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# Common Fixed Point Theorem for $C$-Class Functions in Complete Metric Spaces with Application

**Rakesh Tiwari[a], Shashi Thakur[b], Andreea Fulga[c]**

[a]*Department of Mathematics, Government V.Y.T.Post-Graduate Autonomous College, Durg 491001, Chhattisgarh, India*
[b]*Department of Mathematics, C.L.C.Government Arts and Commerce College, Dhamdha, Chhattisgarh, India*
[c]*Department of Mathematics and Computer Sciences, Transilvania University of Brasov, 500036 Brasov, Romania*

**Abstract.** In this paper, we establish the existence and uniqueness of common fixed point theorem for self mappings satisfying contractive condition of integral type via the concept of $C$-class functions in complete metric spaces. Further, we furnish an example to validate our result. Our result improves various results in the current literature. Towards the end, the existence and uniqueness of common solutions for system of functional equations arising in dynamic programming are discussed as an application of our main result.

## 1. Introduction

Fixed point theory is one of the most fruitful and effective tools in mathematics. It is widely used for the existence to the solutions of many nonlinear problems in many branches of physics and engineering. The Banach contraction principle [13] is the first important result on fixed point for contractive type mapping. This principle has many applications in differential equations, functional equations, integral equations, economics and several others. Banach's contraction principle is mostly used by the researcher for fixed point common fixed point in many types of contraction mapping. Several authors have generalized the Banach's contraction in different ways [2–4, 7, 12, 17].

In 1976, Jungck [27] introduced the concept of commuting maps. He also generalized the Banach fixed point theorem. In 1982, Sessa [42] introduced the concept of weak commutativity and established some common fixed point theorems. After that 1986, Jungck [28] introduced the concept of compatible mappings and then also introduced the weakly compatible mappings [29]. Branciari [16] established an integral version of the Banach contraction principles and proved fixed point theorem for a single-valued contractive mapping of integral type in metric space. After that many authors generalized the result of Branciari and obtained fixed point and common fixed point theorems for various contractive conditions of integral type on different spaces [1, 5, 6, 18–23, 25, 30–32, 40, 41, 44, 45].

Ansari [8] introduced the notion of $C$-class function as a major generalization of Banach contraction principle and obtained some fixed point results. After that many authors were interested to obtained common fixed point theorems for $C$-class function [9, 10, 37]. Recently, some authors obtained fixed point and common fixed point for $C$-class function [11, 15, 24, 26, 38, 39, 43].

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*Email addresses:* rtiwari@govtsciencecollegedurg.ac.in (Rakesh Tiwari), shashithakur89520@gmail.com (Shashi Thakur), afulga@unitbv.ro (Andreea Fulga)

## 2. Preliminaries

We recall some definitions which will be used in the sequel.

**Definition 2.1.** *[36] Let $(X, d)$ be a metric space. Then a sequence $\{x_n\}$ in $X$ is called*

1. *convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.*
2. *Cauchy if and only if $d(x_n, x_m) \to 0$, as $n, m \to \infty$.*

**Definition 2.2.** *[36] The metric space $(X, d)$ is complete if every Cauchy sequence in $X$ is convergent.*

**Definition 2.3.** *[8] A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following axioms:*

1. *$F(s, t) \le s$ for all $(s, t) \in \mathbb{R}_+^2$;*
2. *$F(s, t) = s$ implies that either $s = 0$ or $t = 0$.*

Let us denote $\mathcal{C}$ the family of $C$-class functions.

**Remark 2.1.** *[8] Clearly, for some $F$ we have $F(0, 0) = 0$.*

**Example 2.2.** *[8] The following functions $F : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0, \infty)$:*

1. *$F(s, t) = s - t$, $F(s, t) = s \Rightarrow t = 0$;*
2. *$F(s, t) = ms$, $0 < m < 1$, $F(s, t) = s \Rightarrow s = 0$;*
3. *$F(s, t) = s\beta(s)$, $\beta : [0, \infty) \to [0, 1)$ is continuous, $F(s, t) = s \Rightarrow s = 0$;*
4. *$F(s, t) = s - \varphi(s)$, $F(s, t) = s \Rightarrow s = 0$, here $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$.*

**Definition 2.4.** *[33] A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied.*

1. *$\psi$ is nondecreasing and continuous,*
2. *$\psi(t) = 0$ if and only if $t = 0$.*

**Definition 2.5.** *[8] An ultra altering distance function is a continuous nondecreasing mapping $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \ge 0$.*

In this paper, we establish the existence and uniqueness of common fixed point theorem for self mappings satisfying contractive condition of integral type via the concept of $C$-class functions in complete metric spaces. Further, we furnish an example to validate our result. Our result improves various results in the current literature. We apply our result to the existence and uniqueness of common solutions for system of functional equations arising in dynamic programming.

