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Multidimensional Gauge Theory via Summability Methods

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Abstract. Kurzweil and Henstock presented the notion of Gauge integral, independently. Using their definition Savas and Patterson examined the relationship between Gauge integral and Summability theory. Because of the esoteric of both Gauge and Summability theory, the body of literature is limited. As such the only accessible notion to both theories is Pringsheim limits. The goal of this paper is to present a natural multidimensional extension of Gauge theory via Summability methods. To accomplish this we examine double measurable real-valued functions of the type of $f(x, y)$ in the Gauge sense on $(1, \infty) \times (1, \infty)$. Additionally, we introduce the definition of double $\overline{\gamma}_2-$strongly summable to L with respect to Gauge and present inclusion theorems.

1. Introduction and Background

In the 1957 and 1961 both Kurzweil [4] and Henstock [3] independently presented the definition of Gauge integral. This new integration technique allows one to integrate a larger class of functions than Riemann and Lebesgue can for proper integrals. Before establishing the main result of this paper, let us present a brief introduction to Gauge theory.

Definition 1.1. *[11] A tagged partition of an interval $I = [a, b]$ is a finite set or ordered pairs*

$$D = \{(t_i, I_i) : 1 \leq i \leq m\}$$

where $\{I_i : 1 \leq i \leq m\}$ is a partition of I consisting of closed non overlapping subintervals and t_i is a point belonging to I_i; t_i is called the tag associated with I_i. If $f : I \to \mathbb{R}$, the Riemann sum of f with respect to D is defined to be

$$S(f, D) = \sum_{i=1}^{m} f(t_i)\ell\,(I_i)\,,$$

where $\ell(I_i)$ is the length of the subinterval I_i. If $\delta : I \to (0, \infty)$ is a positive function, we define an open interval valued function on I by setting $\gamma(t) = (t - \delta(t), t + \delta(t))$. If $I_i = [x_i, x_{i+1}]$, we can write $t_i \in I_i \subset \gamma(t_i)$ instead of $t_i - \delta < x_i \leq t_i \leq x_{i+1} < t_i + \delta$. Any interval γ defined on I such that $\gamma(t)$ is an open interval containing t for each $t \in I$ is called a Gauge on I. Let us denote the set of all such interval by Δ_G. If $D = \{(t_i, I_i) : 1 \leq i \leq m\}$ is a tagged partition of I and γ is a Gauge on I, we say that D is $\gamma - fine$ if $t_i \in I_i \subset \gamma(t_i)$ is satisfied.

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Let us now consider the following definition of Gauge integral.

Definition 1.2. *[11] Let $f : [a, b] \to \mathbb{R}$. If $f : [a, b] \to \mathbb{R}$. f is said to be Gauge integrable over $[a, b]$ if there exists $A \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a Gauge γ on $[a, b]$ such that $\left|S(f, D) - A\right| < \varepsilon$ whenever D is a $\gamma - fine$ tagged partition of $[a, b]$. The number A is called the Gauge integral of f over $I = [a, b]$ and is denoted by $\int\limits_a^b f$ or $\int_I f$; when we encounter integrals depending upon parameters, it is also convenient to write $\int\limits_a^b f(t)$ or $\int_I f(t)$.*

To bridge the gap between Gauge and Summability theory the following notion of convergence for double sequences is critical.

Definition 1.3. *[8] A double sequence $x = (x_{k,l})$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in the Pringsheim sense if for $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $\left|x_{k,l} - L\right| < \varepsilon$, whenever $k, l > N_\varepsilon$. In this case, we denote such limit as follow:*

$$P - \lim_{k,l\to\infty} x_{k,l} = L.$$

We shall describe such an x more briefly as "$P -$ convergent", and please note that in contrast to the case for single sequences, a $P-$convergent double sequences needs to be bounded.

In 1967 the space of strongly Cesáro summable sequences i.e. $|\sigma_1|$ and other related spaces of strongly summability sequences were presented by Maddox in [5]. Using Maddox's and Pringsheim's results Moricz presented the following definition for double sequences in [6].

Definition 1.4. *[6] A double sequence $x = (x_{k,l})$ is said to be strongly double Cesáro summable to a number L if*

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left|x_{k,l} - L\right| = 0.$$

The set of all strongly double Cesáro summable sequences shall be denoted by $[\sigma_{1,1}]$.

