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Generalizations of Non-Linear Best Simultaneous Approximation in Köthe Function Spaces

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Abstract. This paper is concerned with the problem of nonlinear best simultaneous approximations in Köthe Bochner function spaces with respect to Minkowski' norms in Euclidean spaces. Characterization results of the generalized best simultaneous approximation are established. These results are considered a generalization of the results concerning the Lebesgue Bochner spaces and the Orlicz Bochner spaces.

1. Introduction

The problem of non-linear best simultaneous approximation in Banach and metric spaces has received considerable attention in the literature over the past 30 years. The main emphasis has been on providing characterization and uniqueness results for particular classes of the problem. Some recent works in this direction are given in [1]-[13], [17]-[21].

This paper is concerned with solutions of best simultaneous approximation of a class of problems. The setting is as follows. Let $(T, \sum, \mu)$ be a measure space and X be a real Banach space. For $x \in \mathbb{R}^n$, and for $1 \leq p < \infty$, let $\|x\|_p$ denote the usual L_p norm in $\mathbb{R}^n$. Let G be a closed subspace of X, and consider d_p for $x_1, x_2, ..., x_n \in X$ to be defined as follows:

$$d_p\left(\{x_i : 1 \leq i \leq n\}, G\right) = \inf_{z \in Y} \left(\sum_{i=1}^{n} \|x_i - z\|^p \right)^{1/p}.$$

An element $g_0 \in G$ is called a generalized best simultaneous approximation (GBA) of the elements $x_1, x_2, ..., x_n$ in X if

$$\left(\sum_{i=1}^{n} \left\|x_i - g_0\right\|^p \right)^{1/p} = d_p\left(\{x_i : 1 \leq i \leq n\},\ G\right).$$

If for the vectors $x_1, x_2, ..., x_n \in X$, there exists $y \in G$ which is the generalized best simultaneous approximation from G, we say that $y \in G$ is GBA of elements $x_1, x_2, ..., x_n$ in X. Moreover, If for any vectors $x_1, x_2, ..., x_n \in X$, there is a generalized best simultaneous approximation from G, then G is said to be GBA

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in X. If this GBA approximation is unique for the finite set of vectors $x_1, x_2, ..., x_n \in X$, then G is said to be Chebyshev GBA in X.

Solutions of simultaneous approximation problems for different spaces and even for different p were investigated by many authors. In the case when G is a closed subspace of X, the problem whether $L_p(T, \sum, G)$, the Banach space of all Bochner p-integrable functions defined on T with values in X, is proximinal in $L_p(T, \sum, X)$ has been studied widely, see for example [3], [7]-[8], and [17]-[20]. In particular, in the case when $(T, \sum, \mu)$ is a finite measure space, was proved in [7] that $L_1(T, \sum, G)$ is proximinal in $L_1(T, \sum, X)$ if G is reflexive and in [8] that $L_p(T, \sum, G)$ is proximinal in $L_p(T, \sum, X)$ if and only if $L_1(T, \sum, G)$ is proximinal in $L_p(T, \sum, X)$. These results have been extended to the case that $(T, \sum, \mu)$ is a σ-finite measure space in [18], where it was further proved for a closed separable subspace G that $L_p(T, \sum, G)$ is proximinal in $L_p(T, \sum, X)$ if and only if G is proximinal in X.

The problem of the best simultaneous approximations in $L_1(T, \sum, X)$ from convex sets was studied in [21]. Saidi et al. proved in [20] that $L_p(T, \sum, G)$ is N-simultaneous proximinal in $L_p(T, \sum, X)$ for each $1 \le p < \infty$, in the case when G be a reflexive subspace of X. Mendoza and Pakhrou [19] proved that $L_1(T, \sum_0, X)$ is N-simultaneous proximinal in $L_1(T, \sum_0, X)$ if X is reflexive, where $\sum_0$ is a sub-σ-algebra of $\sum$. X.F. Luo et al. in [17], generalized the above result and proved that $L_p(T, \sum_0, G)$ is N-simultaneous proximinal in $L_p(T, \sum, X)$ with the additional assumptions that G be a nonempty locally weakly compact convex subset of X such that $\overline{\text{span } Y}$ and its dual have the Radon–Nikodym property.