## 3. Main Result

In our main result we denote $\psi_1 \in \Psi_1$ is an altering distance function, $\psi_2 \in \Psi_2$ is an ultra altering distance function and $\varphi \in \Psi_3$ is a Lebesgue-integrable function.

Now, we prove our main result.

**Theorem 3.1.** *Let $(X, d)$ be a complete metric space and $S, T : X \to X$ be a mapping such that for such $x, y \in X$,*

$$\psi_1\Big(\int_0^{d(Sx,Ty)} \varphi(t)dt\Big) \leq F\Big(\psi_1\Big(\int_0^{M(x,y)} \varphi(t)dt\Big), \psi_2\Big(\int_0^{M(x,y)} \varphi(t)dt\Big)\Big), \tag{1}$$

*where $F$ is a C-class function, $\psi_1 \in \Psi_1, \psi_2 \in \Psi_2$ and $\varphi \in \Psi_3$ where $\varphi : [0, \infty) \to [0, \infty)$ be a Lebesgue-integral function which is summable on each compact subset of $R^+$, non-negative and such that for each $\epsilon > 0$,*

$$\int_0^{\epsilon} \varphi(t)dt > 0,$$

*and*

$$M(x, y) = max\Big\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\Big\}. \tag{2}$$

*Then $S$ and $T$ have a unique common fixed point.*

**Proof.** Let $x_0 \in X$ such that $Sx_0 = x_1$ and $Tx_1 = x_2$. Define a sequence $\{x_n\}$ in $X$ such that $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$, for $n = 0, 1, 2, \cdots$
Let $\{y_n\}$ , $y_n = \int_0^{d(Sx_{2n},Tx_{2n+1})} \varphi(t)dt$
Consider,

$$\begin{aligned}\psi_1\Big(\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt\Big) &= \psi_1\Big(\int_0^{d(Sx_{2n},Tx_{2n+1})} \varphi(t)dt\Big)\\ &\leq F\Big(\psi_1\Big(\int_0^{M(x_{2n},x_{2n+1})} \varphi(t)dt\Big),\ \psi_2\Big(\int_0^{M(x_{2n},x_{2n+1})} \varphi(t)dt\Big)\Big),\end{aligned} \tag{3}$$

where from (2),

$$\begin{aligned}M(x_{2n}, x_{2n+1}) &= max\Big\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}), \frac{1}{2}\{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})\}\Big\}\\ &= max\Big\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{1}{2}\{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})\}\Big\}\\ &= max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}.\end{aligned} \tag{4}$$

If $d(x_{2n+1}, x_{2n+2}) \geq d(x_{2n}, x_{2n+1})$ for some $n$, then from (3) and (4), we get

$$\psi_1\Big(\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt\Big) \leq F\Big(\psi_1\Big(\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt\Big),\ \psi_2\Big(\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt\Big)\Big), \tag{5}$$

Thus by definition of $F \in C$, we get

$$either\ \ \psi_1\Big(\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt\Big) = 0 \ or\ \psi_2\Big(\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt\Big) = 0.$$

From definition of $\psi_1$ and $\psi_2$, it is possible only if

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt = 0.$$

This is a contraction to our hypothesis. Thus $d(x_{2n}, x_{2n+1}) > d(x_{2n+1}, x_{2n+2})$, this implies

$$\psi_1\Big(\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt\Big) < F\Big(\psi_1\Big(\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt\Big), \psi_2\Big(\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt\Big)\Big),$$

by definition of $F \in \mathcal{C}$, we get

$$\psi_1\Big(\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt\Big) < \psi_1\Big(\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt\Big)$$

Since $\psi_1$ is continuous and non-decreasing, therefore

$$\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt < \int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt,$$

continuiting this way

$$\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt < \int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt < \cdots < \int_0^{d(x_0,x_1)}\varphi(t)dt.$$

It follows that $\{y_n\}$ is a monotone decreasing and lower bounded sequence. Therefore there exists a $r \geq 0$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt = r. \tag{6}$$

Suppose that $r > 0$. Taking limit as $n \to \infty$ on both sides of (5) and using (6), we get

$$\psi_1(r) \leq F(\psi_1(r), \psi_2(r)),$$

from definition of $F \in \mathcal{C}$, we get

$$\textit{either}\ \ \psi_1(r) = 0,\ \psi_2(r) = 0.$$

By definition of $\psi_1$ and $\psi_2$, we get $r = 0$.
Hence from (6), we get

$$\lim_{n\to\infty}\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt = 0, \tag{7}$$

and

$$\lim_{n\to\infty}\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt = 0.$$

by property of $\varphi$, we get

$$\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) = 0 \textit{ and } \lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0 \tag{8}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence. To prove this, suppose that $\{x_n\}$ is not a Cauchy sequence. Therefore for an $\epsilon > 0$, there exists two subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$ such that $m_i < n_i < m_{i+1}$ and

$$d(x_{m_i}, x_{n_i}) \geq \epsilon \ \textit{ and }\ d(x_{m_i}, x_{n_{i-1}}) < \epsilon. \tag{9}$$