While the work on sequences continued, strongly summable functions was introduced by Borwein [1]. In 2019, Borwein's definition was extended by presenting the following definitions via multidimensional measurable real valued functions on $(1, \infty) \times (1, \infty)$ by Savas in [9].

Definition 1.5. *[9] A function $f(x, y)$ is said to be strongly double Cesáro summable to L if*

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \int\limits_1^m \int\limits_1^n \left|f(x, y) - L\right| dxdy = 0.$$

The space of all strongly double Cesáro summable functions will be denoted by $[W]_2$.

Definition 1.6. *[9] Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ are two non-decreasing sequences of positive real numbers such that each tending to ∞. Also, let $\lambda_m \le \lambda_m + 1$, $\lambda_1 = 1$ and $\mu_{n+1} \le \mu_n + 1$, $\mu_1 = 1$. The collection of such sequences (λ, μ) will be denoted by Δ. A function $f(x, y)$ is said to be double $\lambda-$strongly summability to L if*

$$P - \lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \int\limits_{m-\lambda_m+1}^{m} \int\limits_{n-\mu_n+1}^{n} \left|f(x, y) - L\right| dxdy = 0,$$

where $I_m = [m - \lambda_m + 1, m]$ and $J_n = [n - \mu_n + 1, n]$ and $\lambda_{m,n} = \lambda_m \mu_n$. Whenever this occurs, we write $f(x, y) \to L([V, \lambda, \mu])$. The set of all double $\lambda-$strongly summable functions will be denoted by simply $[V, \lambda, \mu]$. If we take $\lambda_{m,n} = mn$, then $[V, \lambda, \mu]$ reduced $[V]_2$, the space of all strongly double summable functions.

Also by using Gauge integral Savas and Patterson [10] introduced the new notion of strongly Cesáro type summability method which is as follows.

Definition 1.7. *[10] Let us consider* $\delta : I_i = (t_i - \delta(t_i), t + \delta(t_i)] \to (0, \infty)$ *is a positive function, and* $[a, b] = \cup I_i$ *with* $-\infty < a < b < \infty$. *We define an open interval valued function on I by setting* $\overline{\gamma} = \overline{\gamma}(t_i) = (t_i - \delta(t_i), t + \delta(t_i))$. *If* $J_i = [i - \lambda_i + 1, i]$, *we can write* $t_i \in J_i \subset \overline{\gamma}(t_i)$ *instad of* $t_i - \delta(t_i) < i - \lambda_i + 1 \leq t_i \leq i < t + \delta(t_i)$. *Let* $\overline{\gamma} = \overline{\gamma}(t_i) \in \Delta_G$, *and let* $f(x)$ *be a real valued function which is measurable Gauge sense in the interval* $(1, \infty)$. *Provided that* $\int f(x)$ *and* $\int |f(x)|$ *exist in the gauge sense and*

$$\lim_{t_i \to \infty} \frac{1}{\xi(t_i)} \int_{t_i - \delta(t_i)}^{t_i + \delta(t_i)} |f(x) - L| \, dt = 0$$

where $\xi(t_i) = (t_i + \delta(t_i)) - (t_i - \delta(t_i)) = 2\delta(t_i)$, *then we say that the function* $f(x)$ *is* $\overline{\gamma}$*−strongly summable to L with respect gauge. In this case, we write* $[G, \overline{\gamma}] - \lim f(x) = L$.

2. Main Results

In this section we will extend Gauge theory to functions of two variables and introduce the concept of double $\overline{\gamma}$−strongly summable to L with respect to Gauge. Additionally, using this notion we shall present inclusion theorems to contrast this notion with other integration techniques.

Definition 2.1. *A tagged partition of an interval* $I_2 = [a, b] \times [c, d]$ *is a finite set or ordered pairs*

$$D_{m,n} = \left\{ \left(t_{i,j}, I_{i,j}\right) : 1 \leq i \leq m \text{ and } 1 \leq j \leq n \right\}$$

where $\left\{ I_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n \right\}$ *is a partition of* I_2 *consisting of closed non overlapping subintervals and* $t_{i,j}$ *is a point belonging to* $I_{i,j}$; $t_{i,j}$ *is called the tag associated with* $I_{i,j}$. *If* $f : [a, b] \times [c, d] \to \mathbb{R}$, *the Riemann sum of f with respect to* $D_{m,n}$ *is defined to be*

