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# Existence of Solution for a Singular Fractional Boundary Value Problem of Kirchhoff Type

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**Abstract.** In this work, we investigate the existence of solution for some nonlinear singular problem of Kirchhoff type involving Riemann-Liouville Fractional Derivative and the $p$-Laplacian operator. The main tools are based on the variational method, precisely, we use the minimisation of the corresponding functional in a suitable fractional spaces. Our main result significantly complement and improves the previous ones due to [6] and [31] .

## 1. Introduction

In recent decades, fractional calculus have been investigated extensively. This is due to its importance and applications in many fields such as physics, aerodynamics, chemistry, electro dynamics of complex medium (see [17, 22, 27, 29]). Among all these subjects, there has been significant development boundary value problems involving different fractional operators. For details and examples, one can see the papers [12–14, 18, 19, 21, 24, 28] and references therein.
By using the mountain pass theorem, Torres [31] proved the existence of at least one nontrivial solution for the following problem

$$\begin{cases} -{}_tD_1^{\alpha}\ {}_0D_t^{\alpha}u(t) = f(t,u(t)),\ t \in (0,T) \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

where ${}_tD_1^{\alpha}$ and ${}_0D_t^{\alpha}$ are the right and left Riemann Liouville fractional derivatives. Note that, using the varitional aproach, the first paper studying such problem is the paper of Jiao and Zhou [23]. After this, many authors studied several works by using different methods we refere the readers to [3, 4, 14, 16, 18–21]. In particular, César [16] investigated the following $p$-Laplacian Dirichlet problem with mixed derivatives

$$\begin{cases} -{}_tD_1^{\alpha}\left(\varphi_p\left({}_0D_t^{\alpha}u(t)\right)\right) = f(t,u(t)),\ t \in (0,T) \\ u(0) = u(T) = 0, \end{cases} \tag{2}$$

where $0 < \frac{1}{p} < \alpha < 1$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Under some suitable conditions on the function $f$ and by means of the direct variational method combined with the mountain pass theorem,

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the author has proves that problem (2) admits a nontrivial weak solution.
Kratou in [25] concidered the following problem

$$\begin{cases} \left(a + b\int_0^T |_tD_T^\alpha u(t)|^p\,dt\right)^{p-1} {}_tD_T^\alpha\Big(\Phi_p({}_0D_t^\alpha u(t)) = \lambda \frac{g(t)}{u^\gamma(t)} + f(t,u(t)), t \in (0,T), \\ u(0) = u(T) = 0, \end{cases} \tag{3}$$

where $\lambda$ is a positive parameter, $\frac{1}{p} < \alpha \leq 1 \leq a$, $\gamma \in (0,1)$, and $\Phi_p : \mathbb{R} \to \mathbb{R}$, is the $p-$Laplacian defined as follows:

$$\Phi_p(s) = \begin{cases} |s|^{s-2}s, & \text{if } s \neq 0, \\ 0, & \text{if } s = 0. \end{cases}$$

Using the Nehari manifold method combined with the fibering maps analysis, the author proved that for $\lambda$ small enough, problem (3) possesses at least two nontrivial positive solutions. Problem (3) was studied by Chen et al. [7], in the case when $g \equiv 0$.

Motivated by the above mentioned papers, in this work, we want to contribute with the development of this new area on singular fractional differential equations involving both the Riemann Liouville and the $p$-Laplacian operators . Precisely, we will study the existence of nontrivial weak solutions for the following system:

$$\begin{cases} S(u(t))\Big({}_tD_T^\alpha\Big(\Phi_p({}_0D_t^\alpha u(t))\Big) + M(t)\Phi_p({}_0D_t^\alpha u(t))\Big) = \frac{f(t)}{u^\beta(t)} + \lambda g(t,u(t)), t \in (0,T), \\ u(0) = u(T) = 0, \end{cases} \tag{4}$$

where

$$S(u(t)) = \left(a + b\int_0^T |_tD_T^\alpha u(t)|^p + M(t)|u(t)|^p\,dt\right)^{p-1},$$

