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A Large Class in Köthe-Toeplitz Duals of Generalized Cesàro Difference Sequence Spaces with Fixed Point Property for Nonexpansive Mappings

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Abstract. In 1970, Cesàro sequence spaces was introduced by Shiue. In 1981, Kızmaz defined difference sequence spaces for ℓ^∞, c_0 and c. Then, in 1983, Orhan introduced Cesàro difference sequence spaces. Both works used difference operator and investigated Köthe-Toeplitz duals for the new Banach spaces they introduced. Later, various authors generalized these new spaces, especially the one introduced by Orhan. In this study, first we discuss the fixed point property for these spaces. Then, we recall that Goebel and Kuczumow showed that there exists a very large class of closed, bounded, convex subsets in Banach space of absolutely summable scalar sequences, ℓ^1 with fixed point property for nonexpansive mappings. So we consider a Goebel and Kuczumow analogue result for a Köthe-Toeplitz dual of a generalized Cesàro difference sequence space. We show that there exists a large class of closed, bounded and convex subsets of these spaces with fixed point property for nonexpansive mappings.

1. Introduction and Preliminaries

There is a strong relation between reflexivity and fixed point property for nonexpansive mappings. It is an open question whether or not every non-reflexive fails the fixed point property for nonexpansive mappings but it was shown by Lin [19] that a non-reflexive Banach space failing to have the fixed point property for nonexpansive mappings can be renormed to have the fixed point property for nonexpansive mappings. Lin [19] showed this fact by setting an equivalent norm on Banach space of absolutely summable scalar sequences, ℓ^1. Because of sharing many common properties, it is natural to ask if , Banach space of scalar sequences converging to 0, c_0 can be renormed to have the fixed point property for nonexpansive mappings as another well known classical non-reflexive Banach space. Hernández-Linares and Japón [20] obtained the first example for the class of nonreflexive Banach spaces which can be renormed to have the fixed point property for affine nonexpansive mappins and their space was the Banach space of Lebesgue integrable functions on [0,1], $L_1[0,1]$. It can be said that all these works are inspired by the work of

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Goebel and Kuczumow [15]. Goebel and Kuczumow [15] showed that there exists very large class of non-weakly compact, closed, bounded and convex subsets of ℓ^1 respect to weak* topology of ℓ^1 with fixed point property for nonexpansive mappings. Previously, Everest, in his Ph.D. thesis [12], written under supervision of Chris Lennard, considered large classes in ℓ^1 with fixed point property for nonexpansive mappings by generalizing Goebel and Kuczumow's work.

In this study, we work on Goebel and Kuczumow [15] analogy for a Banach space contained in ℓ^1. The space we consider is a Köthe-Toeplitz dual of a generalized Cesàro difference sequence space. We show that there exists a very large class of closed, bounded and convex subsets of the space with the fixed point property for nonexpansive mappings.

We recall that the Cesàro sequence spaces

$$\mathrm{ces}_p = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

were introduced by Shiue [25] in 1970, where $1 \leq p < \infty$. It has been shown that $\ell^p \subset \mathrm{ces}_p$ for $1 < p \leq \infty$. Moreover, it has been shown that Cesàro sequence spaces ces_p for $1 < p < \infty$ are separable reflexive Banach spaces. Furthermore, it was also proved by Cui and Hudzik [7], Cui, Hudzik and Li [8] and Cui, Meng and Pluciennik [9] that Cesàro sequence spaces ces_p for $1 < p < \infty$ have the fixed point property. They prove this result using different methods. One method is to calculate Garcia-Falset coefficient. It is known that if Garcia-Falset coefficient is less than 2 for a Banach space, then it has the fixed point property for nonexpansive mappings [13]. Using this fact, since they calculate this coefficient for ces_p as $2^{1/p}$ similarly to what it is for ℓ^p, they point the result for the Cesàro sequence spaces. Another fact is that they see that the space has normal structure for $1 < p < \infty$. Then using the fact via Kirk [17] that reflexive Banach spaces with normal structure has the fixed point property, they easily deduce that the space has the fixed point property for $1 < p < \infty$. Their results on Cesàro sequence spaces as a survey can be seen in [6]. Moreover, the books ([1], [2], [4]) contain several useful results concerning various summable spaces for further study in the spirit of this paper.

