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On Banach Algebras Defined by Multipliers

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Abstract. In this paper, we investigate a Banach algebra A_T, where A is a Banach algebra and T is a left (right) multiplier on A. We study some concepts on A_T such as n-weak amenability, cyclic amenability, biflatness, biprojectivity and Arens regularity. For the group algebra $L^1(G)$ of an infinite compact group G, it is shown that there is a multiplier T such that $L^1(G)_T$ has not a bounded approximate identity. For $\ell^1(S)$, where S is a regular semigroup with a finite number of idempotents, we show that there is a multiplier T such that Arens regularity of $\ell^1(S)_T$ implies that S is compact.

1. Introduction

Let A be a Banach algebra and $T \in \mathcal{B}(A)$, where $\mathcal{B}(A)$ is the set of all bounded linear maps on A. Then T is called a left (right) multiplier of A if,

$$T(ab) = T(a)b \quad (T(ab) = aT(b)) \qquad (a, b \in A).$$

The set of all left (right) multipliers on a Banach algebra A is denoted by $\mathcal{M}_l(A)$ ($\mathcal{M}_r(A)$). An operator $T \in \mathcal{B}(A)$ is called a multiplier if

$$T(ab) = T(a)b = aT(b) \qquad (a, b \in A).$$

The set of all multipliers on a Banach algebra A is denoted by $\mathcal{M}(A)$. Let A be a Banach algebra and $T \in \mathcal{M}(A)$. A Banach algebra related to A and T is defined in [20, 21] and it is denoted by A_T with the following multiplication:

$$a \circ b = aT(b),$$

for all $a, b \in A$, where $T \in \mathcal{B}(A)$. The norm on A_T is a norm that is equivalent with the original norm on A i.e., $\|\cdot\|$ which defined as follows:

$$\|a\|_T = \|a\|\|T\| \qquad (a \in A).$$

Some basic results depending on algebraic and analytical properties have been studied in [19]. In this paper, we replace "•" instead of "◦", because we use this notation for a combination of maps. Some results related to the first module cohomology, pseudo amenability, Johnson pseudo-contractibility and module amenability of the above defined Banach algebra A_T are studied in [24, 27].

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Let A be a Banach algebra, and let X be a Banach A-bimodule. A derivation from A into X is a linear map $D : A \longrightarrow X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b,$$

for all $a, b \in A$. The set of derivations from A into X is denoted by $\mathcal{Z}^1(A, X)$; which is a linear subspace of $\mathcal{B}(A, X)$, the space of all bounded linear maps from A into X. If $A = X$, then we write $\mathcal{B}(A)$. For $x \in X$, set $D_x : A \longrightarrow X, a \mapsto a \cdot x - x \cdot a$. Derivations of this form are called inner derivations, and an inner derivation D_x is implemented by x. The set of inner derivations from A into X is a linear subspace $\mathcal{N}^1(A, X)$ of $\mathcal{Z}^1(A, X)$. The quotient space $\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X)$ is called the *first Hochschild cohomology group* of A with coefficients in X.

The concept of amenability for Banach algebras was introduced by Johnson in 1972 [17]. A Banach algebra A is called amenable if $\mathcal{H}^1(A, X^*) = \{0\}$ for any A-bimodule X. An interesting result that Johnson proved stating that $L^1(G)$ is amenable if and only if G is amenable (G is a locally compact group).

Weak amenability of Banach algebras was introduced by Bade et al. in [4]. A Banach algebra A is called weakly amenable if $\mathcal{H}^1(A, A^*) = \{0\}$. Let $n \in \mathbb{N}$; a Banach algebra A is called n-weakly amenable if $\mathcal{H}^1(A, A^{(n)}) = \{0\}$. Dales, Ghahramani and Grønbæk brought the concept of n-weak amenability of Banach algebras in [8].

Let A be Banach algebra. Regarding A as a Banach A-bimodule, the operation $\pi : A \times A \to A$ extends to π^{***} and π^{t***t} defined on $A^{**} \times A^{**}$. These extensions are known as the first (left) and the second (right) Arens products, respectively, and with each of them, the second dual space A^{**} becomes a Banach algebra. The first (left) Arens product of $a'', b'' \in A^{**}$ shall be simply indicated by $a''\Box b''$ and defined by the following three steps:

$$\langle a'a, b\rangle = \langle a', ab\rangle,\ \langle a''a', a\rangle = \langle a'', a'a\rangle,\ \langle a''\Box b'', a'\rangle = \langle a'', b''a'\rangle,$$

for all $a, b \in A$ and $a' \in A^*$. Similarly, the second (right) Arens product of $a'', b'' \in A^{**}$ shall be indicated by $a'' \diamond b''$ and defined as follows:

$$\langle aa', b\rangle = \langle a', ba\rangle,\ \langle a'a'', a\rangle = \langle a'', aa'\rangle,\ \langle a'' \diamond b'', a'\rangle = \langle b'', a'b''\rangle,$$

for all $a, b \in A$ and $a' \in A^*$. If two multiplication coincide, then we say that A is Arens regular [7, 26].

Let A be a Banach algebra and T be an element of $\mathcal{M}(A)$. In this paper, we investigate n-weak amenability, cyclic amenability, biprojectivity and biflatness of A_T. We give a proof for the converse case of [19, Theorem 3.5], where we suppose that T is surjective (it is not invertible). As an interesting result, under some conditions on G and T, we show that $L^1(G)_T$ has not a bounded approximate identity. Finally, we prove that the main result of [19] for regular semigroups with finite idempotent elements. Moreover, we have asked some questions maybe are interesting for readers and future works.

2. n-Weak Amenability and Cyclic Amenability

In this section, for a Banach algebra A and $T \in \mathcal{M}(A)$, we study n-weak amenability of A_T. The dual of A_T carries a natural left and right A_T-module structure defined by

$$\langle b, a^* \odot a\rangle = \langle a \bullet b, a^*\rangle, \quad \langle b, a \odot a^*\rangle = \langle b \bullet a, a^*\rangle,$$

for all $a, b \in A_T$ and $a^* \in A_T^*$. We start with the following Lemmas:

Lemma 2.1. *[19, Lemma 2.3] Let A be a Banach algebra and let T be invertible. Then $T \in \mathcal{M}_l(A)$ $(T \in \mathcal{M}_r(A))$ if and only if $T^{-1} \in \mathcal{M}_l(A)$ $(T^{-1} \in \mathcal{M}_r(A))$.*

Lemma 2.2. *Let $T \in \mathcal{M}(A)$ and $\varphi : A_T \longrightarrow A$ by $\varphi(a) = T(a)$ for all $a \in A_T$.*

(i) $T^{(n)}$ is an A-module morphism.

