \leq 2r} R^{\text{ker} \psi^*}(E_i, E_j, E_t, E_t) \]

\[ -2 \sum_{2 \leq i < j \leq 2r} R^H(E_i, E_j, E_t, E_t). \] 

Also, we know that

\[ 2 \text{scal}^{\text{ker} \psi^*} = 2 \sum_{2 \leq i < j \leq 2r} R^{\text{ker} \psi^*}(E_i, E_j, E_t, E_t) + 2 \sum_{j=1}^{2r} R^{\text{ker} \psi^*}(E_1, E_j, E_t, E_t), \]

If we put the last equality in (36), then we have

\[ 2 \text{Ric}^{\text{ker} \psi^*}(E_1) \geq \frac{\alpha}{4} \left\{ 2r(2r - 1 + 3 \cos^2 \theta) \right\} - 2r \| \mathcal{H} \|^2 - 2 \sum_{2 \leq i < j \leq 2r} R^H(E_i, E_j, E_t, E_t). \]

Since \( H_M(\alpha) \) is a complex space form, curvature tensor \( R^H \) of \( H_M \) provides equation (3), therefore we acquire

\[ \text{Ric}^{\text{ker} \psi^*}(E_1) \geq \frac{\alpha}{4}(2r - 1 + 3 \cos^2 \theta) - r^2 \| \mathcal{H} \|^2. \]

Thus, we can give the following result:

**Theorem 4.1.** Let \( \varphi : H_M(\alpha) \to H_N \) be a slant submersion from a complex space form \( (H_M(\alpha), g_M) \) onto a Riemannian manifold \( (H_N, g_N) \). Then we have

\[ \text{Ric}^{\text{ker} \psi^*}(E_1) \geq \frac{\alpha}{4}(2r - 1 + 3 \cos^2 \theta) - r^2 \| \mathcal{H} \|^2 \]

the equality status of the inequality satisfies if and only if

\[ T_{11} = T_{12} + ... + T_{1r}; \]

\[ T_{1j} = 0, j = 2, ..., 2r. \]

From (23), we have

\[ 2 \text{scal}^{\text{ker} \psi^*} = \frac{\alpha}{4}(2n(2n + 2) + 6n + 3tr(\omega B)) - 3 \sum_{i,j=1}^{2n} g_M(\mathcal{A}_{\text{csc} \theta \omega e}, \text{csc} \theta \omega e), \]

Using (16) and (30), then we have

\[ 2 \text{scal}^{\text{ker} \psi^*} = \frac{\alpha}{4}(2n(2n + 2) + 6n + 3tr(\omega B)) - 3 \sum_{a=1}^{2r} \sum_{i,j=1}^{2n} (\mathcal{A}^a_{ij})^2. \] 

From (6), then (37) turns into

\[ 2 \text{scal}^{\text{ker} \psi^*} = \frac{\alpha}{4}(2n(2n + 2) + 6n + 3tr(\omega B)) - 6 \sum_{a=1}^{2r} \sum_{j=2}^{2n} (X^a_j)^2 - 6 \sum_{a=1}^{2r} \sum_{2 \leq i < j \leq 2n} (\mathcal{A}^a_{ij})^2. \] 

(37)
Moreover, from (11), taking \(X_1 = X_4 = \csc \theta \omega_1, X_2 = X_3 = \csc \theta \omega_1\) and using (30) we obtain

\[
2 \sum_{2 \leq i < j \leq 2n} R^{H_M}(\csc \theta \omega_1, \csc \theta \omega_j, \csc \theta \omega_i, \csc \theta \omega_i) = 2 \sum_{2 \leq i < j \leq 2n} R^{(ker \varphi)^+}(\csc \theta \omega_1, \csc \theta \omega_j, \csc \theta \omega_j, \csc \theta \omega_i) + 6 \sum_{a=1}^{2r} \sum_{2 \leq i < j \leq 2n} (\mathcal{A}_i^a)^2
\]  

(39)

If we consider (34) in (33), then we obtain

\[
2 \text{scal}^{(ker \varphi)^+} = \frac{\alpha}{4} \{(2n(2n + 2) + 6n + 3tr(\omega B)) - 6 \sum_{a=1}^{2n} \sum_{2 \leq i < j \leq 2n} (X_i)^2 - 2 \sum_{2 \leq i < j \leq 2n} R^{H_M}(\csc \theta \omega_1, \csc \theta \omega_j, \csc \theta \omega_j, \csc \theta \omega_i) + 2 \sum_{2 \leq i < j \leq 2n} R^{(ker \varphi)^+}(\csc \theta \omega_1, \csc \theta \omega_j, \csc \theta \omega_j, \csc \theta \omega_i)
\]