Later, in 1981, Kızmaz [16] introduced difference sequence spaces for ℓ^∞, c and c_0 where they are the Banach spaces of bounded, convergent and null sequences $x = (x_n)_n$, respectively. As it is seen below, his definitions for these spaces were given using difference operator applied to the sequence x, $\triangle\, x = (x_k - x_{k+1})_k$.

$$\ell^\infty(\triangle) = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \triangle\, x \in \ell^\infty \right\},$$

$$\mathrm{c}(\triangle) = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \triangle\, x \in \mathrm{c} \right\},$$

$$\mathrm{c}_0(\triangle) = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \triangle\, x \in \mathrm{c}_0 \right\}.$$

Kızmaz [16] investigated Köthe-Toeplitz duals and some properties of these spaces.

Furthermore, Cesàro sequence spaces X^p of non-absolute type were defined by Ng and Lee [22] in 1977 as follows:

$$X^p = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$X^\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| < \infty \right\},$$

where $1 \leq p < \infty$. They prove that X^p is linearly isomorphic and isometric to ℓ^p for $1 \leq p \leq \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Later, in 1983, Orhan [23] introduced Cesàro difference sequence spaces by the following definitions:

$$C_p = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle\, x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$C_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle\, x_k \right| < \infty \right\},$$

where $1 \leq p < \infty$. He noted that their norms are given as below for any $x = (x_n)_n$:

$$\|x\|_p^* = |x_1| + \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle\, x_k \right|^p \right)^{1/p} \quad \text{and} \quad \|x\|_\infty^* = |x_1| + \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle\, x_k \right|$$

respectively.

Orhan [23] showed that there exists a linear bounded operator $S : C_p \to C_p$ for $1 \leq p \leq \infty$ such that Köthe-Toeplitz $\beta-$duals of these spaces are given respectively as follows:

$$S(C_p)^\beta = \left\{ a = (a_n)_n \subset \mathbb{R} \;\middle|\; (na_n)_n \in \ell^q \right\} \text{ where } 1 < p < \infty \text{ and } q = \frac{p}{p-1},$$

$$S(C_1)^\beta = \left\{ a = (a_n)_n \subset \mathbb{R} \;\middle|\; (na_n)_n \in \ell^\infty \right\} \text{ and}$$

$$S(C_\infty)^\beta = \left\{ a = (a_n)_n \subset \mathbb{R} \;\middle|\; (na_n)_n \in \ell^1 \right\}.$$

It might be better to use the notation $X^p(\triangle)$ instead of C_p for $1 \leq p \leq \infty$ since we also recalled the difference sequence spaces and used similar type of notation.

We note that Orhan [23] also proved that $X^p \subset X^p(\triangle)$ for $1 \leq p \leq \infty$ strictly. Also, one can clearly see that $X^p(\triangle)$ is linearly isomorphic and isometric to ℓ^p for $1 \leq p \leq \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Note also that Köthe-Toeplitz dual for $p = \infty$ case in Orhan's study [23] and ℓ^∞ case in Kızmaz study [16] coincides.

Furthermore, Et and Çolak [10] generalized the spaces introduced in Kızmaz's work [16] in the following way for $m \in \mathbb{N}$.

$$\ell^\infty(\triangle^m) = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \triangle^m x \in \ell^\infty \right\},$$

$$\mathrm{c}(\triangle^m) = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \triangle^m x \in \mathrm{c} \right\},$$

$$\mathrm{c}_0(\triangle^m) = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \triangle^m x \in \mathrm{c}_0 \right\}$$

where $\triangle\, x = (\triangle\, x_k) = (x_k - x_{k+1})_k$, $\triangle^0 x = (x_k)_k$, $\triangle^m x = (\triangle^m x_k) = \left(\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1}\right)_k$ and $\triangle^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}$.