$$S(f, D_{m,n}) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(t_{i,j}) \ell\left(I_{i,j}\right)$$

where $f\left(t_{i,j}\right) = f(t_i) \cdot f\left(z_j\right)$ *and* $\ell(I_{i,j}) = \ell\left(I_i \cdot J_j\right) = \ell(I_i) \cdot \ell\left(J_j\right)$ *which is the length of the subinterval* $I_{i,j}$. *If* $\delta_2 : [a, b] \times [c, d] \to (0, \infty)$ *is a positive multidimensional function where we define an open interval valued function on* I_2 *by setting* $\gamma(t, z) = \gamma(t) \cdot \gamma(z) = (t - \delta_2(t), t + \delta_2(t)) \times (z - \delta_2(z), z + \delta_2(z))$. *If* $I_{i,j} = [x_i, x_{i+1}] \times \left[y_j, y_{j+1}\right]$, *we can replace with* $t_i \in I_i \subset \gamma(t_i)$ *instead of* $t_i - \delta_2 < x_i \leq t_i \leq x_{i+1} < t_i + \delta_2$ *and* $z_j \in J_j \subset \gamma(z_j)$ *instead of* $z_j - \delta_2 < y_i \leq z_j \leq z_{j+1} < z_j + \delta_2$. *Any interval* γ *defined on* I_2 *such that* $\gamma(t, z)$ *is an open interval containing* t, z *for each* $(t, z) \in [a, b] \times [c, d]$ *is called a Gauge on* I_2. *Let us denote the set of all such interval by* Δ_{G_2}. *If* $D_{m,n} = \left\{ \left(t_{i,j}, I_{i,j}\right) : 1 \leq i \leq m \text{ and } 1 \leq j \leq n \right\}$ *is a tagged partition of* I_2 *and* γ *is a Gauge on* I_2, *we say that* $D_{m,n}$ *is* $\gamma_2 - fine$ *if* $t_i \in I_i \subset \gamma(t_i)$ *and* $z_j \in J_j \subset \gamma(z_j)$ *are satisfied.*

Example 2.2. *Let us consider the interval* $[0, 1] \times [0, 1]$ *and let* $\gamma(x, y) = \frac{1}{36}$. *Find a* $\gamma_2 - fine$ *tagged partition on* $[0, 1] \times [0, 1]$.

Since the Gauge $\gamma(x, y) = \gamma(x) \cdot \gamma(y) = \frac{1}{36}$ is a constant multidimensional function. Regardless of the choice of tag, $\gamma(c_k, d_l) = \frac{1}{36}$. Thus, any tagged partition $\left(t_{i,j}, I_{i,j}\right)$ in which $x_{i+1} - x_i < \frac{1}{6}$ and $y_{j+1} - y_j < \frac{1}{6}$ is a $\gamma_2 - fine$ tagged partition. Let us consider the following partition by choosing each tag from every interval to be any number in that interval.

$m\left(\left[0, \frac{1}{7}\right] \times \left[0, \frac{1}{7}\right]\right) < \frac{1}{36}$, $m\left(\left[\frac{1}{7}, \frac{2}{7}\right] \times \left[\frac{1}{7}, \frac{2}{7}\right]\right) < \frac{1}{36}$, $m\left(\left[\frac{2}{7}, \frac{3}{7}\right] \times \left[\frac{2}{7}, \frac{3}{7}\right]\right) < \frac{1}{36}$, $m\left(\left[\frac{3}{7}, \frac{4}{7}\right] \times \left[\frac{3}{7}, \frac{4}{7}\right]\right) < \frac{1}{36}$,
$m\left(\left[\frac{4}{7}, \frac{5}{7}\right] \times \left[\frac{4}{7}, \frac{5}{7}\right]\right) < \frac{1}{36}$, $m\left(\left[\frac{5}{7}, \frac{6}{7}\right] \times \left[\frac{5}{7}, \frac{6}{7}\right]\right) < \frac{1}{36}$, $m\left(\left[\frac{6}{7}, 1\right] \times \left[\frac{6}{7}, 1\right]\right) < \frac{1}{36}$. This is an example of a $\gamma(x, y) - fine$ tagged partition.