(ii) If T is invertible and $\alpha(a) : A \longrightarrow A_T$ defined by $\alpha(a) = T^{-1}(a)$ for all $a \in A$, then $\varphi^{(2n)}(\alpha(a) \odot y^{(2n)}) = a \cdot \varphi^{(2n)}(y^{(2n)})$, and $\varphi^{(2n)}(y^{(2n)} \odot \alpha(a)) = \varphi^{(2n)}(y^{(2n)}) \cdot a$, for all $a \in A$, $y^{(2n)} \in (A_T)^{(2n)}$ and $n \in \mathbb{N}$.

(iii) If T is invertible and $\alpha(a) : A \longrightarrow A_T$ defined by $\alpha(a) = T^{-1}(a)$ for all $a \in A$, then $\alpha^{(2n+1)}(\alpha(a) \odot y^{(2n+1)}) = a \cdot \alpha^{(2n+1)}(y^{(2n+1)})$, and $\alpha^{(2n+1)}(y^{(2n+1)} \odot \alpha(a)) = \alpha^{(2n+1)}(y^{(2n+1)}) \cdot a$, for all $a \in A$, $y^{(2n+1)} \in (A_T)^{(2n+1)}$ and $n \in \mathbb{N}$.

Proof. We prove the results by induction on n.

(i) For every $a, b \in A$ and $x \in A^*$, we have

$$\begin{aligned}\langle T^*(a \cdot x), b\rangle &= \langle a \cdot x, T(b)\rangle = \langle x, T(b)a\rangle = \langle x, T(ba)\rangle \\ &= \langle T^*(x), ba\rangle = \langle a \cdot T^*(x), b\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle T^*(x \cdot a), b\rangle &= \langle x \cdot a, T(b)\rangle = \langle x, aT(b)\rangle = \langle x, T(ab)\rangle \\ &= \langle T^*(x), ab\rangle = \langle T^*(x) \cdot a, b\rangle.\end{aligned}$$

By the above argument, we have

$$\begin{aligned}\langle T^{**}(a \cdot x^{(2)}), b\rangle &= \langle a \cdot x^{(2)}, T^*(b)\rangle = \langle x^{(2)}, T^*(b) \cdot a\rangle = \langle x^{(2)}, T^*(b \cdot a)\rangle \\ &= \langle T^{**}(x^{(2)}), b \cdot a\rangle = \langle a \cdot T^{**}(x^{(2)}), b\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle T^{**}(x^{(2)} \cdot a), b\rangle &= \langle x^{(2)} \cdot a, T^*(b)\rangle = \langle x^{(2)}, a \cdot T^*(b)\rangle = \langle x^{(2)}, T^*(a \cdot b)\rangle \\ &= \langle T^{**}(x^{(2)}), a \cdot b\rangle = \langle T^{**}(x^{(2)}) \cdot a, b\rangle,\end{aligned}$$

for every $a \in A$, $b \in A^*$ and $x^{(2)} \in A^{**}$. Then for all $n \geq 1$, $a \in A$ and $x^{(n)} \in A^{(n)}$ we have

$$T^{(n)}(a \cdot x^{(n)}) = a \cdot T^{(n)}(x^{(n)}), \quad \text{and} \quad T^{(n)}(x^{(n)} \cdot a) = T^{(n)}(x^{(n)}) \cdot a.$$

This shows that $T^{(n)}$ is an A-module morphism.

(ii) For all $a \in A$, $b \in A_T$ and $x^* \in A^*$,

$$\begin{aligned}\langle b, \varphi^*(x^*) \odot \alpha(a)\rangle &= \langle \alpha(a) \bullet b, \varphi^*(x^*)\rangle = \langle \alpha(a)T(b), \varphi^*(x^*)\rangle \\ &= \langle \varphi(\alpha(a)T(b)), x^*\rangle = \langle T(\alpha(a)T(b)), x^*\rangle \\ &= \langle aT(b), x^*\rangle = \langle T(b), x^* \cdot a\rangle \\ &= \langle b, \varphi^*(x^* \cdot a)\rangle \end{aligned} \tag{1}$$

and

$$\begin{aligned}\langle b, \alpha(a) \odot \varphi^*(x^*)\rangle &= \langle b \bullet \alpha(a), \varphi^*(x^*)\rangle = \langle ba, \varphi^*(x^*)\rangle \\ &= \langle \varphi(ba), x^*\rangle = \langle T(b)a, x^*\rangle = \langle T(b), a \cdot x^*\rangle \\ &= \langle b, \varphi^*(a \cdot x^*)\rangle. \end{aligned} \tag{2}$$

By (1) we have

$$\begin{aligned}\langle x^*, \varphi^{**}(\alpha(a) \odot y^{**})\rangle &= \langle \varphi^*(x^*), \alpha(a) \odot y^{**}\rangle = \langle \varphi^*(x^*) \odot \alpha(a), y^{**}\rangle \\ &= \langle \varphi^*(x^* \cdot a), y^{**}\rangle = \langle x^* \cdot a, \varphi^{**}(y^{**})\rangle \\ &= \langle x^*, a \cdot \varphi^{**}(y^{**})\rangle,\end{aligned}$$

for all $a \in A$, $y^{**} \in (A_T)^{**}$ and $x^* \in A^*$. This shows that $\varphi^{**}(\alpha(a) \odot y^{**}) = a \cdot \varphi^{**}(y^{**})$, for all $a \in A$ and $y^{**} \in (A_T)^{**}$. Similarly by (2), we have $\varphi^{**}(y^{**} \odot \alpha(a)) = \varphi^{**}(y^{**}) \cdot a$, for all $a \in A$ and $y^{**} \in (A_T)^{**}$. Now, we extend the above results for $2n$, where $n \geq 1$. Then

$$\varphi^{(2n)}(\alpha(a) \odot y^{(2n)}) = a \cdot \varphi^{(2n)}(y^{(2n)}),$$

and

$$\varphi^{(2n)}(y^{(2n)} \odot \alpha(a)) = \varphi^{(2n)}(y^{(2n)}) \cdot a,$$

for all $a \in A$ and $y^{(2n)} \in (A_T)^{(2n)}$.