$\lambda$ is a positive parameter, $0 < \frac{1}{p} < \alpha \leq 1$, $0 < \beta < 1$, $f \in C([0,T])$, and $g \in C([0,T] \times \mathbb{R}, \mathbb{R})$ is positively homogeneous of degree $r-1$, that is $g(x,tu) = t^{r-1}g(x,u)$ holds for all $(x,u) \in [0,T] \times \mathbb{R}$. Moreover, if we put $G(x,s) := \int_0^s g(x,t)dt$, then we assume the following:

**(H$_1$)** $G : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is homogeneous of degree $r$ that is

$$G(x,tu) = t^rG(x,u) \ (t > 0) \ \text{ for all } \ x \in [0,T], \ u \in \mathbb{R}.$$

**(H$_2$)** The function $M \in C([0,T], \mathbb{R})$ is such that

$$0 < \min_{t\in[0,T]} M(t) := M_0 \leq \max_{t\in[0,T]} M(t) := M_\infty.$$

Note that, from **(H$_1$)**, $g$ leads to the so-called Euler identity

$$ug(t,u) = rG(t,u).$$

Moreover, there exists $C_0 > 0$, such that

$$|G(t,u)| \leq C_0|u|^r. \tag{5}$$

In this paper, we want to use the mountain pass geometry combined with the variational method, in order to prove the following result.

**Theorem 1.1.** *Assume that $0 < 1-\beta < 1 < r < p$ and $\frac{1}{p} < \alpha < 1$. If the hypotheses* **(H$_1$)**-**(H$_2$)** *are satisfied. Then there exists $\lambda_0 > 0$, such that for all $\lambda \in (0,\lambda_0)$, problem* (4) *admits a nontrivial weak solution.*

The rest of this paper is organized as follows. In Section 2, we present some preliminaries and results on the fractional calculus. In Section 3, the variational setting of the problem (4) is given. Moreover, in this section, we prove the main result of this work (Theorem 1.1).

## 2. Preliminaries

This section is devoted to present some background theory and results on the concept of fractional Riemann operators. In the following definitions, we introduce the definition of the Riemann-Liouville fractional integral respectively the Riemann-Liouville fractional derivative.

**Definition 2.1.** *Let $\sigma > 0$ and let $\varphi$ be a real function defined a.e. on $(0,T)$. The Left ( resp. right ) Riemann-Liouville fractional integral with inferior limit 0 (resp. superior limit T) of order $\sigma$ of $\varphi$ is given by*

$$ {}_0I_t^{\sigma}\varphi(t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\varphi(s)ds, \quad t \in (0,T], $$

*respectively*

$$ {}_tI_T^{\sigma}\varphi(t) = \frac{1}{\Gamma(\sigma)}\int_t^T (t-s)^{\sigma-1}\varphi(s)ds, \quad t \in [0,T), $$

*provided that the right sides are pointwise defined on $[0,T]$, where $\Gamma$ denotes Euler's Gamma function.*
*We note that If $\varphi \in L^1(0,T)$, then, ${}_0I_t^{\sigma}\varphi$ and ${}_tI_T^{\sigma}\varphi$ are defined a.e. on $(0,T)$.*

**Definition 2.2.** *Let $0 < \sigma < 1$. Then, the Left ( resp. right ) Riemann-Liouville fractional derivative of order $\sigma$ of $\varphi$ is defined as follows:*

$$ {}_0D_t^{\sigma}\varphi(t) = \frac{d}{dt}\left({}_0I_t^{1-\sigma}\varphi\right)(t), \ \forall\ t \in (0,T], $$

*respectively*

$$ {}_tD_T^{\sigma}\varphi(t) = \frac{d}{dt}\left({}_tI_T^{1-\sigma}\varphi\right)(t), \ \forall\ t \in [0,T], $$