Also, Et [11] and Tripathy et. al. [26] generalized the space introduced by Orhan [23] in the following way for $m \in \mathbb{N}$.

$$X^p(\triangle^m) = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle^m x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$X^{\infty}\ (\Delta^m) = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m x_k \right| < \infty \right\},$$

Then, it is seen that that Köthe-Toeplitz dual for $p = \infty$ case in Et's study [11] and ℓ^∞ case in Et and Çolak study [10] coincides such that Köthe-Toeplitz dual was given as below for any $m \in \mathbb{N}$.

$$D_m := \left\{ a = (a_n)_n \subset \mathbb{R} \,\middle|\, (n^m a_n)_n \in \ell^1 \right\}$$

$$= \left\{ a = (a_k)_k \subset \mathbb{R} :\ \|a\| = \sum_{k=1}^{\infty} k^m |a_k| < \infty \right\}.$$

Note that $D_m \subset \ell^1$ for any $m \in \mathbb{N}$. Moreover, there are other difference operators which can also be used to construct more generalized difference sequence spaces (see, for example, [3], [5], [18]).

One can see that corresponding function space for these duals can be given as below:

$$U_m := \left\{ \text{Lebesgue measurable functions } f \text{ on I} = [0,1] :\ \|f\| = \int_0^1 t^m \left| f(t) \right| dt < \infty \right\}.$$

Note that $L_1[0,1] \subset U_m$ and D_m is the space when counting measure is used for U_m.

As we have already stated, in this study, we consider Goebel and Kuczumow [15] analogy for a Köthe-Toeplitz dual of a generalized Cesàro difference sequence space. We show that for any $m \in \mathbb{N}$ there exists a large class of closed, bounded and convex subsets of Köthe-Toeplitz dual for $X^{\infty}(\Delta^m)$ with fixed point property for nonexpansive mappings.

Now we provide some preliminaries before giving our main results.

Definition 1.1. Let (X,$\|\cdot\|$) be a Banach space and C is a non-empty closed, bounded, convex subset.

1. If $T : C \to C$ is a mapping such that for all $\lambda \in [0,1]$ and for all $x, y \in C$, $T((1-\lambda)x + \lambda y) = (1-\lambda)T(x) + \lambda T(y)$ then T is said to be an affine mapping.

2. If $T : C \to C$ is a mapping such that $\|T(x) - T(y)\| \le \|x - y\|$, for all $x, y \in C$ then T is said to be a nonexpansive mapping.

Also, if for every nonexpansive mapping $T : C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for nonexpansive mappings [fpp(ne)].

In 1979, Goebel and Kuczumow [15] showed there exists a large class of closed, bounded and convex subsets of ℓ^1 using a key lemma they obtained. Their lemma says that if $\{x_n\}$ is a sequence in ℓ^1 converging to x in weak-star topology, then for any $y \in \ell^1$,

$$r(y) = r(x) + \|y - x\|_1 \ \text{ where } \ r(y) = \limsup_n \|x_n - y\|_1 \ .$$

Since Köthe-Toeplitz dual for $X^{\infty}(\Delta^m)$ is contained in ℓ^1 for any $m \in \mathbb{N}$ and in fact it is isometrically isomorphic to ℓ^1, for any $m \in \mathbb{N}$, Goebel and Kuczumow's lemma above (Lemma 1 in [15]) applies in Köthe-Toeplitz dual for $X^{\infty}(\Delta^m)$. We will call this fact ♡.

2. Main Result

In this section, we consider Goebel and Kuczumow [15] analogy for a Köthe-Toeplitz dual of a generalized Cesàro difference sequence space. We show that for any $m \in \mathbb{N}$ there exists a large class of closed, bounded and convex subsets of Köthe-Toeplitz dual for $X^{\infty}\ (\Delta^m)$ with fixed point property for nonexpansive mappings. We note that case $m = 1$ has recently been done by Nezir and Cankurt [21]. As we stated, here we present the general case for any $m \in \mathbb{N}$.