Köthe Bochner spaces generalize many important spaces like the L_p spaces, the Orlicz spaces, the Lorentz spaces, and the Marcinkiewiczv spaces. Their theory is developing very fast, being of great actuality. The interested reader can find an up to date and very technical material concerning the Köthe Bochner spaces in [16].

In this paper, we will study the problem of the best simultaneous approximation under the formation of Köthe Bochner spaces $E(X)$, thereby generalizing some results in previous works like that in [9] and [17]. More precisely, for a given closed subspace G of X and n-tuple $F = (f_1, f_2, ..., f_n) \in (E(X))^n$, we show the existence of n-tuple $\Gamma_0 = (g_0, g_0, ..., g_0) \in (E(G))^n$ such that

$$\|F - \Gamma_0\|_p = \inf_{g \in E(G)} \left\| F - (g, g, ..., g) \right\|_p .$$

The n-tuple Γ_0 is called a GBA of $F = (f_1, f_2, ..., f_n)$.

2. Preliminaries

Throughout this paper $(T, \sum, \mu)$ will be a finite complete measure space. A function $f : T \to X$ is called strongly $\mu-$measurable, if there exists a sequence of simple functions $\{f_n\}$ which converges to f $\mu-$almost every where on T (a.e. $t \in T$). The space $L^0 = L^0(T)$ denotes the space of all equivalence classes of $\sum-$measurable real-valued functions, such that $x(t) \le y(t)$ almost everywhere (a.e. $t \in T$), whenever $x \le y$ with $x, y \in L^0$.

A Banach space $(E; \|\cdot\|_E)$ is called a Köthe space, if for every $x, y \in L^0$, with $|x| \le |y|$ and $y \in E$, then x is an element of E and $\|x\|_E \le \|y\|_E$, also if A is a measurable set with $\mu(A)$ is finite, then $\chi_A \in A$. Here χ_A denotes the characteristic function, which is specified to be one on A and zero elsewhere.

If E is a Köthe space on the measure space $(T, \sum, \mu)$, then the space $E(X)$ consists of all equivalence classes of strongly measurable functions $x : T \to X$, where $\|x(\cdot)\|$ is an element in E with the norm:

$$|||x||| = \|(\|x(\cdot)\|)\|_E .$$

Then $(E(X), |||\cdot|||)$ is a Banach spaces called the Köthe Bochner function space induced by E and X. For more details about the Köthe space we refer the reader to [16].

Recall that if $(T, \sum, \mu)$ be a measure space and a function $f : T \longrightarrow X$ is called measurable in the classical sense, if $f^{-1}(O)$ is measurable for every open set $O \subset X$.

The following results are needed to establish some results in the next section.

Lemma 2.1. *[15]. Let $(T, \sum, \mu)$ be a complete measure space and X a Banach space. Then for a function $f : T \longrightarrow X$, we have the following*

(i) If f is strongly measurable, then f is a measurable function in the classical sense.

(ii) If f is a measurable function in the classical sense and has essential separable range, then f is strongly measurable.

Let Θ be the set valued map that takes each point of a measurable space T into a subset of a metric space W. Then, Φ is called weakly measurable if $\Phi^{-1}(O)$ is measurable in T whenever O is open in W, where the set

$$\Theta^{-1}(O) = \{t \in T : \Theta(t) \cap O \neq \phi\}.$$

The following measurable selection theorem is due to Kuratowski and Ryll-Nardzeewskiin in [14].