Consider

$$\psi_1\Big(\int_0^{\epsilon}\varphi(t)dt\Big) \leq \psi_1\Big(\int_0^{d(x_{m_i},x_{n_i})}\varphi(t)dt\Big)$$
$$\leq F\Big(\psi_1\Big(\int_0^{M(x_{m_{i-1}},x_{n_{i-1}})}\varphi(t)dt\Big),\psi_2\Big(\int_0^{M(x_{m_{i-1}},x_{n_{i-1}})}\varphi(t)dt\Big)\Big). \tag{10}$$

From (2)

$$\begin{aligned} M(x_{m_{i-1}},x_{n_{i-1}}) &= max\Big\{d(x_{m_{i-1}},x_{n_{i-1}}),d(x_{m_{i-1}},Sx_{m_{i-1}}),d(x_{n_{i-1}},Tx_{n_{i-1}}),\frac{d(x_{m_{i-1}},Tx_{n_{i-1}})+d(x_{n_{i-1}},Sx_{m_{i-1}})}{2}\Big\} \\ &= max\Big\{d(x_{m_{i-1}},x_{n_{i-1}}),d(x_{m_{i-1}},x_{m_i}),d(x_{n_{i-1}},x_{n_i}),\frac{d(x_{m_{i-1}},x_{n_i})+d(x_{n_{i-1}},x_{m_i})}{2}\Big\} \\ &= max\Big\{d(x_{m_{i-1}},x_{n_{i-1}}),d(x_{m_{i-1}},x_{m_i}),d(x_{n_{i-1}},x_{n_i}),z(m,n)\Big\}, \end{aligned} \tag{11}$$

where

$$z(m,n) = \frac{d(x_{n_{i-1}},x_{m_i})+d(x_{m_{i-1}},x_{n_i})}{2}. \tag{12}$$

Thus

$$\begin{aligned} \int_0^{M(x_{m_{i-1}},x_{n_{i-1}})}\varphi(t)dt &= \int_0^{max\{d(x_{m_{i-1}},x_{n_{i-1}}),d(x_{m_{i-1}},x_{m_i}),d(x_{n_{i-1}},x_{n_i}),z(m,n)\}}\varphi(t)dt \\ &= max\Big\{\int_0^{d(x_{m_{i-1}},x_{n_{i-1}})}\varphi(t)dt,\int_0^{d(x_{m_{i-1}},x_{m_i})}\varphi(t)dt,\int_0^{d(x_{n_{i-1}},x_{n_i})}\varphi(t)dt,\int_0^{z(m,n)}\varphi(t)dt\Big\} \end{aligned} \tag{13}$$

using (9) and triangle inequality, we get

$$\begin{aligned} d(x_{m_{i-1}},x_{n_{i-1}}) &\leq d(x_{m_{i-1}},x_{m_i})+d(x_{m_i},x_{n_{i-1}}) \\ &< d(x_{m_{i-1}},x_{m_i})+\epsilon. \end{aligned}$$

Taking limit as $i \to \infty$ and from (8), we get

$$\lim_{i\to\infty}\int_0^{d(x_{m_{i-1}},x_{n_{i-1}})}\varphi(t)dt \leq \int_0^{\epsilon}\varphi(t)dt. \tag{14}$$

Also

$$\begin{aligned} z(m,n) &= \frac{d(x_{m_{i-1}},x_{n_i})+d(x_{n_{i-1}},x_{m_i})}{2} \\ &\leq \frac{d(x_{m_{i-1}},x_{m_i})+2d(x_{m_{i-1}},x_{n_{i-1}})+d(x_{n_{i-1}},x_{n_i})}{2} \\ &\leq \frac{d(x_{m_{i-1}},x_{m_i})+d(x_{n_{i-1}},x_{n_i})}{2}+d(x_{m_{i-1}},x_{n_{i-1}})+\epsilon. \end{aligned}$$

Taking limit as $i \to \infty$ and from (8), we get

$$\lim_{i\to\infty}\int_0^{z(m,n)}\varphi(t)dt \leq \int_0^{\epsilon}\varphi(t)dt. \tag{15}$$

Taking limit as $i \to \infty$ in equality (10) and using equations (11), (12), (13), (14) and (15) all together in (10) we get

$$\psi_1\Big(\int_0^{\epsilon}\varphi(t)dt\Big) \leq F\Big(\psi_1\Big(\int_0^{\epsilon}\varphi(t)dt\Big), \psi_2\Big(\int_0^{\epsilon}\varphi(t)dt\Big)\Big).$$

From definition of $F \in C$, we get

$$either\ \ \psi_1\Big(\int_0^{\epsilon}\varphi(t)dt\Big) = 0\ \ or\ \ \psi_2\Big(\int_0^{\epsilon}\varphi(t)dt\Big) = 0.$$