Since \(H_M\) is a complex space form, curvature tensor \(R^{H_M}\) of \(H_M\) satisfies (3), hence we have

\[
2 \text{Ric}^{(ker \varphi)^+}(\csc \theta \omega_1) = \frac{\alpha}{4} \{(10(\alpha + 1) + 6tr(\omega B)) - 6 \sum_{a=1}^{2n} \sum_{j=2}^{2n} (\mathcal{A}_j^a)^2
\}

Then, we can write

\[
\text{Ric}^{(ker \varphi)^+}(\csc \theta \omega_1) \leq \frac{\alpha}{4} \{(5(\alpha + 1) + 3tr(\omega B))\}
\]

Thus, we can give the following result:

**Theorem 4.2.** Let \(\varphi : H_M(\alpha) \to H_N\) be a slant submersion from a complex space form \((H_M(\alpha), g_M)\) onto a Riemannian manifold \((H_N, g_N)\). Then we have

\[
\text{Ric}^{(ker \varphi)^+}(\csc \theta \omega_1) \leq \frac{\alpha}{4} \{(5(\alpha + 1) + 3tr(\omega B))\}
\]

the equality status of the inequality satisfies if and only if

\[
\mathcal{A}_j^a = 0, j = 2, \ldots, 2n.
\]

Next, we can calculate the inequality of Chen-Ricci among \((ker \varphi, (ker \varphi)^+). The scal scalar curvature of \(H_M(\alpha)\) is defined as

\[
2 \text{scal} = \sum_{i=1}^{2n} \text{Ric}(\csc \theta \omega_i, \csc \theta \omega_i) + \sum_{k=1}^{2r} \text{Ric}(E_k, \csc \theta \omega_k),
\]

\[
2 \text{scal} = \sum_{j,k=1}^{2n} R^{H_M}(E_j, E_k, E_k) + \sum_{i=1}^{2n} \sum_{k=1}^{2r} R^{H_M}(\csc \theta \omega_i, E_k, E_k, \csc \theta \omega_i) + \sum_{i=1}^{2n} \sum_{j=1}^{2n} R^{H_M}(\csc \theta \omega_i, \csc \theta \omega_j, \csc \theta \omega_j, \csc \theta \omega_i) + \sum_{i=1}^{2n} \sum_{j=1}^{2n} R^{H_M}(E_j, \csc \theta \omega_i, \csc \theta \omega_i, E_j).
\]  

(40)
Since $H_M(\alpha)$ is a complex space form, using (35) and (3), we have

$$2\text{scal} = \frac{\alpha}{4} (2r(2r + 2 + 3\cos^2 \theta) + 2n(2n + 5) + 8nr + 9tr(\omega B)).$$ (41)

On the other hand, using the equations (10),(11) and (12), we obtain also the scalar curvature of $H_M(\alpha)$ as

$$2\text{scal} = 2\text{scal}^{\text{kerp}.} + 2\text{scal}^{\text{kerp}.+} + 4r^2 \|H\|^2 + \sum_{jk=1}^{2r} g_M(T_{E_j} E_j, T_{E_k} E_k)$$

$$+ 3 \sum_{j=1}^{2n} g_M(\mathcal{A}_{\text{csc } \theta \omega E_j, \mathcal{A}_{\text{csc } \theta \omega E_k}, \mathcal{A}_{\text{csc } \theta \omega E_l}) - \sum_{j=1}^{2n} g_M((\nabla^r \mathcal{T})_{E_j} E_k, \text{csc } \theta \omega E_l)$$

$$+ \sum_{i=1}^{2n} \left\{ g_M(T_{E_j} \text{csc } \theta \omega E_i, T_{E_k} \text{csc } \theta \omega E_l) - g_M(\mathcal{A}_{\text{csc } \theta \omega E_i, \mathcal{A}_{\text{csc } \theta \omega E_k}) \right\} - \sum_{i=1}^{2n} g_M((\nabla^r \mathcal{T})_{E_j} E_k, \text{csc } \theta \omega E_l)$$

$$+ \sum_{i=1}^{2n} \left\{ g_M(T_{E_j} \text{csc } \theta \omega E_i, T_{E_k} \text{csc } \theta \omega E_l) - g_M(\mathcal{A}_{\text{csc } \theta \omega E_i, \mathcal{A}_{\text{csc } \theta \omega E_k}) \right\} .$$