Theorem 2.2. *Let Θ be weakly measurable set valued map which carries each point of a measurable space T to a closed non-empty subset of a complete separable metric space W. Then Θ has a measurable selection f: i.e. there is a function $f : T \longrightarrow X$ such that $f(t) \in \Theta(t)$ for each $t \in T$ and $f^{-1}(O)$ is measurable in T whenever O is open in W.*

3. Main Results

Now we'll state and prove the primary result in terms of distance function characterization.

Theorem 3.1. *Let G be a closed subspace of the real Banach space X and let $E(X)$ be a Köthe Bochner function space with absolutely continuous norm. For $f_1, f_2, \ldots, f_n$ in $E(X)$, define the distance function $\Phi : T \to R$ by $\Phi(t) = d_p(\{f_i(t) : 1 \leq i \leq n\}, G)$. Then $\Phi \in E$ and*

$$d_p(\{f_i : 1 \leq i \leq n\}, E(G)) = \|\Phi\|_E.$$

Proof. Let $f_1, f_2, \ldots, f_n \in E(X)$, then for each $i = 1, 2, \ldots, n$, there exist a sequence of simple functions $\{f_{m,i}\}_{i=1}^{n}$, in $E(X)$ which converges to f_i for *a.e.* $t \in T$. That is:

$$\left\| f_{m,i}(t) - f_i(t) \right\|_E \to 0, \text{ for each } i = 1, 2, \ldots, n, \text{ for } a.e.\ t \in T.$$

Suppose that for each $i = 1, 2 \ldots, n$

$$f_{m,i} = \sum_{i=1}^{k(m)} x_{m,i,j}\, \chi_{A_{m,j}}, \quad x_{m,i,j} \in X,$$

where $\sum\limits_{j=1}^{k(m)} \mu(A_{m,j}) = 1$ and $\mu\left(A_{m,j}\right) > 0$. Thus

$$d_p(\{f_{m,i}(t) : 1 \leq i \leq n\}, G) = \sum_{i=1}^{k(m)} d_p\left(\left\{x_{m,i,j} : 1 \leq i \leq n\right\}, G\right) \chi_{A_{m,j}}.$$

By the continuity of the distance function d_p, we have

$$\underset{m \longrightarrow \infty}{Lim}\, d_p(\{f_{m,i}(t) : 1 \leq i \leq n\}, G) = d_p(\{f_i(t) : 1 \leq i \leq n\}, G),\ a.e.\ t \in T.$$

Then Φ is measurable function and since $0 \in G$, we have

$$\Phi(t) \leq \left(\sum_{i=1}^{n} \left\| f_i(t) - z \right\|^p \right)^{\frac{1}{p}}, \text{ for all } z \in G.$$

As a consequence

$$\|\Phi\|_E \le \left\| \left(\sum_{i=1}^{n} \left\| f_i(\cdot) - g(\cdot) \right\|^p \right)^{\frac{1}{p}} \right\|_E, \text{ for all } g \in E(G).$$

Therefore $\Phi \in E$ and

$$\left\|\phi\right\|_E \le d_p\left(\{f_i : 1 \le i \le n\}, E(G)\right). \tag{1}$$

For the reverse inequality. $E(X)$ is a Köthe Bochner function space with absolutely continuous norm, then the simple functions are dense in $E(X)$, see [16]. Let $\varepsilon > 0$ be given, there exist simple functions f_i^* in $E(X)$ such that

$$\left\|\left\| f_i - f_i^* \right\|\right\| < \frac{\varepsilon}{3n}, \ \ i = 1, 2, \ldots, n.$$