Now, we consider the following class of closed, bounded and convex subsets. Note that here we will be using the ideas similar to those in the section 2 of [12], where Everest firstly provides Goebel and Kuczumow's proofs in detailed.

Example 2.1. Fix $m \in \mathbb{N}$ and $b \in (0,1)$. Let Q be an integer larger than 3. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1$, $f_2 := \frac{b\, e_2}{2^m}$, $f_3 := \frac{b\, e_3}{3^m}$, $\dots$, $f_Q := \frac{b\, e_Q}{Q^m}$, and $f_n := \frac{1}{n^m} e_n$ for all integers $n \geq Q+1$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Next, we can define a closed, bounded, convex subset $E^{(m)} = E_b{}^{(m)}$ of the Köthe-Toeplitz dual for X^∞ (Δ^m) for arbitrary $m \in \mathbb{N}$ by

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : \ \forall n \in \mathbb{N},\ t_n \geq 0 \ \ and \ \ \sum_{n=1}^{\infty} t_n = 1 \right\} .$$

Theorem 2.2. *For any $m \in \mathbb{N}$ and $b \in (0,1)$, the set $E^{(m)} \subset D_m$ defined as in the example above has the fixed point property for $\|.\|$-nonexpansive mappings.*

Proof. Fix $m \in \mathbb{N}$ and $b \in (0,1)$. Let $T{:}E^{(m)} \to E^{(m)}$ be a nonexpansive mapping. Then, it is well known that there exists an approximate fixed point sequence $\left(x^{(n)}\right)_{n\in\mathbb{N}} \subset E^{(m)}$ such that $\left\|Tx^{(n)} - x^{(n)}\right\| \underset{n}{\to} 0$. (See, for example, Lemma 3.1 on page 28 of [14].) Without loss of generality, passing to a subsequence if necessary, there exists $x \in D_m$ such that $x^{(n)}$ converges to x in weak* topology. Then, by Goebel Kuczumow analog fact ♡ given in the last part of the previous section, we can define a function $s{:}D_m \to [0,\infty)$ by

$$s(y) = \limsup_n \left\|x^{(n)} - y\right\| \quad , \ \forall y \in D_m$$

and so

$$s(y) = s(x) + \left\|x - y\right\| \ , \ \forall y \in D_m .$$

Now define the weak* closure of the set $E^{(m)}$ as it is seen below.

$$W := \overline{E^{(m)}}^{w^*} = \left\{ \sum_{n=1}^{\infty} t_n\, f_n : \ \ each \ \ t_n \geq 0 \ \ and \ \ \sum_{n=1}^{\infty} t_n \leq 1 \right\}$$

First of all, recall that since T is nonexpansive mapping, $\forall x, y \in E^{(m)}$,

$$\left\|Tx - Ty\right\| \leq \|x - y\|.$$

We will consider two cases.

Case 1: $x \in E^{(m)}$.

Then, $\forall n \in \mathbb{N}$, we have $s(Tx) = s(x) + \|Tx - x\|$ and

$$\begin{aligned} s(Tx) &= \limsup_n \left\|Tx - x^{(n)}\right\| \qquad (1) \\ &\leq \limsup_n \left\|Tx - T\left(x^{(n)}\right)\right\| + \limsup_n \left\|x^{(n)} - T\left(x^{(n)}\right)\right\| \\ &\leq \limsup_n \left\|x - x^{(n)}\right\| + 0 \\ &= s(x). \end{aligned}$$

Therefore, $s(Tx) = s(x) + \|Tx - x\| \leq s(x)$ and so $\|Tx - x\| = 0$. Thus, $Tx = x$.

Case 2: $x \in W \setminus E^{(m)}$.

Then, x is of the form $\sum_{n=1}^{\infty} \gamma_n f_n$ such that $\sum_{n=1}^{\infty} \gamma_n < 1$ and $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$.