(iii) For $a, b \in A$ and $x^* \in A_T^*$,

$$\begin{aligned} \langle b, \alpha^*(\alpha(a) \odot x^*)\rangle &= \langle \alpha(b), \alpha(a) \odot x^*\rangle = \langle \alpha(b) \bullet \alpha(a), x^*\rangle \\ &= \langle \alpha(ba), x^*\rangle = \langle b, a \cdot \alpha^*(x^*)\rangle, \end{aligned} \tag{3}$$

and

$$\begin{aligned} \langle b, \alpha^*(x^* \odot \alpha(a))\rangle &= \langle \alpha(b), x^* \odot \alpha(a)\rangle = \langle \alpha(a) \bullet \alpha(b), x^*\rangle \\ &= \langle \alpha(ab), x^*\rangle = \langle b, \alpha^*(x^*) \cdot a\rangle. \end{aligned} \tag{4}$$

By (4)

$$\begin{aligned} \langle x^*, \alpha(a) \odot \alpha^{**}(y^{**})\rangle &= \langle x^* \odot \alpha(a), \alpha^{**}(y^{**})\rangle = \langle \alpha^*(x^* \odot \alpha(a)), y^{**}\rangle \\ &= \langle \alpha^*(x^*) \cdot a, y^{**}\rangle = \langle \alpha^*(x^*), a \cdot y^{**}\rangle \\ &= \langle x^*, \alpha^{**}(a \cdot y^{**})\rangle, \end{aligned}$$

for all $a \in A$, $x^* \in A_T^*$ and $y^{**} \in A^{**}$. Similarly, by (3), $\alpha^{**}(y^{**}) \odot \alpha(a) = \alpha^{**}(y^{**} \cdot a)$, for all $a \in A$ and $y^{**} \in A^{**}$. Then

$$\begin{aligned} \langle x^{**}, \alpha^{***}(\alpha(a) \odot y^{***})\rangle &= \langle \alpha^{**}(x^{**}), \alpha(a) \odot y^{***}\rangle = \langle \alpha^{**}(x^{**}) \odot \alpha(a), y^{***}\rangle \\ &= \langle \alpha^{**}(x^{**} \cdot a), y^{***}\rangle = \langle x^{**} \cdot a, \alpha^{***}(y^{***})\rangle \\ &= \langle x^{**}, a \cdot \alpha^{***}(y^{***})\rangle \end{aligned}$$

and

$$\begin{aligned} \langle x^{**}, \alpha^{***}(y^{***} \odot \alpha(a))\rangle &= \langle \alpha^{**}(x^{**}), y^{***} \odot \alpha(a)\rangle = \langle \alpha(a) \odot \alpha^{**}(x^{**}), y^{***}\rangle \\ &= \langle \alpha^{**}(a \cdot x^{**}), y^{***}\rangle = \langle a \cdot x^{**}, \alpha^{***}(y^{***})\rangle \\ &= \langle x^{**}, \alpha^{***}(y^{***}) \cdot a\rangle, \end{aligned}$$

for all $a \in A$, $x^{**} \in A^{**}$ and $y^{***} \in (A_T)^{***}$. By extending $n \geq 1$, we complete the proof. □

Now, we are ready to investigate our main result in the current section.

Theorem 2.3. *Let $T \in \mathcal{M}(A)$ and be invertible. Then A is n-weakly amenable if and only if A_T is n-weakly amenable.*

Proof. Let A be n-weakly amenable and define $\varphi : A_T \longrightarrow A$ by $\varphi(a) = T(a)$ for all $a \in A_T$. Consider the $(2n)$-th transpose of φ, $\varphi^{(2n)} : A_T^{(2n)} \longrightarrow A^{(2n)}$ that is a bijective linear map. Let $D : A_T \longrightarrow A_T^{(2n)}$ be a continuous derivation. Define $\alpha : A \longrightarrow A_T$ by $\alpha(a) = T^{-1}(a)$ for all $a \in A$. Clearly, $\mathcal{D} := \varphi^{(2n)} \circ D \circ \alpha : A \longrightarrow A^{(2n)}$ is a continuous map and α is a continuous epimorphism. Then Lemma 2.2(ii) implies that

$$\begin{aligned} \mathcal{D}(ab) &= \varphi^{(2n)} \circ D \circ \alpha(ab) = \varphi^{(2n)}(D(\alpha(ab))) \\ &= \varphi^{(2n)}(D(\alpha(a) \bullet \alpha(b))) \\ &= \varphi^{(2n)}(\alpha(a) \odot D(\alpha(b))) + \varphi^{(2n)}(D(\alpha(a)) \odot \alpha(b)) \\ &= a \cdot \varphi^{(2n)}(D(\alpha(b))) + \varphi^{(2n)}(D(\alpha(a))) \cdot b \\ &= a \cdot \mathcal{D}(b) + \mathcal{D}(a) \cdot b, \end{aligned} \tag{5}$$

for all $a, b \in A$. Thus, $\mathcal{D}$ is a continuous derivation. Therefore, there is an element $x^{(2n)} \in A^{(2n)}$ such that $\mathcal{D}(a) = a \cdot x^{(2n)} - x^{(2n)} \cdot a$ for all $a \in A$. Since $\varphi^{(2n)}$ is an epimorphism, there exists $y^{(2n)} \in (A_T)^{(2n)}$ such that $\varphi^{(2n)}(y^{(2n)}) = x^{(2n)}$. Then by Lemma 2.2(ii) we have

$$\begin{aligned} \varphi^{(2n)} \circ D(a) &= \varphi^{(2n)} \circ D \circ \alpha(\varphi(a)) = \varphi^{(2n)} \circ D \circ \alpha(T(a)) \\ &= \mathcal{D}(T(a)) = T(a) \cdot \varphi^{(2n)}(y^{(2n)}) - \varphi^{(2n)}(y^{(2n)}) \cdot T(a) \\ &= \varphi^{(2n)}(\alpha(T(a)) \odot y^{(2n)}) - \varphi^{(2n)}(y^{(2n)} \odot \alpha(T(a))) \\ &= \varphi^{(2n)}(a \odot y^{(2n)} - y^{(2n)} \odot a), \end{aligned}$$

for every $a \in A_T$. This shows that $D(a) = a \odot y^{(2n)} - y^{(2n)} \odot a$, for every $a \in A_T$ (note that $\varphi^{(2n)}$ is injective). Thus, A_T is $(2n)$-weakly amenable. Therefore, it suffices to show that A_T is $(2n+1)$-weakly amenable. Let $D : A_T \longrightarrow A_T^{(2n+1)}$ be a continuous derivation. Consider the mapping $\mathcal{D} := \alpha^{(2n+1)} \circ D \circ \alpha : A \longrightarrow A^{(2n+1)}$. Clearly, it is linear and continuous. Then Lemma 2.2(iii) implies that