Assume

$$f_i^* = \sum_{j=1}^{k} x_j^i \, \chi_{A_j}, \ i = 1, 2, \ldots, n,$$

where $\sum\limits_{j=1}^{k} \mu(A_j) = 1$, $0 < \mu\left(A_j\right) < \infty$ and $x_j^i \in X$, for each $j = 1, 2, ..., k$. We can assume that $\mu(T) = \beta$ since the measure is finite and hence $|||\chi_T||| = \sum\limits_{j=1}^{k} \mu\left(A_j\right) = \mu\left(\bigcup\limits_{j=1}^{k} A_j\right) = \mu(T) = \beta$. Pick $y_j \in G$ such that

$$\left(\sum_{i=1}^{n} \left\| x_j^i - y_j \right\|^p \right)^{\frac{1}{p}} < d_p\left(\left\{ x_j^i : 1 \le i \le n \right\}, G \right) + \frac{\varepsilon}{3\beta},$$

for each $j = 1, 2, ..., k$. Let

$$g = \sum_{j=1}^{k} y_j \, \chi_{A_j}.$$

Then $g \in E(G)$, and

$$\left\| \left(\sum_{i=1}^{n} \left\| f_i^*(\cdot) - g(\cdot) \right\|^p \right)^{\frac{1}{p}} \right\|_E \tag{2}$$

$$= \left\| \sum_{j=1}^{k} \chi_{A_j}(\cdot) \left[\left(\sum_{i=1}^{n} \left\| x_j^i - y_j \right\|^p \right)^{\frac{1}{p}} \right] \right\|_E$$

$$< \left\| \sum_{j=1}^{k} \chi_{A_j}(\cdot) \left[d_p\left(\left\{ x_j^i : 1 \le i \le n \right\}, G \right) + \frac{\varepsilon}{3\beta} \right] \right\|_E$$

$$\le \left\| d_p\left(\left\{ f_i^*(\cdot) : 1 \le i \le n \right\}, G \right) \right\|_E + \frac{\varepsilon}{3\beta} \left\|\left\| \sum_{j=1}^{k} \chi_{A_j} \right\|\right\|$$

$$= \left\| d_p\left(\left\{f_i^*(\cdot) : 1 \le i \le n\right\}, G\right)\right\|_E + \frac{\varepsilon}{3}.$$

For the following inequality

$$\begin{aligned}
d_p\left(\{f_i : 1 \le i \le n\}, E(G)\right) &\le \left\| \left(\sum_{i=1}^{n} \left\|f_i(\cdot) - g(\cdot)\right\|^p\right)^{\frac{1}{p}} \right\|_E \\
&\le \left\| \left[\sum_{i=1}^{n} \left(\left\|f_i(\cdot) - f_i^*(\cdot)\right\| + \left\|f_i^*(\cdot) - g(\cdot)\right\|\right)^p\right]^{\frac{1}{p}} \right\|_E \\
&\le \left\| \left(\sum_{i=1}^{n} \left\|f_i(\cdot) - f_i^*(\cdot)\right\|^p\right)^{\frac{1}{p}} \right\|_E + \left\| \left(\sum_{i=1}^{n} \left\|f_i^*(\cdot) - g(\cdot)\right\|^p\right)^{\frac{1}{p}} \right\|_E \\
&\le \sum_{i=1}^{n} \left\|\left\|f_i - f_i^*\right\|\right\| + \left\| \left(\sum_{i=1}^{n} \left\|f_i^*(\cdot) - g(\cdot)\right\|^p\right)^{\frac{1}{p}} \right\|_E \\
&< \frac{\varepsilon}{3} + \left\| \left(\sum_{i=1}^{n} \left\|f_i^*(\cdot) - g(\cdot)\right\|^p\right)^{\frac{1}{p}} \right\|_E \\
&< \left\| d_p\left(\left\{f_i^*(\cdot) : 1 \le i \le n\right\}, G\right)\right\|_E + \frac{2\varepsilon}{3}.
\end{aligned} \tag{3}$$