Define $\delta := 1 - \sum_{n=1}^{\infty} \gamma_n$ and for $\alpha \in \left[\frac{-\gamma_1}{\delta}, \frac{\gamma_2}{\delta} + 1\right]$ define

$$\begin{aligned} \mathrm{h}_{\alpha} := & \left(\gamma_1 + \frac{\alpha}{Q-1}\delta\right) f_1 + \left(\gamma_2 + \frac{\alpha}{Q-1}\delta\right) f_2 + \left(\gamma_3 + \frac{\alpha}{Q-1}\delta\right) f_3 + \ldots + \left(\gamma_{Q-1} + \frac{\alpha}{Q-1}\delta\right) f_{Q-1} \\ & + (\gamma_Q + (1-\alpha)\delta) f_Q + \sum_{n=Q+1}^{\infty} \gamma_n f_n. \end{aligned}$$

Then,

$$\begin{aligned} \|\mathrm{h}_{\alpha} - x\| &= \left\| \frac{\alpha}{Q-1} b\delta e_1 + \frac{\alpha}{Q-1} b\delta \frac{e_2}{2^m} + \frac{\alpha}{Q-1} b\delta \frac{e_3}{3^m} + \cdots + \frac{\alpha}{Q-1} b\delta \frac{e_{Q-1}}{(Q-1)^m} + (1-\alpha) b\delta \frac{e_Q}{Q^m} \right\| \\ &= \underbrace{\mathrm{b}\left|\frac{\alpha}{Q-1}\right|\delta + \mathrm{b}\left|\frac{\alpha}{Q-1}\right|\delta + +\mathrm{b}\left|\frac{\alpha}{Q-1}\right|\delta + \cdots + +\mathrm{b}\left|\frac{\alpha}{Q-1}\right|\delta}_{Q-1 \text{ times}} + b|1-\alpha|\delta \\ &= b|\alpha|\delta + b|1-\alpha|\delta. \end{aligned}$$

Therefore, $\|\mathrm{h}_{\alpha} - x\|$ is minimized for $\alpha \in [0,1]$ and its minimum value would be $b\delta$.
Now fix $y \in E^{(m)}$ of the form $\sum_{n=1}^{\infty} t_n f_n$ such that $\sum_{n=1}^{\infty} t_n = 1$ with $t_n \geq 0$, $\forall n \in \mathbb{N}$.
Then,

$$\begin{aligned} \left\|\mathrm{y} - x\right\| &= \left\| \sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k \right\| \\ &= b\left|t_1 - \gamma_1\right| + b\left|t_2 - \gamma_2\right| + b\left|t_3 - \gamma_3\right| + \cdots + b\left|t_{Q-1} - \gamma_{Q-1}\right| + \sum_{k=Q}^{\infty} \left|t_k - \gamma_k\right| \\ &= b\left|t_1 - \gamma_1\right| + b\left|t_2 - \gamma_2\right| + b\left|t_3 - \gamma_3\right| + \cdots + b\left|t_{Q-1} - \gamma_{Q-1}\right| + b\sum_{k=Q}^{\infty} \left|t_k - \gamma_k\right| + (1-b)\sum_{k=Q}^{\infty} \left|t_k - \gamma_k\right| \\ &\geq \mathrm{b}\left| \sum_{k=1}^{\infty} (t_k - \gamma_k) \right| + (1-b)\sum_{k=Q}^{\infty} \left|t_k - \gamma_k\right| \\ &= b\left| \sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \gamma_k \right| + (1-b)\sum_{k=Q}^{\infty} \left|t_k - \gamma_k\right| \\ &= b|1 - (1-\delta)| + (1-b)\sum_{k=Q}^{\infty} \left|t_k - \gamma_k\right| \end{aligned}$$

Hence,

$$\left\|\mathrm{y} - x\right\| \geq b\delta + (1-b)\sum_{k=4}^{\infty} \left|t_k - \gamma_k\right| \geq b\delta$$

and we have the equality if and only if $(1-b)\sum_{k=3}^{\infty} \left|t_k - \gamma_k\right| = 0$ which means we have $\left\|\mathrm{y} - x\right\| = b\delta$ if and only if $t_k = \gamma_k$ for every $k \geq 3$; or say, $\left\|\mathrm{y} - x\right\| = b\delta$ if and only if $\mathrm{y} = \mathrm{h}_{\alpha}$ for some $\alpha \in [0,1]$.