Now let us present the definition of multidimensional Gauge integral which is the following:

Definition 2.3. *Let $f : I_2 = [a,b] \times [c,d] \to \mathbb{R}$. Provided that there exists $A \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a Gauge γ_2 on I_2 such that $\left|S(f, D_{m,n}) - A\right| < \varepsilon$ whenever $D_{m,n}$ is a $\gamma_2 - fine$ tagged partition of I_2, then f is said to be multidimensional Gauge integrable over I_2. The number A is called the Gauge integral of f over I_2 and is denoted by $\int_a^b\int_c^d f$ or $\int_{[a,b]\times[c,d]} f$; when we encounter integrals depending upon parameters, it is also convenient to write $\int_a^b\int_c^d f(x,y)dxdy$ or $\int_{[a,b]\times[c,d]} f(x,y)dxdy$.*

Example 2.4. *Let us consider $f : [a,b] \times [c,d] \to \mathbb{R}$ defined by*

$$f(x,y) = \begin{cases} 1, & \text{if } x \text{ and } y \text{ are rational number} \\ 0, & \text{if } x \text{ or } y \text{ is an irrational number.} \end{cases}$$

We shall refer to f as a multidimensional Dirichlet function. Let $\varepsilon > 0$ and let $\{r_k\}$ and $\{s_l\}$ are enumeration of the rational numbers in $[a,b]$ and $[c,d]$, respectively. Let us define a Gauge γ_2 on $[a,b] \times [c,d]$ by

$$\gamma(t,z) = \begin{cases} \frac{\varepsilon}{3^{k+l+1}}, & t = r_k, z = s_l \\ 1, & (t,z) \notin \mathbb{Q}. \end{cases}$$

If P is a $\gamma_2 - fine$ tagged partition of $[a,b] \times [c,d]$, we can separate P into $P_{r,s}$, those tagged intervals will rational tags, and $P_{i,j}$ with irrational tags. Therefore,

$$\begin{aligned} \left|S(f,P)\right| &= \sum_P\sum f(t_k, z_l)\,\Delta x_k \Delta y_l \\ &= \sum_P\sum f(t_k)\cdot f(z_l)\,\Delta x_k \Delta y_l \\ &= \sum_P f(t_k)\,\Delta x_k \cdot \sum_P f(z_l)\,\Delta y_l \\ &= \sum_{P_{r,s}}\sum 1\Delta x_k \cdot 1\Delta y_l + \sum_{P_{i,j}}\sum 0\Delta x_k \cdot 0\Delta y_l \\ &< \sum_k\sum_l \frac{\varepsilon}{3^{k+l+1}} < \varepsilon. \end{aligned}$$

Hence, the multidimensional Dirichlet function is Gauge integrable.

Example 2.5. *Let us consider the following function:*

$$f(x,y) = \begin{cases} (-1)^{k+l}kl, & x \in \left(\frac{1}{k+1}, \frac{1}{k}\right] \text{ and } y \in \left(\frac{1}{l+1}, \frac{1}{l}\right] \\ 0, & x = 0 \text{ and } y = 0 \end{cases}$$

is gauge integrable over $[0,1] \times [0,1]$ with

$$\int_0^1\int_0^1 f(x,y)\,dxdy = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{(-1)^{k+l}}{(k+1)(l+1)} = (\ln 2 - 1)^2.$$

Let us begin by noting that f is constant on the interval $\left(\frac{1}{k+1}, \frac{1}{k}\right] \times \left(\frac{1}{l+1}, \frac{1}{l}\right]$ and

$$\begin{aligned} \int_{\frac{1}{k+1}}^{\frac{1}{k}}\int_{\frac{1}{l+1}}^{\frac{1}{l}} f(x,y)\,dxdy &= (-1)^{k+l}kl\left(\frac{1}{k} - \frac{1}{k+1}\right)\left(\frac{1}{l} - \frac{1}{l+1}\right) \\ &= \frac{(-1)^{k+l}}{(k+1)(l+1)} \end{aligned}$$

with this fact in mind, we need to find a gauge such that the union of the intervals with tags in $\left(\frac{1}{k+1},\frac{1}{k}\right]\times\left(\frac{1}{l+1},\frac{1}{l}\right]$ *approximates* $\left(\frac{1}{k+1},\frac{1}{k}\right]\times\left(\frac{1}{l+1},\frac{1}{l}\right]$ *and also select appropriate numbers of the intervals* $\left(\frac{1}{k+1},\frac{1}{k}\right]\times\left(\frac{1}{l+1},\frac{1}{l}\right]$ *such that we are granted an approximation to* $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^{k+1}}{(k+1)(l+1)}$. *To this end, let* $\varepsilon>0$ *and define our gauge by*