$$\begin{aligned}
\mathcal{D}(ab) &= \alpha^{(2n+1)} \circ D \circ \alpha(ab) = \alpha^{(2n+1)}(D(\alpha(ab))) \\
&= \alpha^{(2n+1)}(D(\alpha(a) \bullet \alpha(b))) \\
&= \alpha^{(2n+1)}(\alpha(a) \odot D(\alpha(b))) + \alpha^{(2n+1)}(D(\alpha(a)) \odot \alpha(b)) \\
&= a \cdot \alpha^{(2n+1)}(D(\alpha(b))) + \alpha^{(2n+1)}(D(\alpha(a))) \cdot b \\
&= a \cdot \mathcal{D}(b) + \mathcal{D}(a) \cdot b, \qquad (6)
\end{aligned}$$

for all $a, b \in A$. Thus, $\mathcal{D}$ is a continuous derivation. Thus, there exists $x^{(2n+1)} \in A^{(2n+1)}$ such that $\mathcal{D}(a) = a \cdot x^{(2n)} - x^{(2n)} \cdot a$ for all $a \in A$. Since $\alpha^{(2n+1)}$ is surjective, there exists $y^{(2n+1)} \in (A_T)^{(2n+1)}$ such that $\alpha^{(2n+1)}(y^{(2n+1)}) = x^{(2n+1)}$. Again the item (iii) of Lemma 2.2 implies that

$$\begin{aligned}
\alpha^{(2n+1)} \circ D(a) &= \alpha^{(2n+1)} \circ D \circ \alpha(\varphi(a)) = \alpha^{(2n+1)} \circ D \circ \alpha(T(a)) \\
&= \mathcal{D}(T(a)) = T(a) \cdot \alpha^{(2n+1)}(y^{(2n+1)}) - \alpha^{(2n+1)}(y^{(2n+1)}) \cdot T(a) \\
&= \alpha^{(2n+1)}(\alpha(T(a)) \odot y^{(2n+1)}) - \alpha^{(2n+1)}(y^{(2n+1)} \odot \alpha(T(a))) \\
&= \alpha^{(2n+1)}(a \odot y^{(2n+1)} - y^{(2n+1)} \odot a),
\end{aligned}$$

for every $a \in A_T$. Injectivity of $\alpha^{(2n+1)}$ implies that $D(a) = a \odot y^{(2n+1)} - y^{(2n+1)} \odot a$, for all $a \in A_T$ and this means that A_T is $(2n+1)$-weakly amenable.

For the converse, Lemma 2.1 implies that $T^{-1} \in \mathcal{M}(A)$. Set $B = A_T$ and let $B_{T^{-1}}$ be a Banach algebra is defined by T^{-1} on B. By the argumentation above we get $B_{T^{-1}}$ is n-weakly amenable. But $B_{T^{-1}} = A$ and this means that A is n-weakly amenable. □

Example 2.4. *(i) Let G be a locally compact group and $T \in \mathcal{M}(L^1(G))$ be invertible. By [5], $L^1(G)$ is n-weakly amenable, $n \in \mathbb{N}$, then by Theorem 2.3, $L^1(G)_T$ is n-weakly amenable.*

(ii) A Rees semigroup has the form $S = \mathcal{M}(G, P, m, n)$; here $P = (a_{ij}) \in M_{n,m}(G)$, the collection of $n \times m$ matrices with components G, where G is a group and $m, n \in \mathbb{N}$. We denote the zero adjoined to G by o and by $G^o = G \cup \{o\}$. Let $(x)_{ij}$ be an element of $M_{m,n}(G^o)$ with x in the (i, j)-th place and o elsewhere, where $x \in G$, $1 \le i \le m$ and $1 \le j \le n$. By the following formula S becomes a semigroup $(x)_{ij}(y)_{kl} = (xa_{jk}y)_{il}$, for $x, y \in G, 1 \le i, k \le m$, $1 \le j, l \le n$. The semigroup $\mathcal{M}^o(G, P, m, n)$, where the elements of this semigroup are those of $\mathcal{M}(G, P, m, n)$, together with the element o, identified with the matrix that has o in each place (so that o is the zero of $\mathcal{M}^o(G, P, m, n)$), and the components of P are belong to G^o. The matrix P is called the sandwich matrix in each case. The semigroup $\mathcal{M}^o(G, P, m, n)$ is a Rees matrix semigroup with a zero over G. We write $\mathcal{M}^o(G, P, n)$ for $\mathcal{M}^o(G, P, n, n)$ in the case where $m = n$. As well as, P is called regular if every row and column contains at least one entry in G. The semigroup $\mathcal{M}^o(G, P, m, n)$ is regular as a semigroup if and only if the sandwich matrix P is regular. According to [9] we have

$$\ell^1(S) = \mathcal{M}^o(\ell^1(G), P, m, n) = \mathcal{M}(\ell^1(G), P, m, n) \oplus \mathbb{C}\delta_0.$$

Mewomo in [23], proved that $\ell^1(S)$ is $(2k+1)$-weakly amenable where $S = \mathcal{M}^o(G, P, n)$, for $k, n \in \mathbb{N}$ and it is proved that $\ell^1(S)$ is k-weakly amenable, for all $k \in \mathbb{N}$ [16, Theorem 3.1]. Now, let $T \in \mathcal{M}(\ell^1(S))$ be invertible. Then by Theorem 2.3, $\ell^1(S)_T$ is k-weakly amenable, for all $k \in \mathbb{N}$.

(iii) Let S be a semigroup such that has a zero o. Then S is called a o-simple if $S_{[2]} \neq \{o\}$ and the only ideals in S are $\{o\}$ and S. The semigroup S is called completely o-simple if it is o-simple and contains a primitive idempotent. By [16, Corollary 3.1], an infinite, completely o-simple semigroup S with finitely many idempotents, is n-weakly amenable. Then by Theorem 2.3, $\ell^1(S)_T$ is n-weakly amenable, for all $n \in \mathbb{N}$ and any invertible $T \in \mathcal{M}(\ell^1(S))$.

Let A be a Banach algebra and $D : A \longrightarrow A^*$ be a derivation. Then D is called *cyclic*, if,

$$\langle b, D(a)\rangle + \langle a, D(b)\rangle = 0,$$

for all $a, b \in A$. The Banach algebra A is called *cyclic amenable* (resp. *approximately cyclic amenable*, see [28], for more details) if every cyclic continuous derivation $D : A \longrightarrow A^*$ is inner (resp. approximately inner).