From definition of $\psi_1$ and $\psi_2$, it is possible only if $\int_0^{\epsilon}\varphi(t)dt = 0$. This is a contradiction to our hypothesis. Hence the sequence $\{x_n\}$ is a Cauchy sequence. Since $(X, d)$ is a complete metric space, it yields that $\{x_n\}$ and hence any subsequence thereof, converge to $z \in X$. So, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ converge to $z \in X$, which implies

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\int_0^{d(Sx_{2n},Tx_{2n+1})}\varphi(t)dt = 0.$$

and

$$\lim_{n\to\infty} Sx_{2n} = z\ and\ \lim_{n\to\infty} Tx_{2n+1} = z \tag{16}$$

Now, we show that $z$ is a fixed point of $S$ and $T$. First claim that $z$ is a fixed point of $S$. Suppose, it is not. Then $d(Sz, z) > 0$
Let $\delta = d(Sz, z)$ Consider,

$$\begin{aligned}
\psi_1\Big(\int_0^{\delta}\varphi(t)dt\Big) &= \psi_1\Big(\int_0^{d(Sz,z)}\varphi(t)dt\Big)\\
&= \lim_{n\to\infty}\psi_1\Big(\int_0^{d(Sz,Tx_{2n+1})}\varphi(t)dt\Big)\\
&\leq \lim_{n\to\infty} F\Big(\psi_1(\int_0^{M(z,x_{2n+1})}\varphi(t)dt),\ \psi_2(\int_0^{M(z,x_{2n+1})}\varphi(t)dt)\Big),
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
M(z, x_{2n+1}) &= max\Big\{d(z, x_{2n+1}), d(z, Sz), d(x_{2n+1}, Tx_{2n+1}), \frac{1}{2}\{d(z, Tx_{2n+1}) + d(x_{2n+1}, Sz)\}\Big\}\\
\lim_{n\to\infty} M(z, x_{2n+1}) &= max\Big\{d(z, z), d(z, Sz), d(z, z), \frac{1}{2}\{d(z, z) + d(z, Sz)\}\Big\}\\
&= max\{0, d(z, Sz), 0, \frac{d(z, Sz)}{2}\}\\
&= d(z, Sz).
\end{aligned}$$

Taking limit as $n \to \infty$ in (17), we get

$$\begin{aligned}
\psi_1\Big(\int_0^{\delta}\varphi(t)dt\Big) &\leq F\Big(\psi_1(\int_0^{d(z,Sz)}\varphi(t)dt), \psi_2(\int_0^{d(z,Sz)}\varphi(t)dt)\Big)\\
&\leq F\Big(\psi_1(\int_0^{\delta}\varphi(t)dt), \psi_2(\int_0^{\delta}\varphi(t)dt)\Big).
\end{aligned}$$

Thus we obtain,

$$either\ \psi_1(\int_0^{\delta}\varphi(t)dt) = 0\ \ or\ \ \psi_2(\int_0^{\delta}\varphi(t)dt) = 0.$$

It is possible only if $\int_0^{\delta}\varphi(t)dt)=0$. This is a contradiction to our hypothesis. Hence $z$ is a fixed point of $S$. Now we show that $z$ is also fixed point of $T$. Suppose, it is not. Then $d(z,Tz)>0$
Let $\delta=d(z,Tz)$ Consider,

$$\begin{aligned}\psi_1\Big(\int_0^{\delta}\varphi(t)dt\Big)&=\psi_1\Big(\int_0^{d(z,Tz)}\varphi(t)dt\Big)\\&=\lim_{n\to\infty}\psi_1\Big(\int_0^{d(Sx_{2n},Tz)}\varphi(t)dt\Big)\\&\le\lim_{n\to\infty}F\Big(\psi_1(\int_0^{M(x_{2n},z)}\varphi(t)dt),\ \psi_2(\int_0^{M(x_{2n},z)}\varphi(t)dt)\Big),\end{aligned}\tag{18}$$

where

$$\begin{aligned}M(x_{2n},z)&=max\Big\{d(x_{2n},z),d(x_{2n},Sx_{2n}),d(z,Tz),\frac{1}{2}\{d(x_{2n},Tz)+d(z,Sx_{2n})\}\Big\}\\\lim_{n\to\infty}M(x_{2n},z)&=max\Big\{d(z,z),d(z,z),d(z,Tz),\frac{1}{2}\{d(z,Tz)+d(z,z)\}\Big\}\\&=max\{0,0,d(z,Tz),\frac{d(z,Tz)}{2}\}\\&=d(z,Tz).\end{aligned}$$

Taking limit as $n\to\infty$ in (18), we get

$$\begin{aligned}\psi_1\Big(\int_0^{\delta}\varphi(t)dt\Big)&\le F\Big(\psi_1(\int_0^{d(z,Tz)}\varphi(t)dt),\psi_2(\int_0^{d(z,Tz)}\varphi(t)dt)\Big)\\&\le F\Big(\psi_1(\int_0^{\delta}\varphi(t)dt),\psi_2(\int_0^{\delta}\varphi(t)dt)\Big).\end{aligned}$$