$$\gamma(t,z)=\begin{cases}\left(\frac{1}{(k+1)(l+1)},\left(\frac{1}{kl}+\frac{\varepsilon}{2kl\cdot 2^{l}}+\frac{\varepsilon}{2kl\cdot 2^{k}}+\frac{\varepsilon^2}{4kl\cdot 2^{k+l}}\right)\right), & t\in\left(\frac{1}{k+1},\frac{1}{k}\right] \text{ and } z\in\left(\frac{1}{l+1},\frac{1}{l}\right]\\ \left(-\frac{\varepsilon}{4},\frac{\varepsilon}{4}\right), & t=z=0.\end{cases}$$

Now suppose that $P=\{(t_k,[x_{k-1},x_k])\text{ and }(z_l,[y_{l-1},y_l])\}_{k,l=1,1}^{m,n}$ *is a* γ_2-fine *partition of* $[0,1]\times[0,1]$. *Let r and s be the natural number satifying*

$$\frac{1}{r+1}<x_1<\frac{1}{r} \text{ and } \frac{1}{s+1}<y_1<\frac{1}{s}.$$

Note that the tag for $[0,x_1]$ *and* $[0,y_1]$ *must be* $t_1=0$ *and* $z_1=0$ *because no other value can tag an interval with a left endpoint of* 0. *If we express the union of the intervals from P with tags in* $\left(\frac{1}{k+1},\frac{1}{k}\right]\times\left(\frac{1}{l+1},\frac{1}{l}\right]$ *as* $[\omega_{k+1},\omega_k]\times[u_{l+1},u_l]$, *then*

$$S(f,P)=\sum_{k=1}^{r}\sum_{l=1}^{s}(-1)^{k+l}kl(\omega_k-\omega_{k+1})(u_l-u_{l+1}).$$

Since $\omega_1=1$, $u_1=1$, $\frac{1}{k+1}<\omega_{k+1}$, *and* $\frac{1}{l+1}<u_{l+1}$ *for* $1\le k\le r$ *and* $1\le l\le s$,

$$\frac{1}{k}\le\omega_k<\frac{1}{k}+\frac{\varepsilon}{2k\cdot 2^k} \text{ and } \frac{1}{l}\le u_l<\frac{1}{l}+\frac{\varepsilon}{2l\cdot 2^l}$$

for $1\le k\le r+1$ *and* $1\le l\le s+1$. *Therefore, for* $1\le k\le r$ *and* $1\le l\le s$ *we can use the fact that*

$$kl\left(\frac{1}{k}-\frac{1}{(k+1)}\right)\left(\frac{1}{l}-\frac{1}{l+1}\right)=\frac{1}{(k+1)(l+1)}$$

to see that

$$\begin{aligned}
&\left(\frac{1}{(k+1)}-\frac{\varepsilon}{2}\frac{1}{2^{k+l}}\right)\left(\frac{1}{l+1}-\frac{\varepsilon}{2}\frac{1}{2^{l+l}}\right)\\
<\;&\left(\frac{1}{k+1}-\frac{\varepsilon}{2}\frac{k}{(k+1)2^{k+1}}\right)\left(\frac{1}{l+1}-\frac{\varepsilon}{2}\frac{k}{(l+1)2^{l+1}}\right)\\
=\;&\left(k\left(\frac{1}{k}-\frac{1}{k+1}\right)-k\frac{\varepsilon}{2(k+1)2^{k+1}}\right)\left(l\left(\frac{1}{l}-\frac{1}{l+1}\right)-l\frac{\varepsilon}{2(l+1)2^{l+1}}\right)\\
=\;&\left(k\left(\frac{1}{k}-\frac{1}{k+1}-\frac{\varepsilon}{2(k+1)\cdot 2^{k+1}}\right)\right)\left(l\left(\frac{1}{l}-\frac{1}{l+1}-\frac{\varepsilon}{2(l+1)2^{l+1}}\right)\right)\\
=\;&k\left(\frac{1}{k}-\left(\frac{1}{k+1}+\frac{\varepsilon}{2(k+1)\cdot 2^{k+1}}\right)\right)l\left(\frac{1}{l}-\left(\frac{1}{l+1}+\frac{\varepsilon}{2(l+1)\cdot 2^{l+1}}\right)\right)\\
<\;&k(\omega_k-\omega_{k+1})\,l(u_l-u_{l+1})\\
<\;&k\left(\frac{1}{k}+\frac{\varepsilon}{2k\cdot 2^k}-\frac{1}{k+1}\right)l\left(\frac{1}{l}+\frac{\varepsilon}{2l\cdot 2^l}-\frac{1}{l+1}\right)\\
=\;&\left(\frac{1}{k+1}+\frac{\varepsilon}{2}\frac{1}{2^k}\right)\left(\frac{1}{l+1}+\frac{\varepsilon}{2}\frac{1}{2^l}\right).
\end{aligned}$$