Using (27) and (29), we obtain

$$2\text{scal} = 2\text{scal}^{\text{kerp}.} + 2\text{scal}^{\text{kerp}.+}$$

$$+ 2r^2 \|H\|^2 - \frac{1}{2} \sum_{i=1}^{2n} \left[ T_{11}^i - T_{22}^i - \ldots - T_{rr}^i \right]^2$$

$$- 2 \sum_{i=1}^{2n} \sum_{j=2}^{2r} (\mathcal{A}^i_{11})^2 + 2 \sum_{i=1}^{2n} \sum_{j=2}^{2r} \left( \sum_{k=1}^{2r} \mathcal{A}^i_{jk} \right)^2 - 2\delta(N)$$

$$+ 2 \sum_{i=1}^{2n} \sum_{j=1}^{2r} \left\{ g_M(T_{E_j} \text{csc } \theta \omega E_i, T_{E_k} \text{csc } \theta \omega E_l) - g_M(\mathcal{A}_{\text{csc } \theta \omega E_i, \mathcal{A}_{\text{csc } \theta \omega E_k}) \right\}$$

$$+ 2 \sum_{i=1}^{2n} \sum_{j=1}^{2r} \left\{ g_M(T_{E_j} \text{csc } \theta \omega E_i, T_{E_k} \text{csc } \theta \omega E_l) - g_M(\mathcal{A}_{\text{csc } \theta \omega E_i, \mathcal{A}_{\text{csc } \theta \omega E_k}) \right\} .$$

Using (31), (39) and (41) in (42) then we have

$$\frac{\alpha}{2} (3r + 3n + 4nr + 3tr(\omega B)) = \text{Ric}^{\text{kerp}.}(E_1) + \text{Ric}^{\text{kerp}.+}(\text{csc } \theta \omega E_1) + 2r^2 \|H\|^2$$

$$- \frac{1}{4} \sum_{i=1}^{2n} \left[ T_{11}^i - T_{22}^i - \ldots - T_{rr}^i \right]^2 - 2 \sum_{i=1}^{2n} \sum_{j=1}^{2r} \left( \mathcal{A}^i_{11} \right)^2$$

$$+ 3 \sum_{i=1}^{2n} \sum_{j=2}^{2r} \left( \mathcal{A}^i_{11} \right)^2 - 2\delta(N) + \|T^{\text{kerp}.}\|^2 - \|\mathcal{A}^{\text{kerp}.}\|^2$$

where $\|T^{\text{kerp}.}\|^2 = \sum_{i=1}^{2n} \sum_{j=1}^{2r} g_M(T_{E_j} \text{csc } \theta \omega E_i, T_{E_k} \text{csc } \theta \omega E_l)$ and $\|\mathcal{A}^{\text{kerp}.}\|^2 = \sum_{i=1}^{2n} \sum_{j=1}^{2r} g_M(\mathcal{A}_{\text{csc } \theta \omega E_i, \mathcal{A}_{\text{csc } \theta \omega E_k})$. Since $H_M(\alpha)$ is a complex space form, from (3), we have following result readily:
Theorem 4.3. Let $\varphi : H_M(\alpha) \to H_N$ be a slant submersion from a complex space form $(H_M(\alpha), g_M)$ onto a Riemannian manifold $(H_N, g_N)$. Then we have

$$\frac{\alpha^2}{2} (3r + 3n + 4nr + 3tr(\omega B)) \leq \operatorname{Ric}^{ker\varphi}(E_1) + \operatorname{Ric}^{ker\varphi,+}(csc \theta \omega_1) + \frac{1}{4} s^2 \|H\|^2 + 3 \sum_{s=1}^{n} \sum_{s=2}^{m} (\mathcal{A}^\alpha_{ij})^2$$

$$-\delta(N) + \|\mathcal{T}^{ker\varphi,\alpha}\|^2 - \|\mathcal{A}^{ker\varphi,+}\|^2$$

the equality status of the inequality satisfies if and only if

$$\mathcal{T}^1 = \mathcal{T}^2 + \ldots + \mathcal{T}^r$$

$$\mathcal{T}^j_j = 0, j = 2, \ldots, 2r.$$

Now, we give some examples which satisfy inequalities.

Example 4.4. Let $(R^6, g_{R^6}, J_{R^6})$ be an almost Hermitian manifold and $(R^3, g_{R^3})$ a Riemannian manifold. Define a map $\varphi : R^6 \to R^3$ by $\varphi(u_1, u_2, u_3, u_4, u_5, u_6) = (u_1 \sin u_3 + u_2 \sin u_3, u_4, u_5, u_6)$ where $u_1 \in (0, \frac{\pi}{2})$, $u_1, u_2 \neq 0$, $\frac{u_1}{u_2} = \tan u_3$.