The inequalities (2) and (3) give the following result,

$$d_p\left(\{f_i : 1 \le i \le n\}, E(G)\right) < \left\| d_p\left(\left\{f_i^*(\cdot) : 1 \le i \le n\right\}, G\right)\right\|_E + \frac{2\varepsilon}{3}$$

$$\le \left\| d_p\left(\{f_i(\cdot) : 1 \le i \le n\}, G\right) + \left(\sum_{i=1}^{n} \left\|f_i(\cdot) - f_i^*(\cdot)\right\|^p\right)^{\frac{1}{p}} \right\|_E + \frac{2\varepsilon}{3}$$

$$\le \left\| d_p\left(\{f_i(\cdot) : 1 \le i \le n\}, G\right)\right\|_E + \sum_{i=1}^{n} \left\|\left\|f_i - f_i^*\right\|\right\| + \frac{2\varepsilon}{3}$$

$$\begin{aligned}
&\le \left\| d_p\left(\{f_i(\cdot) : 1 \le i \le n\}, G\right)\right\|_E + \sum_{i=1}^{n} \left\|\left\|f_i - f_i^*\right\|\right\| + \frac{2\varepsilon}{3} \\
&< \left\| d_p\left(\{f_i(\cdot) : 1 \le i \le n\}, G\right)\right\|_E + \varepsilon \\
&= \left\|\phi\right\|_E + \varepsilon.
\end{aligned}$$

Therefore

$$d_p\left(\{f_i : 1 \le i \le n\}, E(G)\right) < \left\|\phi\right\|_E + \varepsilon.$$

Since ε arbitrary let $\varepsilon \to 0$, then

$$d_p\left(\{f_i : 1 \le i \le n\}, E(G)\right) \le \left\|\phi\right\|_E. \tag{4}$$

Inequalities (1) and (4) give the result. □

As a consequence of Theorem 3.1, we get the following result.

Corollary 3.2. *Let $f_1, f_2, \ldots, f_n \in E(X)$ and $g : T \to G$ be a strongly measurable function such that $g(t)$ is GBA for elements $f_1(t), f_2(t), \ldots, f_n(t)$ in G for a.e. $t \in T$. Then $g \in E(G)$ and the element g is GBA of the elements $f_1, f_2, \ldots, f_n$ in $E(X)$.*

Proof. By the assumption that $g(t)$ is *GBA* for the n- elements $f_1(t), f_2(t), \ldots, f_n(t)$ in G for *a.e.* $t \in T$. Then

$$\left(\sum_{i=1}^{n} \left\|f_i(t) - g(t)\right\|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} \left\|f_i(t) - z\right\|^p\right)^{\frac{1}{p}},$$

for all $z \in G$. Since $0 \in G$, then for *a.e.* $t \in T$, we have

$$\begin{aligned}\left(\sum_{i=1}^{n} \left\|f_i(t) - g(t)\right\|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^{n} \left\|f_i(t)\right\|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{i=1}^{n} \left\|f_i(t)\right\|\right).\end{aligned} \tag{5}$$

Also

$$\begin{aligned} n\left\|g(t)\right\|^p &= \left\|g(t)\right\|^p + \left\|g(t)\right\|^p + \cdots + \left\|g(t)\right\|^p \\ &\leq \sum_{i=1}^{n} \left(\left\|g(t) - f_i(t)\right\| + \left\|f_i(t)\right\|\right)^p .\end{aligned} \tag{6}$$

Using inequalities (5) and (6), we get

$$\begin{aligned} n^{\frac{1}{p}}\left\|g(t)\right\| &\leq \left(\sum_{i=1}^{n} \left(\left\|g(t) - f_i(t)\right\| + \left\|f_i(t)\right\|\right)^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{i=1}^{n} \left\|g(t) - f_i(t)\right\|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} \left\|f_i(t)\right\|^p\right)^{\frac{1}{p}} \\ &\leq 2\sum_{i=1}^{n} \left\|f_i(t)\right\| .\end{aligned}$$