*provided that the right sides are pointwise defined on $[0,T]$.*

**Remark 2.3.** *From [24], if $\varphi$ is an absolutely continuous function in $[0,T]$. Then ${}_0D_t^{\sigma}\varphi$ and ${}_tD_T^{\sigma}\varphi$ are defined a.e. on $(0,T)$. Moreover, we have*

$$ {}_0D_t^{\sigma}\varphi(t) = {}_0I_t^{1-\sigma}\varphi'(t) + \frac{\varphi(0)}{t^{\sigma}\Gamma(1-\sigma)}, \tag{6} $$

*and*

$$ {}_tD_T^{\sigma}\varphi(t) = -{}_tI_T^{1-\sigma}\varphi'(t) + \frac{\varphi(T)}{(T-t)^{\sigma}\Gamma(1-\sigma)}. \tag{7} $$

*Moreover, if $\varphi(0) = \varphi(T) = 0$, then*

$$ {}_0D_t^{\sigma}\varphi(t) ={}_0 I_t^{1-\sigma}\varphi'(t) \ \text{ and } \ {}_tD_T^{\sigma}\varphi(t) = -{}_tI_T^{1-\sigma}\varphi'(t). $$

*We notes that from the above equations, we have the equality of Riemann-Liouville fractional derivative and Caputo derivative.*

In the following, we collect from [5], some properties concerning the left Riemann-Liouville fractional operators. One can easily derive the analogous version for the right one.

**Proposition 2.4.** *If $\sigma_1,\ \sigma_2 > 0$, then for any $\varphi \in L^1(0,T)$, we have*

$$ {}_0I_t^{\sigma_1} \circ {}_0I_t^{\sigma_2}\varphi = {}_0I_t^{\sigma_1+\sigma_2}\varphi. $$

From Proposition 2.4, Equations (6) and (7), it is not difficult to deduce the following results concerning the composition between fractional integral and fractional derivative. That is, if $0 < \sigma < 1$, and $\varphi \in L^1(0,T)$, then we have

$$ {}_0D_t^\sigma \circ {}_0I_t^\sigma \varphi = \varphi, $$

and if $\varphi$ is absolutely continuous with $\varphi(0) = 0$. Then, we get

$$ {}_0I_t^\sigma \circ {}_0D_t^\sigma \varphi = \varphi. $$

**Proposition 2.5.** *For each $\sigma > 0$ and for any $p \geq 1$, the operator ${}_0I_t^\sigma : L^p(0,T) \to L^p(0,T)$, is linear and continuous. Moreover for all $\varphi \in L^p(0,T)$, we have*

$$ \|{}_0I_t^\sigma \varphi\|_p \leq \frac{T^\sigma}{\Gamma(1+\sigma)} \|\varphi\|_p. $$

Now, we give another classical result on the boundness of the left fractional integral in the sens of the supremum norm.

**Proposition 2.6.** *Let $0 < \frac{1}{p} < \sigma < 1$ and $q = \frac{p}{p-1}$. Then, for each $\varphi \in L^p(0,T)$, ${}_0I_t^\sigma \varphi$ is Hölder continuous on $(0,T]$ with exponent $\sigma - \frac{1}{p} > 0$, moreover, ${}_0I_t^\sigma \varphi$ can be continuously extended by $0$ at $t = 0$. Also, ${}_0I_t^\sigma \varphi \in C_0(0,T)$, and*

$$ \|{}_0I_t^\sigma \varphi\|_\infty \leq \frac{T^{\sigma - \frac{1}{p}}}{\Gamma(\sigma)\left((\sigma-1)q+1\right)^{\frac{1}{q}}} \|\varphi\|_p. \tag{8} $$

Finally, in order to introduce the variational setting associated to the main problem, we will need the following formula for integration by parts:

**Proposition 2.7.** *Let $0 < \sigma < 1$ and $p, q$ are such that*

$$ p \geq 1,\ q \geq 1 \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \sigma \quad \text{or} \quad p \neq 1,\ q \neq 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1 + \sigma. $$