Then, the map $\varphi$ is a Riemannian submersion such that

$$\ker \varphi = \langle V_1 = -\cos u_3 \frac{\partial}{\partial u_1} + \sin u_3 \frac{\partial}{\partial u_2}, V_2 = \frac{\partial}{\partial u_3}, V_3 = \frac{\partial}{\partial u_6} \rangle$$

and

$$(ker\varphi)^\perp = \langle Y_1 = \sin u_3 \frac{\partial}{\partial u_1} + \cos u_3 \frac{\partial}{\partial u_2}, Y_2 = \frac{\partial}{\partial u_3}, Y_3 = \frac{\partial}{\partial u_6} \rangle.$$

Moreover, $\varphi$ is a slant Riemannian submersion with slant angle $\theta = \frac{\pi}{2}$. Where $J_{R^6}(u_1, u_2, u_3, u_4, u_5, u_6) = (u_2, -u_1, u_4, -u_3, u_6, -u_5)$. By straightforward computations, we obtain $T^{ker\varphi}_{V_1} Y_1 = -V_3$ and $T^{ker\varphi,+}_{V_1} Y_2 = Y_1$.

Other components of operators $T^{ker\varphi}_{V_1}, T^{ker\varphi,+}_{V_1}, \mathcal{A}^{ker\varphi}_{V_1}$ and $\mathcal{A}^{ker\varphi,+}_{V_1}$ vanish identically. Moreover, $sca^{ker\varphi} = 0$, $Ric^{ker\varphi}(Y_1) = 0$ and $Ric^{ker\varphi}(V_1) = 1$. Also, since $J(ker \varphi) = (ker \varphi)^\perp$, we have:

Remark 4.5. It is clear that the Lagrangian submersion $\varphi$, in Example 4.4, satisfies the inequalities in Theorem 4.2 and Theorem 4.3.

In order to have a Lagrangian submersion $\varphi : (M, J, g) \to N$ dimension must be related in the following way: $\dim(M) = 2 \dim(N)$, the most natural examples of manifolds having this relation are given by the tangent bundle of $M = TN \to N$. In the seminal paper ([31]), Dombrowski introduces the almost complex structure $J$ on the tangent bundle $TN$ of a manifold $N$ having a linear connection which is given by the conditions $J(X^{ker\varphi}) = X^{ker\varphi}$; $J(X^{ker\varphi,+}) = -X^{ker\varphi}$, $\ker \varphi$, and $(ker \varphi)^\perp$ being the vertical and horizontal lifts. On the other hand, Sasaki ([32]) introduced the diagonal lift $g^D$, or Sasaki metric, over the tangent bundle of a Riemannian manifold $(N, g)$, given by $g(X^{ker\varphi}), Y^{ker\varphi}) = g(X^{ker\varphi}), Y^{ker\varphi}) = g(X, Y)$; $g(X^{ker\varphi}), Y^{ker\varphi}) = 0$. Thus, the tangent $(TN, J, g^D)$ of a Riemannian manifold $(N, g)$ is an almost Hermitian manifold. Then, one can easily gets:

Example 4.6. With the above notation, $\varphi : (TN, J, g^D) \to (N, g)$ is a Lagrangian submersion ([33]).

Remark 4.7. Example 4.6 satisfies the inequality in Theorem 3.13.

Let $\varphi : (H_M, J_M, g_M) \to (H_N, g_N)$ be a Lagrangian Riemannian submersion with totally geodesic from a Hermitian manifold to Riemannian manifold. Then, the total space $(H_M, J_M, g_M)$ is Einstein if and only if the Riemannian manifold $B$ and fibers of such a submersion are Einstein ([34]).

Example 4.8. A Lagrangian Riemannian submersion $\varphi : (H_M, J_M, g_M) \to (H_N, g_N)$ with totally geodesic from a Hermitian manifold to Riemannian manifold, such that the total space $H_M$ is an Einstein.
Remark 4.9. From the above notation, one can see that such a submersion $\varphi: H_M \rightarrow H_N$ satisfies the inequality in Corollary 3.3.

Denote the equivalence class of a point $(z_0, ..., z_n) \in K^{n+1} \setminus \{0\}$ by $[z_0, ..., z_n]$ or $[z_0, ..., z_n]_K$; then we have a canonical projection

$$K^{n+1} \setminus \{0\} \rightarrow KP^n, (z_0, ..., z_n) \rightarrow [z_0, ..., z_n]_K.$$ 

For $K = C$, we obtain the complex $n$–space $CP^n$.

Example 4.10. $(36)/(Maps to CP^n)$ Let $n \in \{1, 2, ...,\}$. For $K = C$, the canonical projection (43) restricts to a map

$$S^{2n+1} \rightarrow CP^n, (z_0, ..., z_n) \rightarrow [z_0, ..., z_n]_C, (z_i \in C, \sum_{i=0}^{n} |z_i|^2 = 1)$$

called a Hopf fibration. We give $CP^n$ the unique metric for which is a Riemannian submersion.

Remark 4.11. Since the Hopf map in Example 4.10 has minimal (in fact totally geodesic) fibres, then Example 4.10 satisfies the inequality in Theorem 4.1.

References