Therefore

$$|||g||| \leq \frac{2}{n^{1/p}}\left(\sum_{i=1}^{n} |||f_i|||\right).$$

This shows that $g \in E(G)$. Also by Theorem 3.1

$$\left\|\left(\sum_{i=1}^{n} \left\|f_i(\cdot) - g(\cdot)\right\|^p\right)^{\frac{1}{p}}\right\|_E \leq \left\|\left(\sum_{i=1}^{n} \left\|f_i(\cdot) - h(\cdot)\right\|^p\right)^{\frac{1}{p}}\right\|_E,$$

for all $h \in E(G)$. This means that g is *GBA* of the elements $f_1, f_2, \ldots, f_n$ in $E(X)$. □

Theorem 3.3. *If G is GBA in the real Banach space X. If the elements $f_1, f_2, \ldots, f_n$ are simple functions in $E(X)$, then there exist an element in $E(G)$ which is GBA of these simple functions.*

Proof. Let $\{f_i\}_{i=1}^{n}$be a finite set of simple functions in $E(X)$. Then for each $i = 1, 2, \ldots, n$, we have

$$f_i = \sum_{j=1}^{k} u_j^i \, \chi_{A_j},$$

where $\sum_{j=1}^{k} \mu(A_j) = 1$ and $\mu\left(A_j\right) > 0$. By the assumption we know that for each $j = 1, 2, ..., k$, there exists y_j in G, such that y_j is *GBA* of the elements $u_j^1, \ldots, u_j^n$ in X respectively. Thus

$$\sum_{i=1}^{n} d_p\left(\left\{u_j^i : 1 \le i \le n\right\}, G\right) = \left(\sum_{i=1}^{n} \left\|u_j^i - y_j\right\|^p\right)^{\frac{1}{p}}.$$

Set

$$g = \sum_{j=1}^{k} y_j \, \chi_{A_j},$$

then $g \in E(G)$. Also for any $\alpha > 0$ and $h \in E(G)$, we obtain that

$$\begin{aligned}
&\left\| \left(\sum_{i=1}^{n} \left\|f_i(\cdot) - h(\cdot)\right\|^p\right)^{\frac{1}{p}} \right\|_E \\
&\ge \left\| \sum_{j=1}^{k} \chi_{A_j}(\cdot) \left(\sum_{i=1}^{n} \left\|u_j^i - y_j\right\|^p\right)^{\frac{1}{p}} \right\|_E \\
&= \left\| \left(\sum_{i=1}^{n} \left\|f_i(\cdot) - g(\cdot)\right\|^p \right)^{\frac{1}{p}} \right\|_E .
\end{aligned}$$

Taking the infimum over all $h \in E(G)$, we have

$$d_p\left(\{f_i : 1 \le i \le n\}, E(G)\right) = \left\| \left(\sum_{i=1}^{n} \left\|f_i(\cdot) - g(\cdot)\right\|^p \right)^{\frac{1}{p}} \right\|_E .$$

This means that g is *GBA* of the simple functions $f_1, f_2, \ldots, f_n$ in $E(X)$. □

Theorem 3.4. *Let $E(X)$ be a Köthe Bochner function space with absolutely continuous and strictly monotone norm. If $E(G)$ is GBA in $E(X)$, then G is GBA in X.*

Proof. Let $x_1, x_2, \ldots, x_n \in X$ and set $f_i = \chi_T(\cdot)\, x_i$, for each $i = 1, 2, \ldots, n$ for *a.e.* $t \in T$. Since

$$|||f_i||| = \left\|\left(\left\|f_i(\cdot)\right\|\right)\right\|_E = ||||x_i \, \chi_T(\cdot)||||_E = ||x_i|| \quad |||\chi_T|||,$$

which is finite, then $f_i \in E(X)$ for each $i = 1, 2, \ldots, n$. By the assumption, there exists $g_0 \in E(G)$ such that

$$\left\| \left(\sum_{i=1}^{n} \left\|f_i(\cdot) - g_0(\cdot)\right\|^p \right)^{\frac{1}{p}} \right\|_E \le \left\| \left(\sum_{i=1}^{n} \left\|f_i(\cdot) - h(\cdot)\right\|^p \right)^{\frac{1}{p}} \right\|_E ,$$

for all $h \in E(G)$.