Thus we obtain,

$$either\ \psi_1(\int_0^{\delta}\varphi(t)dt)=0\ \ or\ \ \psi_2(\int_0^{\delta}\varphi(t)dt)=0.$$

It is possible only if $\int_0^{\delta}\varphi(t)dt)=0$. This is a contradiction to our hypothesis. Hence $z$ is also a fixed point of $T$. Thus $z$ is a common fixed point of $S$ and $T$.
For uniqueness, suppose that there exists an other common fixed point $w\neq z$ such that $Sw=Tw=w$, from (1)

$$\begin{aligned}\psi_1\Big(\int_0^{d(w,z)}\varphi(t)dt\Big)&=\psi_1\Big(\int_0^{d(Sw,Tz)}\varphi(t)dt\Big)\\&\le F\Big\{\psi_1(\int_0^{M(w,z)}\varphi(t)dt),\psi_2(\int_0^{M(w,z)}\varphi(t)dt)\Big\}\end{aligned}$$

where

$$\begin{aligned}M(w,z)&=max\Big\{d(w,z),d(w,Sw),d(z,Tz),\frac{1}{2}\{d(w,Tz)+d(z,Sw)\}\Big\}\\&=max\Big\{d(w,z),d(w,w),d(z,z),\frac{1}{2}\{d(w,z)+d(z,w)\}\Big\}\\&=max\Big\{d(w,z),0,0,d(w,z)\Big\}\\&=d(w,z)\end{aligned}$$

Hence

$$\psi_1\Big(\int_0^{d(w,z)}\varphi(t)dt\Big) \le F\Big(\psi_1(\int_0^{d(w,z)}\varphi(t)dt), \psi_2(\int_0^{d(w,z)}\varphi(t)dt)\Big),$$

by definition of $F \in C$, we get

$$either\ \psi_1(\int_0^{d(w,z)}\varphi(t)dt) = 0\ or\ \psi_2(\int_0^{d(w,z)}\varphi(t)dt) = 0.$$

This implies $\int_0^{d(w,z)}\varphi(t)dt = 0$. This is a contradiction to our hypothesis. Thus $z$ is a unique common fixed point of $S$ and $T$.

**Remark 3.1.** *If we take $\psi_1(t) = t$ in Theorem 3.1, we get the following result.*

**Corollary 3.2.** *Let $(X, d)$ be a complete metric space and $S, T : X \to X$ be a mapping such that for such $x, y \in X$,*

$$\int_0^{d(Sx,Ty)}\varphi(t)dt \le F\Big(\int_0^{M(x,y)}\varphi(t)dt, \psi_2\Big(\int_0^{M(x,y)}\varphi(t)dt\Big)\Big),$$

*where $F$ is a C-class function, $\psi_2 \in \Psi_2$ and $\varphi \in \Psi_3$ where $\varphi : [0,\infty) \to [0,\infty)$ be a Lebesgue-integrable function which is summable on each compact subset of $R^+$, non-negative and such that for each $\epsilon > 0$,*

$$\int_0^{\epsilon}\varphi(t)dt > 0$$

*and*

$$M(x, y) = max\Big\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\Big\}.$$

*Then $S$ and $T$ have a unique common fixed point.*

**Remark 3.2.** *If we take $S = T$ in Theorem 3.1, we get the Theorem 8 of Gupta et al. [23].*

**Corollary 3.3.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ be a mapping such that for such $x, y \in X$,*

$$\psi_1\Big(\int_0^{d(Tx,Ty)}\varphi(t)dt\Big) \le F\Big(\psi_1\Big(\int_0^{M(x,y)}\varphi(t)dt\Big), \psi_2\Big(\int_0^{M(x,y)}\varphi(t)dt\Big)\Big),$$

*where $F$ is a C-class function, $\psi_1 \in \Psi_1, \psi_2 \in \Psi_2$ and $\varphi \in \Psi_3$ where $\varphi : [0,\infty) \to [0,\infty)$ be a Lebesgue-integrable function which is summable on each compact subset of $R^+$, non-negative and such that for each $\epsilon > 0$,*

$$\int_0^{\epsilon}\varphi(t)dt > 0$$

*and*

$$M(x, y) = max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\Big\}.$$

*Then $T$ has a unique fixed point.*

Now, we illustrate an example to validate our main Theorem 3.1.