Hence,

$$\left| S(f,P) - \sum_{k,l=1,1}^{r,s} \frac{(-1)^{k+l}}{(k+1)(l+1)} \right| \le \sum_{k,l=1,1}^{r,s} \left(k(\omega_k - \omega_{k+1}) - \frac{1}{k+1} \right) \left(\left(l(u_l - u_{l+1}) - \frac{1}{l+1} \right) \right)$$
$$< \sum_{k=1}^{r} \frac{\varepsilon}{2} \frac{1}{2^k} \sum_{l=1}^{s} \frac{\varepsilon}{2} \frac{1}{2^l} < \frac{1}{4}\varepsilon$$

As $\frac{1}{r+1} < x_1 < \frac{\varepsilon}{2}$ *and* $\frac{1}{s+1} < y_1 < \frac{\varepsilon}{2}$,

$$\left| \sum_{k,l=1,1}^{\infty,\infty} \frac{(-1)^{k+l}}{(k+1)(l+1)} - \sum_{k,l=1,1}^{r,s} \frac{(-1)^{k+l}}{(k+1)(l+1)} \right| < \frac{\varepsilon}{4}.$$

Hence,

$$\left| S(f,P) - (\ln 2 - 1)^2 \right| \le \left| S(f,P) - \sum_{k,l=1,1}^{r,s} \frac{(-1)^{k+l}}{(k+1)(l+1)} \right| + \left| \sum_{k,l=1,1}^{r,s} \frac{(-1)^{k+l}}{(k+1)(l+1)} - \sum_{k,l=1,1}^{\infty,\infty} \frac{(-1)^{k+l}}{(k+1)(l+1)} \right|$$
$$< \varepsilon$$

We conclude that f *is gauge integrable over* $[0,1] \times [0,1]$ *with*

$$\int_0^1 \int_0^1 f = \sum_{k,l=1,1}^{\infty,\infty} \frac{(-1)^{k+l}}{(k+1)(l+1)} = (\ln 2 - 1)^2 .$$

Now we shall consider the following generalization of bounded variation result in [2].

Definition 2.6. *Let* $a,b,c,d \in \mathbb{R}$ *with* $a \le b$ *and* $c \le d$, *and let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be any function. Denote by* R_f *the subset of* $\mathbb{R}$ *consisting of finite sums of the form*

$$\sum_{i=1}^{m} \left| f(x_i, y_i) - f(x_{i-1}, y_{i-1}) \right|,$$

where $m \in \mathbb{N}$ *and* $(x_0, y_0), \ldots, (x_m, y_m)$ *are any points in* $\mathbb{R}^2$ *satisfying*

$$(a,c) = (x_0, y_0) \le (x_1, y_1) \le \cdots \le (x_{n-1}, y_{n-1}) \le (x_n, y_n) = (b,d).$$

If the set R_f *is bounded above, then* f *is said to be bounded variation. In this case, we denote the supremum of* R_f *by* $V(f)$, *and call it the total variation of* f *on* $[a,b] \times [c,d]$.

The following new definition was presented by using Patterson's result in [7].

Definition 2.7. *A multidimensional function* $f(x,y)$ *is of bounded variation provided that there exists a positive integer* B *such that*

$$\int_{x,y=1,1}^{\infty,\infty} \left| f(x,y) - f(x-r, y-s) \right| dxdy < B$$

where r *and* $s = 0$ *and/or* 1.

Let us now present the following notation.