Theorem 2.5. *Let $T \in \mathcal{M}(A)$ and be invertible. Then A is cyclic (resp. approximately cyclic) amenable if and only if A_T is cyclic (resp. approximately cyclic) amenable.*

Proof. We prove the cyclic amenability and the case approximately cyclic amenability is similar. Assume that A is cyclic amenable and define $\alpha : A \longrightarrow A_T$ by $\alpha(a) = T^{-1}(a)$ for all $a \in A$. Clearly, α is a continuous epimorphism. From (3) and (4) we have

$$\langle b, \alpha^*(\alpha(a) \odot a^*)\rangle = \langle b, a \cdot \alpha^*(a^*)\rangle \quad \text{and} \quad \langle b, \alpha^*(a^* \odot \alpha(a))\rangle = \langle b, \alpha^*(a^*) \cdot a\rangle, \tag{7}$$

for all $a, b \in A$ and $a^* \in A_T^*$. Thus (7) implies that

$$\alpha^*(\alpha(a) \odot a^*) = a \cdot \alpha^*(a^*) \quad \text{and} \quad \alpha^*(a^* \odot \alpha(a)) = \alpha^*(a^*) \cdot a, \tag{8}$$

for all $a \in A$ and $a^* \in A_T^*$. Let $D : A_T \longrightarrow A_T^*$ be a continuous cyclic derivation. Define $\mathcal{D} : A \longrightarrow A^*$ by $\mathcal{D}(a) = \alpha^* \circ D \circ \alpha(a)$ for all $a \in A$. Then (8) implies that $\mathcal{D}$ is a continuous derivation. Also,

$$\begin{aligned} \langle b, \mathcal{D}(a)\rangle + \langle a, \mathcal{D}(b)\rangle &= \langle b, \alpha^* \circ D \circ \alpha(a)\rangle + \langle a, \alpha^* \circ D \circ \alpha(b)\rangle \\ &= \langle \alpha(b), D(\alpha(a))\rangle + \langle \alpha(a), D(\alpha(b))\rangle \\ &= 0, \end{aligned}$$

for all $a, b \in A$, because D is a cyclic derivation. This shows that $\mathcal{D}$ is cyclic. Hence, there exists $x^* \in A^*$ such that $\mathcal{D}(a) = a \cdot x^* - x^* \cdot a$ for all $a \in A$. Since, α^* is bijective, there exists $y^* \in A_T^*$ such that $\alpha^*(y^*) = x^*$. Then by (8),

$$\begin{aligned} \alpha^* \circ D(a) &= \alpha^* \circ D \circ \alpha(T(a)) = \mathcal{D}(T(a)) \\ &= T(a) \cdot x^* - x^* \cdot T(a) \\ &= T(a) \cdot \alpha^*(y^*) - \alpha^*(y^*) \cdot T(a) \\ &= \alpha^*(\alpha(T(a)) \odot y^*) - \alpha^*(y^* \odot \alpha(T(a))) \\ &= \alpha^*(a \odot y^* - y^* \odot a), \end{aligned} \tag{9}$$

for all $a \in A_T$. Then (9) implies that $D(a) = a \odot y^* - y^* \odot a$ for all $a \in A_T$, because, α^* is bijective.

Conversely, assume that A_T is cyclic amenable. Lemma 2.1 implies that $T^{-1} \in \mathcal{M}(A)$. Set $B = A_T$ and let $B_{T^{-1}}$ be a Banach algebra is defined by T^{-1} on B. By the discussion above, we get $B_{T^{-1}}$ is cyclic amenable. But $B_{T^{-1}} = A$ and this means that A is cyclic amenable. □

Example 2.6. *Let X be a nonempty set and $\mathbb{F}_X$ be the free semigroup on X, then $\ell^1(\mathbb{F}_X)$ is cyclicly amenable [15]. Then for any invertible $T \in \mathcal{M}(\ell^1(\mathbb{F}_X))$, Theorem 2.5 implies that $\ell^1(\mathbb{F}_X)_T$ is cyclicly amenable.*

3. Biflatness and Biprojectivity

Let A be a Banach algebra and $\Delta_A : A\widehat{\otimes}A \longrightarrow A$ be a multiplication map for the Banach algebra A. Then A is called *biprojective* if Δ_A has a bounded right inverse which is an A-bimodule map. The Banach algebra A is *biflat* if the adjoint $\Delta_A^* : A^* \longrightarrow (A\widehat{\otimes}A)^*$ has a bounded left inverse which is an A-bimodule map.

Lemma 3.1. *Let A be a Banach algebra and $T \in \mathcal{M}_l(A)$. We have the following assertions:*

(i) If $\varphi : A_T \longrightarrow A$ is define by $\varphi(a) = T(a)$ for all $a \in A_T$. Then $(\varphi \otimes \varphi)^* \circ \Delta_A^* = \Delta_{A_T}^* \circ \varphi^*$.

(ii) If T is invertible and $\alpha : A \longrightarrow A_T$ is defined by $\alpha(a) = T^{-1}(a)$ for all $a \in A$. Then $(\alpha \otimes \alpha)^* \circ \Delta^*_{A_T} = \Delta^*_A \circ \alpha^*$.

Proof. (i) For all $x, y \in A_T$ and $a^* \in A^*$,

$$\begin{aligned}\langle x \otimes y, (\varphi \otimes \varphi)^* \circ \Delta^*_A(a^*)\rangle &= \langle(\varphi \otimes \varphi)(x \otimes y), \Delta^*_A(a^*)\rangle \\ &= \langle \Delta_A\left(T(x) \otimes T(y)\right), a^*\rangle = \langle T(x)T(y), a^*\rangle \\ &= \langle T\left(xT(y)\right), a^*\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle x \otimes y, \Delta^*_{A_T} \circ \varphi^*(a^*)\rangle &= \langle \Delta_{A_T}(x \otimes y), \varphi^*(a^*)\rangle = \langle x \bullet y, \varphi^*(a^*)\rangle \\ &= \langle \varphi\left(xT(y)\right), a^*\rangle,\end{aligned}$$

for all $x, y \in A_T$ and $a^* \in A^*$. This shows that (i) holds.