*Then, for all $\varphi \in L^p(0,T)$ and all $\psi \in L^q(0,T)$, we have*

$$ \int_0^T \psi(t)\, {}_0I_t^\sigma \varphi(t)dt = \int_0^T \varphi(t)\, {}_0I_t^\sigma \psi(t)dt, \tag{9} $$

*and*

$$ \int_0^T \varphi(t)\, {}_0^cD_t^\sigma \psi(t)dt = \psi(t)_t I_T^{1-\sigma} \varphi(t)\Big|_{t=0}^{t=T} + \int_0^T \psi(t)\, {}_0D_t^\sigma \varphi(t)dt. \tag{10} $$

*Moreover, if $\psi(0) = \psi(T) = 0$, then, one we get*

$$ \int_0^T \varphi(t)\, {}_0D_t^\sigma \psi(t)dt = \int_0^T \psi(t)\, {}_0D_t^\sigma \varphi(t)dt. \tag{11} $$

Now, we are in a position to discuss the variational setting associated with the problem (4). We denote by $C_0^\infty([0,T],\mathbb{R})$ the set of all functions $v \in C^\infty([0,T],\mathbb{R})$ such that $v(0) = v(T) = 0$. For, $\sigma > 0$ and $p > 1$, we denoted by $E_0^\sigma$ the closure of $C_0^\infty([0,T],\mathbb{R})$ under the norm

$$ \|u\| = \left(\|u\|_p^p + \|{}_0^cD_t^\sigma u\|_p^p\right)^{\frac{1}{p}}. \tag{12} $$

**Remark 2.8.** *The following properties are useful for the rest of the paper:*

*(i) The fractional derivative space $E_0^\sigma$ is the space of functions $u \in L^p([0,T])$ having an $\sigma$-order Caputo fractional derivative ${}_0^cD_t^\sigma u \in L^p([0,T])$ and $u(0) = u(T) = 0$.*

(ii) *If $v \in E_0^\sigma$ is such that $u(0) = 0$, then the left and right Riemann-Liouville fractional derivatives of order $\sigma$ are equivalent to the left and right Caputo fractional derivatives of order $\sigma$. That is*

$$ {}_0^cD_t^\sigma u(t) = {}_0D_t^\sigma u(t),\ t \in [0,T]. $$

(iii) *The fractional space $E_0^\sigma$ is a reflexive and a separable Banach space.*

**Lemma 2.9.** *For any $v \in E_0^\sigma$, we have*

$$ \|v\|_p \leq \frac{T^\sigma}{\Gamma(\sigma+1)}\|{}_0D_t^\sigma v\|_p. \tag{13} $$

*Moreover, if $\frac{1}{p} < \sigma < 1$, then we get*

$$ \|u\|_\infty \leq \frac{T^{\sigma-\frac{1}{p}}}{\Gamma(\sigma)\left((\sigma-1)\widetilde{p}+1\right)^{\frac{1}{\widetilde{p}}}}\|{}_0D_t^\sigma u\|_p, \tag{14} $$

*where $\widetilde{p} = \frac{p}{p-1}$.*

**Remark 2.10.** *From Equation (13), we can consider $E_0^\sigma$ with respect to the following equivalent norm*

$$ \|u\|_{\sigma,p} = \|{}_0D_t^\sigma u\|_p. $$

*Also from hypothesis* **(H$_2$)**, *can be equipped with the following equivalent norm*

$$ \|u\|_M = \left(\|{}_0^cD_t^\sigma u\|_p^p + \|M^{\frac{1}{p}}u\|_p^p\right)^{\frac{1}{p}}. $$

*Moreover, we have*

$$ \min(1,M_0)\|u\|_{\sigma,p} \leq \|u\|_M \leq \max(1,M_\infty)\|u\|_{\sigma,p}. \tag{15} $$