Definition 2.8. *$\delta : I_2 = [a,b] \times [c,d] \to (0,\infty)$ is a positive function we define an open interval valued function on I_2 by setting $\gamma(t,z) = \gamma(t)\cdot\gamma(z) = (t-\delta(t), t+\delta(t)) \times (z-\delta(z), z+\delta(z))$. If $I_{i,j} = I_i \cdot J_j$, where $I_i = [i-\lambda_i+1, i]$ and $J_j = \left[j-\mu_j+1, j\right]$, then we can write $t_i \in I_i \subset \gamma(t_i)$ instead of $t_i - \delta < i - \lambda_i + 1 \le t_i \le i < t_i + \delta$ and $z_j \in J_j \subset \gamma(z_j)$ instead of $z_j - \delta < j - \mu_j + 1 \le z_j \le j < z_j + \delta$. Let $\overline{\gamma} = \overline{\gamma}(t_{i,j}) = \overline{\gamma}(t_i)\cdot\overline{\gamma}\left(z_j\right) \in \Delta_{G_2}$, and let $f(x,y)$ be a real valued multidimensional function which is measurable in the Gauge sense on $(1,\infty)\times(1,\infty)$. Provided that $\int f(x,y)$ and $\int \left|f(x,y)\right|$ exist in the Gauge sense and*

$$\lim_{t_i, z_j \to \infty} \frac{1}{\xi\left(t_{i,j}\right)} \int_{t_i-\delta(t_i)}^{t_i+\delta(t_i)} \int_{z_j-\delta(z_j)}^{z_j+\delta(z_j)} \left|f(x,y) - L\right| dxdy = 0,$$

where $\xi\left(t_{i,j}\right) = \xi(t_i)\cdot\xi\left(z_j\right) = [(t_i+\delta(t_i)) - (t_i-\delta(t_i))]\cdot\left[\left(z_j+\delta\left(z_j\right)\right) - \left(z_j-\delta\left(z_j\right)\right)\right] = 2\xi(t_i)\cdot 2\delta\left(z_j\right) = 4\delta\left(t_{i,j}\right)$, then we say that the function $f(x,y)$ is double $\overline{\gamma}_2-$strongly summable to L with respect to Gauge. In this case, we write $\left[G, \overline{\gamma}_2\right] - \lim f(x,y) = L$.

This notion leads us to the following theorem.

Theorem 2.9. *Let $I_{i,j} = [t_i - \delta(t_i), t + \delta(t_i)] \times \left[z_j - \delta(z_j), z_j + \delta\left(z_j\right)\right]$ and $[a,b]\times[c,d] = \cup I_{i,j}$ with $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$. If $f(x,y)$ is $\overline{\gamma}_2-$strongly summable to L with respect to Gauge, then $f(x,y)$ is $\overline{\gamma}_2-$ summable to L with respect to Gauge.*

Proof. Let $f(x,y)$ be a function on $I_{i,j} = [t_i - \delta(t_i), t + \delta(t_i)] \times \left[z_j - \delta(z_j), z_j + \delta\left(z_j\right)\right]$. Given a partition $P = \left\{[i-\lambda_i+1, i] \times \left[j-\mu_j+1, j\right]\right\}$ of $I_{i,j}$, we are granted from the properties of Gauge integral that f is absolutely integrable over I_i. Thus, $\left|\int_I f_1\right| \le \int_I \left|f_1\right|$. Since $f(x,y)$ is double $\overline{\gamma}_2-$strongly summable to L with respect to Gauge, we obtain the following:

$$\left| \int_{t_i-\delta(t_i)}^{t+\delta(t_i)} \int_{z_j-\delta(z_j)}^{z_j+\delta(z_j)} (f(x,y) - L)\, dxdy \right| \le \int_{t_i-\delta(t_i)}^{t+\delta(t_i)} \int_{z_j-\delta(z_j)}^{z_j+\delta(z_j)} \left|f(x,y) - L\right| dxdy.$$

Moreover,

$$\lim_{t_i \to \xi(t_i),\, z_j \to \xi(z_j) \text{ and } \|\Delta T_1\| \to 0,\, \|\Delta T_2\| \to 0} \sum_i \sum_j \frac{1}{\xi\left(t_{i,j}\right)} \int_{t_i-\delta(t_i)}^{t+\delta(t_i)} \int_{z_j-\delta(z_j)}^{z_j+\delta(z_j)} \left|f(x,y) - L\right| dx = 0$$

where $\|\Delta T_1\| = \|t_i - \xi(t_i)\|$ and $\|\Delta T_2\| = \left\|z_j - \xi\left(z_j\right)\right\|$. Hence $f(x,y)$ is $\overline{\gamma}_2-$summable to L with respect to Gauge. □

We conclude this paper with the following connection between double strongly summability in the Lebesgue and Gauge sense.