(ii) From Section 2, we know that α is a continuous epimorphism. For all $x, y \in A$ and $a^* \in A^*_T$,

$$\begin{aligned}\langle x \otimes y, (\alpha \otimes \alpha)^* \circ \Delta^*_{A_T}(a^*)\rangle &= \langle(\alpha \otimes \alpha)(x \otimes y), \Delta^*_{A_T}(a^*)\rangle \\ &= \langle \Delta_{A_T}\left(T^{-1}(x) \otimes T^{-1}(y)\right), a^*\rangle = \langle T^{-1}(x) \bullet T^{-1}(y), a^*\rangle \\ &= \langle \alpha(xy), a^*\rangle.\end{aligned}$$

Also,

$$\begin{aligned}\langle x \otimes y, \Delta^*_A \circ \alpha^*(a^*)\rangle &= \langle \Delta_A(x \otimes y), \alpha^*(a^*)\rangle = \langle xy, \alpha^*(a^*)\rangle \\ &= \langle \alpha(xy), a^*\rangle,\end{aligned}$$

for all $x, y \in A$ and $a^* \in A^*_T$. Hence, (ii) holds. □

Theorem 3.2. *Let $T \in \mathcal{M}(A)$ and be invertible. Then A is biflat if and only if A_T is biflat.*

Proof. Define $\varphi : A_T \longrightarrow A$ by $\varphi(a) = T(a)$ and $\alpha : A \longrightarrow A_T$ by $\alpha(x) = T^{-1}(x)$ for all $a \in A_T$ and $x \in A$. We shall show that these maps are A_T-bimodule mappings. Consider A as a Banach A_T-bimodule with the following module actions: $a \cdot x = a \bullet x$ and $x \cdot a = x \bullet a$ for all $a \in A_T$ and $x \in A$. For all $a, b \in A_T$,

$$\begin{aligned}\varphi(a \bullet b) &= T(a \bullet b) = T(aT(b)) = aT(T(b)) = aT(\varphi(b)) \\ &= a \bullet \varphi(b),\end{aligned} \tag{10}$$

and

$$\begin{aligned}\varphi(a \bullet b) &= T(a \bullet b) = T(aT(b)) = T(a)T(b) = \varphi(a)T(b) \\ &= \varphi(a) \bullet b.\end{aligned} \tag{11}$$

Hence, (10) and (11) imply that φ is an A_T-bimodule. For all $a \in A_T$ and $x \in A$ we have

$$\begin{aligned}\alpha(a \bullet x) &= T^{-1}(a \bullet x) = T^{-1}(aT(x)) = aT^{-1}(T(x)) \\ &= ax.\end{aligned} \tag{12}$$

On the other hand,

$$a \bullet \alpha(x) = a \bullet T^{-1}(x) = aT\left(T^{-1}(x)\right) = ax, \tag{13}$$

for all $a \in A_T$ and $x \in A$. Thus, (12) and (13) show that α is a left A_T-module map. Similarly, one can show that α is a right A_T-module map.

Assume that A is biflat, thus there is an A-bimodule map $\lambda_A : \left(A\widehat{\otimes}A\right)^* \longrightarrow A^*$ such that $\lambda_A \circ \Delta_A^* = i_{A^*}$. Define $\lambda_{A_T} : \left(A_T\widehat{\otimes}A_T\right)^* \longrightarrow A_T^*$ by $\lambda_{A_T} := \varphi^* \circ \lambda_A \circ (\alpha \otimes \alpha)^*$ that is an A_T-bimodule map. Then by Lemma 2.1(ii) we have

$$
\begin{aligned}
\lambda_{A_T} \circ \Delta_{A_T}^* &= \varphi^* \circ \lambda_A \circ (\alpha \otimes \alpha)^* \circ \Delta_{A_T}^* \\
&= \varphi^* \circ \lambda_A \circ \Delta_A^* \circ \alpha^* = \varphi^* \circ \alpha^* \\
&= i_{A_T^*}.
\end{aligned}
$$

This shows that A_T is biflat.

Conversely, suppose that A_T is biflat, then there is an A_T-bimodule map $\lambda_{A_T} : \left(A_T\widehat{\otimes}A_T\right)^* \longrightarrow A_T^*$ such that $\lambda_{A_T} \circ \Delta_{A_T}^* = i_{A_T^*}$. For biflatness of A, there are two methods. The first case is similar to the proof of Theorem 2.5 i.e., set $B = A_T$, then $B_{T^{-1}} = A$. This implies that A is biflat. The second case is direct method. In this method we see A_T as a Banach A-bimodule with the left and right actions $a \cdot x = a \bullet x$ and $x \cdot a = x \bullet a$ for all $x \in A_T$ and $a \in A$. Then define an A-bimodule map $\lambda_A : \left(A\widehat{\otimes}A\right)^* \longrightarrow A^*$ by $\lambda_A := \alpha^* \circ \lambda_{A_T} \circ (\varphi \otimes \varphi)^*$ and apply Lemma 2.1(i). □

Theorem 3.3. *Let $T \in \mathcal{M}(A)$ and be invertible. Then A is biprojective if and only if A_T is biprojective.*

Proof. Similar to the previous Theorem, it is suffices to show that if A is biprojective, then A_T is biprojective. Assume that A is biprojective, then there is an A-bimodule map $\rho_A : A \longrightarrow A\widehat{\otimes}A$ such that $\Delta_A \circ \rho_A = i_A$. Define $\rho_{A_T} : A_T \longrightarrow A_T\widehat{\otimes}A_T$ by $\rho_{A_T} := (\alpha \otimes \alpha) \circ \rho_A \circ \varphi$. Then by similar discussions in the proof of Theorem 3.2, ρ_{A_T} is an A_T-bimodule map. Then it is easy to see that $\Delta_{A_T} \circ (\alpha \otimes \alpha) = \alpha \circ \Delta_A$. This implies that $\Delta_{A_T} \circ \rho_{A_T} = i_{A_T}$. □

4. Arens Products and Bounded Approximate Identity

This section deals with the Arens products on the Banach algebras A and A_T. One of the main results in [19] is Theorem 3.5, author, in this result assumed that T is invertible. First, we show that the converse of [19, Theorem 3.5] is true, when T is surjective. Second, we give an example of a Banach algebra such as A_T that is defined by a multiplier T that has not a bounded approximate identity but A has a bounded approximate identity.

Let A be a Banach algebra, let $\square$ and $\Diamond$ be the first and second Arens product on A^{**}, respectively. Let $T \in \mathcal{M}(A)$, similar to A, let $\underline{\square}$ and $\underline{\Diamond}$ be the first and second Arens product on A_T^{**}, respectively.