**Lemma 2.11.** *If $\frac{1}{p} < \sigma < 1$, and the sequence $\{u_n\} \rightharpoonup u$ weakly in $E_0^\sigma$. Then $\{u_n\} \to u$ strongly in $C([0,T])$, that is*

$$ \|u_n - u\|_\infty \to 0 \quad as\ n \to \infty. $$

## 3. Proof of the main result

In this section we will prove the main result of this paper (Theorem 1.1). So we begin by introduce the variational setting for problem (4). Associated to the problem (4), we define the functional $\Phi_\lambda : E_0^\sigma \to \mathbb{R}$, as follows:

$$ \Phi_\lambda(\varphi) = \frac{1}{bp^2}\left(a + b\|\varphi\|_M^p\right)^p - \frac{\lambda}{r}\int_0^T G(t,\varphi(t))dt - \frac{1}{1-\beta}\int_0^T f(t)|\varphi(t)|^{1-\beta}dt - \frac{a^p}{bp^2}. $$

Note that, a function $\varphi \in E_0^\sigma$ is said to be a weak solution of problem (4), if for any $\psi \in E_0^\sigma$ we have:

$$ \begin{aligned} S(\varphi(t))\int_0^T |\,{}_0D_t^\sigma\varphi(t)|^{p-2}\,{}_0D_t^\sigma\varphi(t)\,{}_0D_t^\sigma\psi(t) + |\varphi(t)|^{p-2}\varphi(t)\psi(t)\,dt &= \int_0^T f(t)\varphi(t)^{-\gamma}\psi(t)dt \\ &+ \lambda\int_0^T g(t,\varphi(t))\psi(t)dt. \end{aligned} $$

Now, in order to prove Theorem 1.1, we need to prove two lemmas.

**Lemma 3.1.** *Under assumptions* **(H$_1$)**-**(H$_2$)**. *If $0 < 1-\beta < 1 < r < p^2$ and $\frac{1}{p} < \alpha < 1$, then the functional $\Phi_\lambda$ is coercive in $E_0^\sigma$.*

*Proof.* let $\varphi \in E_0^\sigma$ with $\|\varphi\|_{\sigma,p} > 1$, then from Equations (5), (13), (14), and Remark 2.10, we have

$$\begin{aligned}\int_0^T G(t,\varphi(t))dt &\leq C_0 \int_0^T |\varphi(t)|^r\, dt \\ &\leq C_0 T \|\varphi\|_\infty^r \\ &\leq \frac{C_0 T^{1+r(\alpha-\frac{1}{p})}}{\left(\Gamma(\alpha)\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1}{\widetilde{p}}}\right)^r} \|\varphi\|_{\alpha,p}^r. \end{aligned} \tag{16}$$

On the other hand, from Equations (13), (14), and Remark 2.10, we get

$$\begin{aligned}\int_0^T f(t)|\varphi(t)|^{1-\beta}dt &\leq \|f\|_\infty \int_0^T |\varphi(t)|^{1-\beta}\, dt \\ &\leq \|f\|_\infty T \|\varphi\|_\infty^{1-\beta} \\ &\leq \frac{\|f\|_\infty T^{1+(1-\beta)(\alpha-\frac{1}{p})}}{\left(\Gamma(\alpha)\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1}{\widetilde{p}}}\right)^{1-\beta}} \|\varphi\|_{\alpha,p}^{1-\beta}. \end{aligned} \tag{17}$$

Finally, by combining (16) with (17), and using Equation (15), we obtain

$$\begin{aligned}\Phi_\lambda(\varphi) &= \frac{1}{bp^2}\left(a + b\|\varphi\|_M^p\right)^p - \frac{\lambda}{r}\int_0^T G(t,\varphi(t))dt - \frac{1}{1-\beta}\int_0^T f(t)|\varphi(t)|^{1-\beta}dt - \frac{a^p}{bp^2} \\ &\geq \frac{b^{p-1}}{p^2}\|\varphi\|_M^{p^2} - \frac{\lambda}{r}\int_0^T G(t,\varphi(t))dt - \frac{1}{1-\beta}\int_0^T f(t)|\varphi(t)|^{1-\beta}dt \\ &\geq \frac{b^{p-1}}{p^2}\min(1, M_0)\|\varphi\|_{\alpha,p}^{p^2} - \frac{C_0 \lambda T^{1+r(\alpha-\frac{1}{p})}}{r\left(\Gamma(\alpha)\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1}{\widetilde{p}}}\right)^r}\|\varphi\|_{\alpha,p}^r \\ &\quad - \frac{\|f\|_\infty T^{1+(1-\beta)(\alpha-\frac{1}{p})}}{(1-\beta)\left(\Gamma(\alpha)\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1}{\widetilde{p}}}\right)^{1-\beta}}\|\varphi\|_{\alpha,p}^{1-\beta}. \end{aligned}$$