Now, define $\Lambda := \{\mathrm{h}_{\alpha} \ : \ \alpha \in [0,1]\}$. Clearly, Λ is the contiunous image of a compact set and so it is a compact subset of $E^{(m)}$. It is also easy to see that it is convex.

Now for any $h \in \Lambda$, since $\left\|\mathrm{y} - x\right\|$ achieves its minimum value at $\mathrm{y} = \mathrm{h}_{\alpha}$, firstly we have

$$
\begin{aligned}
s(h) = s(x) + \|\mathrm{h}-x\| &\leq s(x) + \|T\mathrm{h}-x\| \\
&= s(T\mathrm{h}) \text{ but this follows} \\
&= \limsup_n \left\|Th - x^{(n)}\right\| \text{ then similarly to the inequality (1)} \\
&\leq \limsup_n \left\|Th - T\left(x^{(n)}\right)\right\| + \limsup_n \left\|x^{(n)} - T\left(x^{(n)}\right)\right\| \\
&\leq \limsup_n \left\|h - x^{(n)}\right\| + \limsup_n \left\|x^{(n)} - T\left(x^{(n)}\right)\right\| \\
&\leq \limsup_n \left\|h - x^{(n)}\right\| + 0 \\
&= s(h).
\end{aligned}
$$

Hence, $s(h) \leq s(T\mathrm{h}) \leq s(h)$ and so $s(T\mathrm{h}) = s(h)$. Thus, $s(x) + \|T\mathrm{h}-x\| = s(x) + \|\mathrm{h}-x\|$.

Therefore, $\|T\mathrm{h}-x\| = \|\mathrm{h}-x\|$ and so $Th \in \Lambda$ but this means $T(\Lambda) \subseteq \Lambda$ and since T is continuous, Schauder's fixed point theorem [24] tells us that T has a fixed point such that h is the unique minimizer of $\left\|\mathrm{y}-x\right\| \ : \ y \in E^{(m)}$ and $Th = h$.

Thus, $E^{(m)}$ has fpp(ne) as desired. □

Now, we can give two quick corollaries below by taking $Q = 2$ and next $Q = 3$ in the previous theorem, respectively.

Corollary 2.3. *Fix $m \in \mathbb{N}$ and $b \in (0,1)$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1$, $f_2 := \frac{b\, e_2}{2^m}$, and $f_n := \frac{1}{n^m} e_n$ for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Next, define a closed, bounded, convex subset $E^{(m)} = E_b{}^{(m)}$ of the Köthe-Toeplitz dual for $X^\infty(\Delta^m)$ for arbitrary $m \in \mathbb{N}$ by*

$$
E^{(m)} := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : \ \forall n \in \mathbb{N},\ t_n \geq 0 \ \text{and} \ \sum_{n=1}^{\infty} t_n = 1 \right\} .
$$

Then, the set $E^{(m)} \subset D_m$ has the fixed point property for $\|.\|$-nonexpansive mappings.

Corollary 2.4. *Fix $m \in \mathbb{N}$ and $b \in (0,1)$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b\, e_1$, $f_2 := \frac{b\, e_2}{2^m}$, $f_3 := \frac{b\, e_3}{3^m}$ and $f_n := \frac{1}{n^m} e_n$ for all integers $n \geq 4$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Next, define a closed, bounded, convex subset $E^{(m)} = E_b{}^{(m)}$ of the Köthe-Toeplitz dual for $X^\infty(\Delta^m)$ for arbitrary $m \in \mathbb{N}$ by*

$$
E^{(m)} := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : \ \forall n \in \mathbb{N},\ t_n \geq 0 \ \text{and} \ \sum_{n=1}^{\infty} t_n = 1 \right\} .
$$

Then, the set $E^{(m)} \subset D_m$ has the fixed point property for $\|.\|$-nonexpansive mappings.

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