**Example 3.3.** *Consider $X = [0, 2]$ and $d(x, y) = |x - y|$. Define the mappings $S$ and $T$ on $X$ such that*

$$S(x) = \frac{x}{2}, \quad T(x) = \frac{x}{4} \text{ for all } x \in X.$$

*Define a function $F : [0, \infty)^2 \to \mathbb{R}$ as*

$$F(r, t) = mr, \quad \text{for all } 0 < m = \frac{8}{9} < 1.$$

*Then F is a C-class function.*

*Let us define $\psi_1, \varphi : [0, \infty) \to [0, \infty)$ by $\psi_1(t) = t$ and $\varphi(t) = \frac{t}{3}$, then for each $\epsilon > 0$, clearly*

$$\int_0^{\epsilon} \varphi(t)dt = \frac{\epsilon^2}{6} > 0.$$

*We can verify the contraction condition* (1) *by a simple calculation for the case $x, y \in X$ as follows:*
*Case(1). If $x = 0, y = 1$. Then*

$$d(x, y) = 1,\ d(x, Sx) = 0,\ d(y, Ty) = \frac{3}{4},\ \frac{d(x, Ty) + d(y, Sx)}{2} = \frac{3}{8}.$$
$$M(x, y) = max\Big\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\Big\} = 1.$$

$$L.H.S. = \psi_1\Big(\int_0^{d(Sx,Ty)} \varphi(t)dt\Big) = \frac{|2x - y|^2}{96} = \frac{1}{96}$$

*and*

$$R.H.S. = F\Big(\psi_1\Big(\int_0^{M(x,y)} \varphi(t)dt\Big), \psi_2\Big(\int_0^{M(x,y)} \varphi(t)dt\Big)\Big) = m\int_0^{M(x,y)} \varphi(t)dt$$
$$= m\int_0^{1} \varphi(t)dt = \frac{8}{54}.$$

*Case(2). If $x = 0, y = 2$. Then*

$$d(x, y) = 2,\ d(x, Sx) = 0,\ d(y, Ty) = \frac{3}{2},\ \frac{d(x, Ty) + d(y, Sx)}{2} = \frac{3}{4}.$$
$$M(x, y) = max\Big\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\Big\} = 2.$$

$$L.H.S. = \psi_1\Big(\int_0^{d(Sx,Ty)} \varphi(t)dt\Big) = \frac{|2x - y|^2}{96} = \frac{1}{24}$$

*and*

$$R.H.S. = F\Big(\psi_1\Big(\int_0^{M(x,y)} \varphi(t)dt\Big), \psi_2\Big(\int_0^{M(x,y)} \varphi(t)dt\Big)\Big) = m\int_0^{M(x,y)} \varphi(t)dt$$
$$= m\int_0^{2} \varphi(t)dt = \frac{16}{27}.$$

*case(3). if $x = 1, y = 2$. Then*

$$d(x,y) = 1,\ d(x,Sx) = \frac{1}{2},\ d(y,Ty) = \frac{3}{2},\ \frac{d(x,Ty) + d(y,Sx)}{2} = 1.$$
$$M(x,y) = max\Big\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty) + d(y,Sx)}{2}\Big\} = \frac{3}{2}.$$

$$L.H.S. \;=\; \psi_1\Big(\int_0^{d(Sx,Ty)} \varphi(t)dt\Big) = \frac{|2x-y|^2}{96} = 0$$

*and*

$$R.H.S. \;=\; F\Big(\psi_1\Big(\int_0^{M(x,y)} \varphi(t)dt\Big), \psi_2\Big(\int_0^{M(x,y)} \varphi(t)dt\Big)\Big) = m\int_0^{M(x,y)} \varphi(t)dt$$
$$= m\int_0^{\frac{3}{2}} \varphi(t)dt = \frac{1}{3}.$$

*Hence from the above three cases it follows that*

$$\psi_1\Big(\int_0^{d(Sx,Ty)} \varphi(t)dt\Big) \le F\Big(\psi_1\Big(\int_0^{M(x,y)} \varphi(t)dt\Big), \psi_2\Big(\int_0^{M(x,y)} \varphi(t)dt\Big)\Big),$$

*Similarly, we can verify for other cases. Thus all the conditions of Theorem* 3.1 *are satisfied and* 0 *is the unique common fixed point of the mappings $S$ and $T$.*

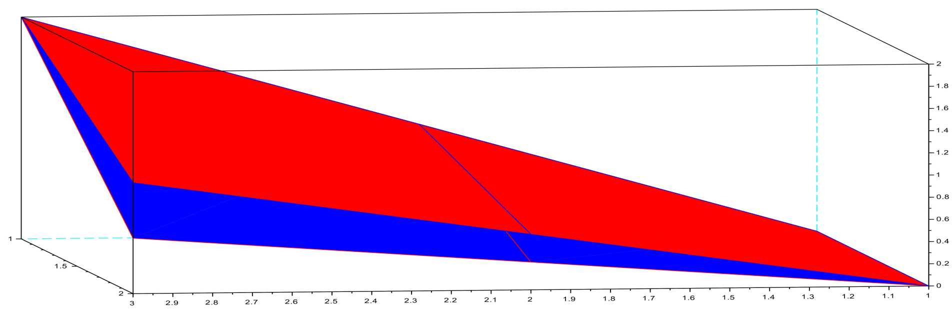

Figure 1: (3D-view) The red surface and blue surface demonstrate the functions $S(X)$ and $T(X)$, respectively, which has common fixed point at 0.