Since $0 < 1-\beta < 1 < r < p^2$, then we infer that $\lim\limits_{\|\varphi\|_{\alpha,p}\to\infty} \Phi_\lambda(\varphi) = \infty$. That is $\Phi_\lambda$ is coercive and bounded bellow on $E_0^\sigma$. □

**Lemma 3.2.** *Assume that assumptions* **(H$_1$)-(H$_2$)** *hold. If* $0 < 1-\beta < 1 < r < p$ *and* $\frac{1}{p} < \alpha < 1$, *then there exist* $\lambda_0 > 0$ *and* $\psi \in E_0^\alpha$ *with* $\psi \geq 0$, $\psi \neq 0$, *such that for each* $\lambda \in (0, \lambda_0)$ *we have*

$$\Phi_\lambda(t\psi) < 0, \quad \textit{for} \ \ t > 0, \quad \textit{small enough}.$$

*Proof.* Let $\psi \in C_0^\infty([0,T], \mathbb{R})$ such that $supp(\psi) \subset [0,T]$ with $\psi = 1$ in a sub-interval of $[0,T]$ and $0 \leq \psi \leq 1$ in $[0,T]$. Let $0 < t < 1$, then we have

$$\begin{aligned}\Phi_\lambda(t\psi) &= \frac{1}{bp^2}\left(a + bt^p\|\psi\|_M^p\right)^p - \frac{\lambda t^r}{r}\int_0^T G(t,\psi(t))dt - \frac{t^{1-\beta}}{1-\beta}\int_0^T f(t)|\psi(t)|^{1-\beta}dt - \frac{a^p}{bp^2} \\ &\leq \frac{1}{bp^2}\left(a + bt^p\|\psi\|_M^p\right)^p - \frac{\lambda t^r}{r}\int_0^T G(t,\psi(t))dt - \frac{t^{1-\beta}}{1-\beta}\int_0^T f(t)|\psi(t)|^{1-\beta}dt - \frac{a^p}{bp^2} \\ &\leq h(t) - \frac{\lambda t^r}{r}\int_0^T G(t,\psi(t))dt - \frac{t^{1-\beta}}{1-\beta}\int_0^T f(t)|\psi(t)|^{1-\beta}dt, \end{aligned} \tag{18}$$

where $h(t) = \frac{1}{bp^2}\left(a + bt^p\|\psi\|_M^p\right)^p - \frac{a^p}{bp^2}$.
Using the convexity of the function $h$, it is not difficult to prove that

$$h(t) \leq th'(t).$$

So from (18), we obtain

$$\begin{aligned}
\Phi_\lambda(t\psi) &\leq h(t) - \frac{\lambda t^r}{r}\int_0^T G(t,\psi(t))dt - \frac{t^{1-\beta}}{1-\beta}\int_0^T f(t)|\psi(t)|^{1-\beta}dt \\
&\leq t^{p-1}\|\psi\|_M^p\left(a + bt^p\|\psi\|_M^p\right)^p - \frac{\lambda t^r}{r}\int_0^T G(t,\psi(t))dt - \frac{t^{1-\beta}}{1-\beta}\int_0^T f(t)|\psi(t)|^{1-\beta}dt \\
&\leq t^r\left(\|\psi\|_M^p\left(a + b\|\psi\|_M^p\right)^p - \frac{\lambda}{r}\int_0^T G(t,\psi(t))dt\right) \\
&\quad -\frac{t^{1-\beta}}{1-\beta}\int_0^T f(t)|\psi(t)|^{1-\beta}dt. \qquad (19)
\end{aligned}$$