Theorem 4.1. *Let A be a Banach algebra and $T \in \mathcal{M}_l(A)$ is surjective. If A_T is Arens regular, then A is Arens regular.*

Proof. We investigate the first Arens product and the second Arens product has a similar argument. Let $a^{**}, b^{**} \in A^{**}$. Then there are nets $(a_\alpha), (b_\beta) \subseteq A$ such that $a_\alpha \xrightarrow{w^*} a^{**}$ and $b_\beta \xrightarrow{w^*} b^{**}$. Also, there is a net $(c_\beta) \subseteq A$ such that $T(c_\beta) = b_\beta$ for all β, because T is surjective. Then

$$
\begin{aligned}
a^{**}\square b^{**} &= w^* - \lim_\alpha w^* - \lim_\beta a_\alpha b_\beta = w^* - \lim_\alpha w^* - \lim_\beta a_\alpha T(c_\beta) \\
&= w^* - \lim_\alpha w^* - \lim_\beta a_\alpha \bullet c_\beta = w^* - \lim_\beta w^* - \lim_\alpha a_\alpha \bullet c_\beta \\
&= w^* - \lim_\beta w^* - \lim_\alpha a_\alpha T(c_\beta) = w^* - \lim_\beta w^* - \lim_\alpha a_\alpha b_\beta \\
&= a^{**}\Diamond b^{**},
\end{aligned}
$$

as required. □

Let A, A_T and $T \in M_l(A)$ be as the above. Let $\pi_l : A \times A_T \longrightarrow A_T$ be the left module action such that $\pi_l(a, b) = a \bullet b$ for all $a \in A$ and $b \in A_T$. Similarly, for the right module action, we denote this action by $\pi_r : A_T \times A \longrightarrow A_T$ with $\pi_r(b, a) = b \bullet a$ for all $a \in A$ and $b \in A_T$. It is easy to check that $\pi_l^{***}(a^{**}, b^{**}) = a^{**} \Box T^{**}(b^{**})$ and $\pi_r^{***}(b^{**}, a^{**}) = T^{**}(b^{**}) \Box a^{**}$ for all $a^{**}, b^{**} \in A^{**}$. The maps π_l and π_r are called Arens regular, if $\pi_l^{***} = \pi_l^{t***t}$ and $\pi_r^{***} = \pi_r^{t***t}$, respectively for more details see [2, 3, 10]. As a result of the above Theorem, we have the following:

Corollary 4.2. *Let A be a Banach algebra and $T \in \mathcal{M}_l(A)$.*

(i) If A is Arens regular, then the left and right module actions π_l and π_r are Arens regular.

(ii) If A_T is Arens regular and T is surjective, then the left and right module actions π_l and π_r are Arens regular.

Now, we investigate the second aim of the current section. We begin with the following result:

Lemma 4.3. *Let A be a Banach algebra and $T \in \mathcal{M}(A)$. If $(A_T^{**}, \underline{\Box})$ or $(A_T^{**}, \underline{\Diamond})$ has an identity, then $(A^{**}, \Box)$ or $(A^{**}, \Diamond)$ has an identity.*

Proof. Let E_T be an identity of $(A_T^{**}, \underline{\Box})$. Then

$$a \Box T^{**}(E_T) = a \underline{\Box} E_T = a, \tag{14}$$

and

$$T^{**}(E_T) \Box a = T^{**}(E_T \Box a) = E_T \Box T^{**}(a) = E_T \underline{\Box} a = a \tag{15}$$

for all $a \in A^{**}$. Thus, $T^{**}(E_T)$ is an identity for $(A^{**}, \Box)$. Similarly, one can show that $(A^{**}, \Diamond)$ has an identity, when $(A_T^{**}, \underline{\Diamond})$ has an identity. $\Box$

Let G be a locally compact group and $M(G)$ be the space of bounded regular Borel measures with the following norm:

$$\|\mu\| = \int_G d|\mu| = |\mu|(G) \qquad (\mu \in M(G)).$$

The Banach algebra $L^1(G)$ is a two-sided ideals of $M(G)$, consisting of all absolutely continuous measures with respect to a left Haar measure. We denote the space of all $f \in L^1(G)$ such that $f \geq 0$ by $L^1(G)^+$. Let $T \in \mathcal{M}(L^1(G))$, Wendel proved that there is a unique regular (real or complex) measure μ of bounded variation such that T is given by $T(f) = \mu * f$, for all $f \in L^1(G)$ and $\|T\| = \|\mu\|$ [31, Theorem 1]. It is well-known that $L^1(G)$ has a bounded approximate identity. In [19] Laali studied the Arens regularity of $L^1(G)_T$, where $T : M(G) \longrightarrow M(G)$ is a left multiplier.

Theorem 4.4. *Let G be an infinite compact group and $0 \neq \eta \in L^1(G)^+$ with compact support. If $T : M(G) \longrightarrow M(G)$ is defined by $T = L_\eta$, then $L^1(G)_T$ has not a bounded approximate identity.*

Proof. Assume towards a contradiction that $L^1(G)_T$ has a bounded approximate identity. Since G is compact, $L^1(G)_T$ is Arens regular [19, Theorem 4.4]. Thus $L^1(G)_T^*$ factors (on the left and right) [30, Theorem 3.1] and this implies that $L^1(G)_T^{**}$ has an identity [22, Proposition 2.2]. By Lemma 4.3, $L^1(G)^{**}$ has an identity. Again by [22, Proposition 2.2], $L^1(G)^*$ factors. This follows that $L^1(G)$ is unital and consequentially G is discrete. This means that G is finite, a contradiction. $\Box$

The above Theorem gives some cohomological results related to the existence of bounded approximate identities as follows:

Corollary 4.5. *Let G be an infinite compact group and $0 \neq \eta \in L^1(G)^+$ with compact support. If $T : M(G) \longrightarrow M(G)$ is defined by $T = L_\eta$, then $L^1(G)_T$*

(i) is not amenable.

(ii) is not contractible.

(iii) is not biflat.

Let $\mathcal{A}$ be Banach algebra, I and J be arbitrary nonempty index sets and P be a $I \times J$ matrix over $\mathcal{A}$ such that $\|P\|_\infty = \sup\{\|P_{ji}\| : j \in J, i \in I\} \leq 1$. The set $\mathfrak{L}\mathcal{M}(\mathcal{A}, P)$ of $I \times J$ matrices A over $\mathcal{A}$ such that $\|A\|_1 = \sum_{i \in I, j \in J} \|A_{ij}\| < \infty$ with ℓ^1-norm and product $A \circ B = APB$ for all $A, B \in \mathfrak{L}\mathcal{M}(\mathcal{A}, P)$ is a Banach algebra. These Banach algebras are called ℓ^1-Munn algebras that they are widely considered by Esslamzadeh in [11]. Let $T \in \mathcal{M}(\mathfrak{L}\mathcal{M}(\mathcal{A}, P))$, then there exists $B = [T_{ij}] \in \ell^\infty(I \times J, \mathcal{M}(\mathcal{A}))$ such that

$$T(A) = [T_{ij}(A_{ij})] = B \odot A = A \odot B,$$

for every $A \in \mathfrak{L}\mathcal{M}(\mathcal{A}, P)$ [14, Theorem 3.4]. Now, let $\mathcal{T} \in \mathcal{M}(A)$, then $T = [T_{ij} = \mathcal{T}]$ is in $\mathcal{M}(\mathfrak{L}\mathcal{M}(\mathcal{A}, P))$. Thus, for such multipliers, we have

$$\mathfrak{L}\mathcal{M}(\mathcal{A}, P)_T = \mathfrak{L}\mathcal{M}(\mathcal{A}_{\mathcal{T}}, P). \tag{16}$$