Theorem 2.10. *Let $\lambda = (\lambda_n) \in \Delta$, $\mu = (\mu_m) \in \Delta$, $\overline{\gamma}(t_i) \in \Delta_G$, $\overline{\gamma}(z_j) \in \Delta_G$, and $I_{i,j} = \left[t_i - \delta(t_i), t_i + \delta(t_i) \times z_j - \delta(z_j), z_j + \delta\left(z_j\right)\right]$ and $[a,b]\times[c,d] = \cup I_{i,j}$ with $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$. Additionally, let $f(x,y)$ be a real valued function in the Gauge sense in the interval $(1,\infty)\times(1,\infty)$, then*

1. $[V, \lambda, \mu] \subset \left[G, \overline{\gamma}_2\right]$

2. If $f(x,y)$ is bounded variation and f is double $\overline{\gamma}_2-$strongly summable to L with respect to Gauge sense over every measurable subset of

$$\left[t_i-\delta(t_i), t+\delta\left(t_i\right)\times z_j-\delta(z_j), z_j+\delta\left(z_j\right)\right],$$

then f is $[V]_2-\lim f(x,y)=L$.

Proof. 1. Since all multidimensional functions that are integrable in the Lebesgue sense are also integrable in the Gauge sense, if

$$\frac{1}{\lambda_{m,n}}\int\limits_{(x,y)\in I_{m,n}}\left|f(x,y)-L\right|dxdy$$

exists then

$$\frac{1}{4\delta\left(t_{i,j}\right)}\int\limits_{(x,y)\in\gamma\left(t_{i,j}\right)}\left|f(x,y)-L\right|dxdy$$

exists. Therefore $[V,\lambda,\mu]-\lim f(x,y)=L$ implies $\left[G,\overline{\gamma}_2\right]-\lim f(x,y)=L$.

In a manner similar to [11] we consider the following example,

$$f(x,y)=\begin{cases} 2xy\cos\left(\frac{\pi^2}{x^2y^2}\right)+\left(\frac{2\pi^2}{xy}\right)\sin\left(\frac{\pi^2}{x^2y^2}\right), & 0<x\leq 1 \text{ and } 0<y\leq 1\\ 0, & x=0 \text{ and } y=0.\end{cases}$$

f is neither Riemann integral nor Lebesgue integral. However, f is integrable in the Gauge sense. i.e $f(x,y)\notin[V,\lambda,\mu]$.

2. If $f(x,y)$ be a bounded variation, since $\frac{8\delta\left(t_{i,j}\right)}{mn}\leq 1$ for all (m,n), we obtain the following

$$\begin{aligned}
\left|\frac{1}{mn}\int_1^m\int_1^n\left(f(x,y)-L\right)dxdy\right| &= \left|\frac{1}{mn}\int_1^{m-\lambda_m}\int_1^{n-\mu_n}\left(f(x,y)-L\right)dxdy+\frac{1}{mn}\int\limits_{(x,y)\in I_{m,n}}\left(f(x,y)-L\right)dxdy\right|\\
&\leq \frac{1}{mn}\int_1^{n-\lambda_n}\int_1^{n-\mu_n}\left|f(x,y)-L\right|dxdy+\frac{1}{mn}\int\limits_{(x,y)\in I_{m,n}}\left|f(x,y)-L\right|dxdy\\
&\leq \frac{2}{mn}\int\limits_{(x,y)\in I_{m,n}}\left|f(x,y)-L\right|dxdy\\
&\leq \frac{2}{8\delta\left(t_i\right)}\int\limits_{(x,y)\in\gamma\left(t_{i,j}\right)}\left|f(x,y)-L\right|dxdy\\
&= \frac{1}{4\delta\left(t_{i,j}\right)}\int\limits_{(x,y)\in\gamma\left(t_{i,j}\right)}\left|f(x,y)-L\right|dxdy.
\end{aligned}$$

Since $\left[G,\overline{\gamma}_2\right]-\lim f(x,y)=L$, we can say that $[V]_2-\lim f(x,y)=L$. □

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