Put

$$\lambda_0 = \frac{r\|\psi\|_M^p\left(a + b\|\psi\|_M^p\right)^p}{\int_0^T G(t,\psi(t))dt},$$

and

$$\delta = \left(\frac{\int_0^T f(t)|\psi(t)|^{1-\beta}dt}{(1-\beta)\left(\|\psi\|_M^p\left(a + b\|\psi\|_M^p\right)^p - \frac{\lambda}{r}\int_0^T G(t,\psi(t))dt\right)}\right)^{\frac{1}{r+\beta-1}}.$$

From (19), we see that if $0 < \lambda < \lambda_0$ and $0 < t < \delta$, then $\Phi_\lambda(t\psi) < 0$. This ends the proof of Lemma 3.2. □

From Lemmas 3.1, 3.2, we can define $m_\lambda < 0$ as follows:

$$m_\lambda = \inf_{v\in E_0^\alpha}\Phi_\lambda(v).$$

**Proposition 3.3.** *Under assertions* **(H$_1$)-(H$_2$)**. *If* $0 < 1-\beta < 1 < r < p$ *and* $\frac{1}{p} < \alpha < 1$. *Then the function* $\Phi_\lambda$ *reaches its global minimizer in* $E_0^\alpha$*, which means that there there exists* $\psi_* \in E_0^\alpha$*, such that,* $\Phi_\lambda(\psi_*) = m_\lambda < 0$.

*Proof.* Let $\{\psi_n\}$ be a minimising sequence, so $\Phi_\lambda(\psi_n) \to m_\lambda < 0$.
First, we clame that $\{\psi_n\}$ is bounded in $E_0^\alpha$. If not, up to a subsequence, we can assume that $\|\psi_n\| \to \infty$ as $n$ tends to infinity. From Lemma 3.1, we get $\Phi_\lambda(\psi_n) \to \infty$ which is a contradiction. Since $\{\psi_n\}$ is bounded then, from Remark 2.8, there exist a subsequence still denoted by $\{\psi_n\}$, and $\psi_* \in E_0^\alpha$ such that

$$\begin{cases} \psi_n \rightharpoonup \psi_*, & \text{weakly in } E_0^\alpha, \\ \psi_n \to \psi_*, & \text{strongly in } L^r([0,T],\mathbb{R}), \\ \psi_n \to \psi_*, & \text{a.e. in } [0,T]. \end{cases}$$

Now, let us prove that

$$\lim_{n\to\infty}\int_0^T f(t)|\psi_n|^{1-\beta}dt = \int_0^T f(t)|\psi_*|^{1-\beta}dt. \qquad (20)$$

From Lemma 2.11, as $n$ large enough we have

$$\begin{aligned}\int_0^T f(t)|\psi_n|^{1-\beta}dt &\leq \int_0^T f(t)|\psi_*|^{1-\beta}dt + \int_0^T f(t)|\psi_n-\psi_*|^{1-\beta}dt\\ &\leq \int_0^T f(t)|\psi_*|^{1-\beta}dt + T\|f\|_\infty\|\psi_n-\psi_*\|_\infty^{1-\beta}dt\\ &\leq \int_0^T f(t)|\psi_*|^{1-\beta}dt + \circ(1).\end{aligned}$$

where $\circ(1)$ satisfies $\lim\limits_{n\to\infty}\circ(1)=0$.
On the other hand, as in the above inequality, we get

$$\begin{aligned}\int_0^T f(t)|\psi_*|^{1-\beta}dt &\leq \int_0^T f(t)|\psi_n|^{1-\beta}dt + \int_0^T f(t)|\psi_n-\psi_*|^{1-\beta}dt\\ &\leq \int_0^T f(t)|\psi_n|^{1-\beta}dt + T\|f\|_\infty\|\psi_n-\psi_*\|_\infty^{1-\beta}dt\\ &\leq \int_0^T f(t)|\psi_n|^{1-\beta}dt + \int_0^T f(t)|\psi_n-\psi_*|^{1-\beta}dt\\ &\leq \int_0^T f(t)|\psi_n|^{1-\beta}dt + \circ(1).\end{aligned}$$