Let S be a semigroup and E_S be the set of all idempotent elements of S. If T is an ideal of S, then the Rees factor semigroup S/T is the result of collapsing T into a single element 0 and retaining the identity of elements of $S \backslash T$, also we suppose that $S/\emptyset = S$. If S has an identity, then $S^1 = S$ otherwise $S^1 = S \cup \{1\}$ where 1 is the identity joined to S. An ideal series $S = S_1 \supset S_2 \supset \cdots \supset S_m \supset S_{m+1} = \emptyset$ that has no proper refinement is called a principal series. A semigroup S is called regular, if for every $a \in S$, there exists $b \in S$ such that $a = aba$ and S is called an inverse semigroup if for every $a \in S$ there is a unique $a^* \in S$ such that $aa^*a = a$ and $a^*aa^* = a^*$.

Let G be a group, I and J be arbitrary nonempty sets, and $G^o = G \cup \{o\}$ be the group with zero arising from G by adjunction of a zero element. An $I \times J$ matrix A over G^o that has at most one nonzero entry $a = A(i, j)$ is called a Rees $I \times J$ matrix over G^o and is denoted by $(a)_{ij}$. Let P be a $J \times I$ matrix over G. The set $S = G \times I \times J$ with the composition $(a, i, j) \circ (b, l, k) = (aP_{jl}b, i, k), (a, i, j), (b, l, k) \in S$ is a semigroup that we denote by $M(G, P)$. Similarly, if P is a $J \times I$ matrix over G^o, then $S = G \times I \times J \cup \{0\}$ is a semigroup under the following composition operation which is denoted by $\mathcal{M}^o(G, P)$:

$$(a, i, j) \circ (b, l, k) = \begin{cases} (aP_{jl}b, i, k) & P_{jl} \neq 0 \\ 0 & P_{jl} = 0, \end{cases}$$

$$(a, i, j) \circ o = o \circ (a, i, j) = o \circ o = o.$$

An $I \times J$ matrix P over G^o is called regular (invertible) if every row and every column of P contains at least (exactly) one nonzero entry.

If S is a regular semigroup such that E_S is finite, then S has a principal series $S = S_1 \supset S_2 \supset \cdots \supset S_m \supset S_{m+1} = \emptyset$. Moreover, for all $k = 1, ..., m-1$, there are natural numbers n_k, l_k, a group G_k and a regular $l_k \times n_k$ matrix P_k on G_k^o such that $S_k/S_{k+1} = \mathcal{M}^o(G_k, P_k)$. Also, $S_m = \mathcal{M}(G_m, P_m)$ for some $l_m \times n_m$ matrix P_m over a group G_m [12, Lemma 5.2]. If $S = M^o(G, P)$ and the zero of G^o is identified with the zero of the ℓ^1-Munn algebra $\mathfrak{L}\mathcal{M}(\ell^1(G), P)$, where P is considered as a matrix over $\ell^1(G)$, then $\ell^1(S)/\ell^1(o)$ is isometrically algebra isomorphic to $\mathfrak{L}\mathcal{M}(\ell^1(G), P)$ [12, Proposition 5.6].

Theorem 4.6. *Let S be a regular semigroup with a finite number of idempotents, and $T \in \mathcal{M}(\ell^1(S))$ (defined as above) such that $T_{i,j} = L_\eta$ where $0 \neq \eta \in \ell^1(G)^+$. If $\ell^1(S)_T$ is Arens regular, then S is compact.*

Proof. Suppose that $\ell^1(S)_T$ is Arens regular, then by [25, Corollary 1.4.12], any quotient of $\ell^1(S)_T$ is Arens regular. By the discussion above (i.e. [12, Proposition 5.6]) and (16), $\mathfrak{L}\mathcal{M}(\ell^1(G_k)_T, P_k)$ is Arens regular. Then [13, Theorem 4.2(ii)] implies that $\ell^1(G_k)_T$ is Arens regular for $k = 1, 2, \ldots, m$. By [19, Theorem 4.4] G_k is compact for $k = 1, 2, \ldots, m$. Since S has a principal series, the factors of this series are isomorphic in some order to the principal factors of S [6, Theorem 2.40], so, each principal factor of S is compact. This means that S is compact. □

5. Problems

We close this paper with the following problems:

1. Let A be a commutative Banach algebra and $\Gamma : A \longrightarrow C_0(\Delta(A))$ the Gelfand representation of A. A subset R of $\Delta(A)$ (character space of A) is called a boundary for A if R is boundary for $\Gamma(A)$, the range of the Gelfand homomorphism. In particular $\partial(\Gamma(A))$ is called the Shilov boundary of A and denoted by $\partial(A)$ [18, Definition 3.3.6]. Now, let $T \in \mathcal{M}(A)$. From, [21, Theorem 1.3.1 and Corollary 1.3.1], $\Delta(A) = \Delta(A_T)$. Is there a relationship between Shilov boundary of A and A_T?
2. A submultiplicative (not necessarily complete) norm $|\cdot|$ on a Banach algebra A is called uniform norm if it satisfies the square property $|a^2| = |a|^2$, for all $a \in A$. If A has the unique uniform norm property and $T \in \mathcal{M}(A)$, has A_T the unique uniform norm property and vice versa?
3. The BSE-property (Bochner-Schoeberg-Eberlein property) on commutative Banach algebras is a property that is related to multiplier algebras and character space of Banach algebras. This notion was introduced by Takahasi and Hatori in [29]. Let $T \in \mathcal{M}(A)$. Are there any relationships between being BSE-algebra of A and A_T?
4. A commutative Banach algebra A is called a Tauberian Banach algebra if the set of all $a \in A$ such that $\hat{a}$ has compact support is dense in A, see [18, Definition 5.1.7]. If A is a Tauberian Banach algebra and $T \in \mathcal{M}(A)$, is A_T Tauberian? What about the converse of this question?
5. Is there a relationship between projectivity (flatness, injectivity) of A and A_T?
6. Let A be a Banach algebra and X be a Banach A-bimodule. Consider the module extensions Banach algebra $A \oplus_1 X$. If $T \in \mathcal{M}(A \oplus_1 X)$, how we can characterize $(A \oplus_1 X)_T$? The multiplies algebra of $A \oplus_1 X$ in commutative case is characterized in [1].

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