## 4. Application to existence theorem for functional equations arising in dynamic programming

In this section, we find the existence and uniqueness of a common solutions for a system of functional equations arising in dynamic programming which was initiated by Bellman and Lee [14] through the help of our main Theorem 3.1.

Let $P$ and $Q$ denote the two Banach spaces, $S \subseteq P$ and $D \subseteq Q$. Taking $S$ and $D$ signify the state and decision spaces, respectively. Let $B(S)$ denotes the set of all bounded real-valued functions on S. For an arbitrary $h \in B(S)$, define

$$||h|| = sup\{|h(x)| : x \in S\}.$$

Define $d : (B(S))^2 \to \mathbb{R}^+$ by $d(h,k) = sup|h(x) - k(x)|$ for all $h, k \in B(S)$.
Then $(B(S), d)$ is a complete metric space. As proposed in Bellman and Lee [14], the basic form of the functional equation in dynamic programming is

$$f(x) = opt_{y\in D}H(x, y, f(T(x, y))),\ x \in S,$$

where $x$ and $y$ denote the state and decision vectors, respectively. $T$ denotes the transformation of the process, $f(x)$ denotes the optimal return function with the initial state $x$ and $opt$ represents $sup$ or $inf$.

In particular, Liu et al. [34] established fixed point theorems satisfying a contractive condition of integral type and applied their results for the existence and uniqueness of a solution to the following functional equation arising in dynamic programming.

$$f(x) = opt_{y\in D}\{u(x, y) + H_1(x, y, f(a_1(x, y)))\},\ \ for\ all\ \ x \in S,$$

Further, Liu et al. [35] established common fixed point theorems satisfying contractive condition of integral type and applied their results for the existence and uniqueness of common solutions to the following system of functional equations arising in dynamic programming.

$$\begin{aligned} f(x) &= opt_{y\in D}\{u(x, y) + H_1(x, y, f(a_1(x, y)))\},\ \ for\ all\ \ x \in S,\\ g(x) &= opt_{y\in D}\{v(x, y) + H_2(x, y, g(a_2(x, y)))\},\ \ for\ all\ \ x \in S, \end{aligned} \tag{19}$$

where $u : S \times D \to \mathbb{R}$, $a_1, a_2 : S \times D \to S$ and $H_1, H_2 : S \times D \times \mathbb{R} \to \mathbb{R}$.

Let $S, T : B(S) \to B(S)$ be the mappings defined by

$$Sh(x) = opt_{y\in D}\{u(x, y) + H_1(x, y, h(a_1(x, y)))\},\ \ for\ all\ \ (x, h) \in S \times B(S), \tag{20}$$
$$Th(x) = opt_{y\in D}\{u(x, y) + H_2(x, y, h(a_2(x, y)))\},\ \ for\ all\ \ (x, h) \in S \times B(S). \tag{21}$$

**Theorem 4.1.** *Let $S, T : B(S) \to B(S)$ be the mappings as above for which the following conditions hold:*

1. *$u$ and $H_i$ are bounded for $i = 1, 2$,*
2. *for all $(x, y, h, w) \in S \times D \times B(S) \times B(S)$.*

$$\psi_1(\int_0^{H_1(x,y,h(a_1(x,y)))-H_2(x,y,w(a_2(x,y)))} \varphi(t)dt) \le F(\psi_1\Big(\int_0^{M(h,w)} \varphi(t)dt), \psi_2(\int_0^{M(h,w)} \varphi(t)dt)), \tag{22}$$

*where $\psi_1 \in \Psi_1, \psi_2 \in \Psi_2$, $\varphi \in \Psi_3$ and*

$$M(h, w) = max\Big\{||h - w||, ||h - Sh||, ||w - Tw||, \frac{||h - Tw|| + ||w - Sh||}{2}\Big\}. \tag{23}$$

*Then the system of functional equations* (19) *have a unique common solution in $B(S)$.*