Since $E(X)$ is a Köthe Bochner function space which is strictly monotone, then we have

$$\sum_{i=1}^{n} \left\| f_i(t) - g_0(t) \right\|^p \leq \sum_{i=1}^{n} \left\| f_i(t) - h(t) \right\|^p ,$$

for *a.e.* $t \in T$.

Fix $t_0 \in T$ and $y_0 = g_0(t_0)$, then $y_0 \in G$ and

$$\sum_{i=1}^{n} \left\| x_i - y_0 \right\|^p \leq \sum_{i=1}^{n} \| x_i - h(t) \|^p , \text{ for all } h \in E(G).$$

Since G is embedded isometrically into $E(G)$, then

$$\sum_{i=1}^{n} \left\| x_i - y_0 \right\|^p \leq \sum_{i=1}^{n} \| x_i - z \|^p ,$$

for all $z \in G$. Thus y_0 is *GBA* of the elements $x_1, x_2, \ldots, x_n$ in X, which completes the proof. □

Next, we give the characterization of the *GBA* of $E(G)$ in $E(X)$.

Theorem 3.5. *Let G be a closed separable subspace of the real Banach space X and let $E(X)$ be the Köthe Bochner function space with absolutely continuous and strictly monotone norm. Then the following are equivalent:*

(*i*) *$E(G)$ is GBA in $E(X)$.*

(*ii*) *G is GBA in X.*

Proof. (i) $\Longrightarrow$ (ii) included in Theorem 3.4.

For (ii) $\Longrightarrow$ (i). Suppose $f_1, f_2, \ldots, f_n \in E(X)$. For each $t \in T$, denote

$$\Theta(t) = \{ g \in G : \left(\sum_{i=1}^{n} \left\| f_i(t) - g \right\|^p \right)^{\frac{1}{p}} = d_p\left(\{ f_i(t) : 1 \leq i \leq n \}, G \right) \}.$$

Then for each $t \in T$, $\Theta(t)$ is a closed, bounded, and non-empty subset of G. Now, we shall prove that Θ is weakly measurable. Let O be an open set in X, the set

$$\Theta^{-1}(O) = \left\{ t \in T : \Theta(t) \cap O \neq \phi \right\}$$

can be represented as

$$\Theta^{-1}(O) = \{ t \in T : \inf_{g \in G} \sum_{i=1}^{n} \left\| f_i(t) - g \right\|^p = \inf_{v \in O} \sum_{i=1}^{n} \left\| f_i(t) - v \right\|^p \}.$$

Since $(T, \sum, \mu)$ is a complete space, then by Lemma 2.1, part (i), f_i are measurable functions in the classical sense for each $i = 1, 2, \ldots, n$. By the continuity of the norm, the map

$$t \longrightarrow \inf_{v \in O} \sum_{i=1}^{n} \left\| f_i(t) - v \right\|^p$$

is measurable for each set $Q \subset T$. It follows that $\Theta^{-1}(O)$ is measurable. Thus by Theorem 2.2, Θ has a measurable selector $g : T \to G$ such that $g(t) \in \Theta(t)$, for each $t \in T$, where g is measurable in the classical sense. By Lemma 2.1, part(ii), g is strongly measurable. Thus by Corollary 3.2, g is *GBA* for $f_1, f_2, \ldots, f_n$ in $E(X)$. Therefore, we conclude that $E(G)$ is *GBA* in $E(X)$. This completes the proof. □

4. Conclusion

The generalized best simultaneous approximation GBA of a finite number of functions in Köthe Bochner function spaces were studied in this paper. The relationship between the GBA of G, the closed subspace of X and the GBA of $E(G)$ in $E(X)$ was also investigated. These characterization can be viewed as a generalization of a number of related theorems concerning the Lebesgue Bochner space and the Orlicz Bochner spaces.

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