Consequently, we obtain

$$\int_0^T f(t)|\psi_n|^{1-\beta}dt = \int_0^T f(t)|\psi_*|^{1-\beta}dt + \circ(1).$$

Hence, (20) is valid.
On the other hand, from [15], there exists $h\in L^r([0,T],\mathbb{R})$ such that for $n$ large enough $|\psi_n(t)|\leq h(t)$. Therefore, the Dominated convergence Theorem implies that

$$\lim_{n\to\infty}\int_0^T g(t,\psi_n(t))\,dt = \int_0^T g(t,\psi_*(t))\,dt. \tag{21}$$

Now, by combining Equations (20), (21) with the weakly lower semi-continuity of the norm, we deduce that $\Phi_\lambda$ is weakly lower semi-continuous. Hence, we get

$$\Phi_\lambda(\psi_*)\leq \lim_{n\to\infty}\Phi_\lambda(\psi_n)=m_\lambda.$$

Also from the definition of $m_\lambda$, we have $\Phi_\lambda(\psi_*)\geq m_\lambda$. Finally, the above informations imply that $\Phi_\lambda(\psi_*)=m_\lambda$, which ends the proof of Lemma 3.3. □

Now, we are ready to present and prove the main result of this paper.
**Proof of Theorem 1.1**
From Proposition 3.3, there exists $\psi_*$, such that $\Phi_\lambda(\psi_*)=m_\lambda$. Since $\psi_*$ is a global minimizer for $\Phi_\lambda$ in $E_0^\alpha$, then for any $t>0$ and any $\varphi\in E_0^\alpha$ we have

$$\frac{\Phi_\lambda(\psi_*+t\varphi)-\Phi_\lambda(\psi_*)}{t}\geq 0.$$

So by letting $t$ tends to zero, we obtain

$$\begin{aligned}S(\psi_*(t))\int_0^T |\,{}_0D_t^\alpha\psi_*(t)|^{p-2}\,{}_0D_t^\alpha\psi_*(t)\,{}_0D_t^\alpha\varphi(t)\,dt \quad &+ \quad S(\psi_*(t))\int_0^T |\psi_*(t)|^{p-2}\psi_*(t)\varphi(t)\,dt\\ -\int_0^T f(t)\psi_*(t)^{-\gamma}\varphi(t)dt \quad &- \quad \lambda\int_0^T g(t,\psi_*(t))\varphi(t)dt\geq 0.\end{aligned}$$

Since the function $\varphi$ is arbitrary in $E_0^\alpha$. Then we can replace $\varphi$ by $-\varphi$ in the least inequality. That is we can replace the inequality by the equality and deduce that $\psi_*$ is a weak solution of problem (4). Finally, the fact that $\Phi_\lambda(\psi_*) = m_\lambda < 0$, implies that $\psi_*$ is nontrivial. This concludes the proof.

**Remark 3.4.** *The result in Theorem 1.1 can be extended to more general problems involving the fractional Riemann Liouville operator with respect to another function. That is, under suitable conditions, we can give the existence result for the following problem*

$$\begin{cases} S(u(t))\, {}_tD_T^{\alpha,\psi}\left(\Phi_p({}_0D_t^{\alpha,\psi}u(t))\right) + M(t)\Phi_p({}_0D_t^{\alpha,\psi}u(t)) = \frac{f(t)}{u^\beta(t)} + \lambda g(t,u(t)), t \in (0,T); \\ \\ u(0) = u(T) = 0, \end{cases}$$

*where ${}_0D_t^{\alpha,\psi}$ and ${}_tD_T^{\alpha,\psi}$ are respectively the left and the right fractional derivative with respect to a function $\psi$. The interested readers can see [24, 30] for more details about these operators.*

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