**Proof.** Since $u$ and $H_i$ are bounded for $i = 1, 2$, there exists $M > 0$ such that

$$sup\{|u(x, y)|, |H_i(x, y, t)| : (x, y, t) \in S \times D \times R\} \le M, \tag{24}$$

from (20) and (21) we obtain $Sh$ and $Th$ are bounded for each $h \in B(S)$, which yields that $S$ and $T$ are self mappings in $B(S)$. It follows from $\varphi \in \Psi_3$ that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_C \varphi(t)dt < \epsilon\ for\ all\ C \subset \mathbb{R}^+\ with\ m(C) \le \delta,$$

where $m$ denotes the Lebesgue measure. Let $(x,h,w) \in S \times B(S) \times B(S)$. Suppose that $opt_{y\in D} = sup_{y\in D}$. Then using (20) and (21) we can find $y_1, y_2 \in D$ such that

$$Sh(x) < u(x,y_1) + H_1(x,y_1,h(a_1(x,y_1))) + \delta, \tag{25}$$
$$Tw(x) < u(x,y_2) + H_2(x,y_2,w(a_2(x,y_2))) + \delta, \tag{26}$$
$$Sh(x) \geq u(x,y_2) + H_1(x,y_2,h(a_1(x,y_2))), \tag{27}$$
$$Tw(x) \geq u(x,y_1) + H_2(x,y_1,w(a_2(x,y_1))). \tag{28}$$

From (25) and (28), we get

$$\begin{aligned} Sh(x) - Tw(x) &< H_1(x,y_1,h(a_1(x,y_1))) - H_2(x,y_1,w(a_2(x,y_1))) + \delta \\ &\leq |H_1(x,y_1,h(a_1(x,y_1))) - H_2(x,y_1,w(a_2(x,y_1)))| + \delta. \end{aligned} \tag{29}$$

From (26) and (27), we get

$$\begin{aligned} Tw(x) - Sh(x) &< H_2(x,y_2,w(a_2(x,y_2))) - H_1(x,y_2,h(a_1(x,y_2))) + \delta \\ &\leq |H_2(x,y_2,w(a_2(x,y_2))) - H_1(x,y_2,h(a_1(x,y_2)))| + \delta. \end{aligned} \tag{30}$$

From (29) and (30), we get

$$|Sh(x) - Tw(x)| < max\{T_1, T_2\} + \delta, \tag{31}$$

where

$$\begin{aligned} T_1 &= |H_1(x,y_1,h(a_1(x,y_1))) - H_2(x,y_1,w(a_2(x,y_1)))|, \\ T_2 &= |H_2(x,y_2,w(a_2(x,y_2))) - H_1(x,y_2,h(a_1(x,y_2)))|. \end{aligned} \tag{32}$$

It follows from (22) and (31) and $\psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2$, $\varphi \in \Psi_3$ that

$$\begin{aligned} \psi_1(\int_0^{|Sh(x)-Tw(x)|} \varphi(t)dt) &\leq \psi_1\Big(\int_0^{max\{T_1,T_2\}+\delta} \varphi(t)dt\Big) \\ &= max\Big\{\psi_1(\int_0^{T_1+\delta} \varphi(t)dt), \psi_1(\int_0^{T_2+\delta} \varphi(t)dt)\Big\} \\ &= max\Big\{\psi_1(\int_0^{T_1} \varphi(t)dt + \int_{T_1}^{T_1+\delta} \varphi(t)dt), \\ &\qquad \psi_1(\int_0^{T_2} \varphi(t)dt + \int_{T_2}^{T_2+\delta} \varphi(t)dt)\Big\} \\ &\leq max\Big\{\psi_1(\int_0^{T_1} \varphi(t)dt) + \psi_1(\int_{T_1}^{T_1+\delta} \varphi(t)dt), \\ &\qquad \psi_1(\int_0^{T_2} \varphi(t)dt) + \psi_1(\int_{T_2}^{T_2+\delta} \varphi(t)dt)\Big\} \\ &\leq max\Big\{\psi_1(\int_0^{T_1} \varphi(t)dt), \psi_1(\int_0^{T_2} \varphi(t)dt)\Big\} \\ &\qquad + max\Big\{\psi_1(\int_{T_1}^{T_1+\delta} \varphi(t)dt), \psi_1(\int_{T_2}^{T_2+\delta} \varphi(t)dt)\Big\} \\ &\leq F(\psi_1\Big(\int_0^{M(h,w)} \varphi(t)dt), \psi_2(\int_0^{M(h,w)} \varphi(t)dt)) + \psi_1(\epsilon) \end{aligned}$$

Taking $\epsilon \to 0^+$ in the above inequality and using $\psi_1 \in \Psi_1$, we get

$$\psi_1\Big(\int_0^{\|Sh-Tw\|}\varphi(t)dt\Big) \leq F\Big(\psi_1\Big(\int_0^{M(h,w)}\varphi(t)dt\Big), \psi_2\Big(\int_0^{M(h,w)}\varphi(t)dt\Big)\Big).$$

Thus all the conditions of Theorem 3.1 are satisfied. Hence the mappings $S$ and $T$ have a unique common fixed point in $B(S)$, that is, the system of functional equations (19) has a unique common solution.

## 5. Conclusion

From our investigations, we conclude that the self mappings on a complete metric space satisfying contractive condition of integral type via the concept of $C$-class functions have a unique common fixed point. As an application, we find the existence and uniqueness of common solution for system of functional equations arising in dynamic programming. An example is given in support of our main result. Our result provides new path for the researchers in the concerned field.

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