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Explicit Formulae for the Drazin Inverse of Anti-Triangular Block Matrices

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Abstract. In this paper, we analyze the index relation of anti-triangular block matrices and their entries to separately obtain new and strict expressions for the Drazin inverse of them under certain circumstances. As applications, we utilize the relationship between the anti-triangular block matrix and a 2×2 block matrix to establish several formulae. Our results generalize and unify a series of results in the literature.

1. Introduction

There are some original applications of Drazin inverse of block matrices in systems of linear differential equations and liner difference equations [8], finite Markov chains [26], iterative methods [27] and so on [1, 14, 15, 17, 23, 24, 29, 31–34], precisely because it has important spectral properties.

The *Drazin inverse* of A is the unique matrix A^d satisfying the equations applicable only to square matrices as follows

$$AA^d = A^dA, \quad A^dAA^d = A^d, \quad A^k = A^{k+1}A^d,$$

in these equations k is the smallest non-negative integer such that $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$, called index of A and denoted by $\mathrm{ind}(A)$. The spectral idempotent A^π of A corresponding to $\{0\}$ is given by $A^\pi = I - A^e$, where $A^e = AA^d$. If $\mathrm{ind}(A) = 1$, then $A^d = A^\#$, where $A^\#$ is a special case of the Drazin inverse, the group inverse.

Especially, the expression problem of the Drazin inverse of anti-triangular block matrices occurred in [7], is to obtain the solution of the second-order singular differential equations. This problem was firstly proposed by Campbell and Meyer [8], and it is still an open problem without additional assumptions upon the blocks herein. Consider two anti-triangular block matrices

$$N = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{1}$$

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and

$$\bar{N} = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}. \tag{2}$$

Up to now, many formulae for the Drazin inverse of a 2×2 block matrix under several certain restrictions were considered (see [18, 20, 37, 38]). Let a 2×2 block complex matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{3}$$

We note former fornulae for M^d studied under appropriate restrictions, and list them as follows:

1. in [16], $BC = 0, BD = 0$ and $DC = 0$;
2. in [18], $BC = 0, BDC = 0$ and $BD^2 = 0$;
3. in [20], $BC = 0, DC = 0$ (or $BD = 0$) and D is nilpotent;
4. in [6], $A = 0$ and $D = 0$;
5. in [11], $ABC = 0, DC = 0$ and $BD = 0$ (or BC is nilpotent, or D is nilpotent);
6. in [3], $ABC = 0$ and $DC = 0$ (or $BD = 0$);
7. in [35], $BCA = 0, BCB = 0, DCA = 0$ and $DCB = 0$.

The additional research focus on the research of the Drazin inverse of the anti-triangular block matrix N partitioned as in the form (1), is also widely investigated in [2, 5, 9, 12, 19, 22, 25].

Note that, in [13], the representation for the Drazin inverse of the anti-triangular block matrix N as in (1) was shown respectively under different assumptions as follows

1. $AB = 0$;
2. $ABC = 0$.

It is worth mentioning that the Drazin inverse of matrices partitioned as $\hat{N} = \begin{bmatrix} A & I \\ B & 0 \end{bmatrix}$ were also concerned, for instance, [4, 30, 36].

In the paper, we note the relationship between $\bar{N}$ of the form (2) and $\hat{N}$, through simple calculation,

$$\hat{N}^d = (R\bar{N}R^{-1})^d = R\bar{N}^dR^{-1},$$

where $R = \begin{bmatrix} 0 & I \\ I & -A \end{bmatrix}$.

The research about the Drazin inverse of $\bar{N}$ is significantly less than the same research of M or N, but it is original and equally important. Our aim is, by analyzing the index relationship of anti-triangular block matrices and their entries, to give accurate representations for the Drazin inverse of $\bar{N}$ on new restrictions, and establish a relationship among $\bar{N}$, N and M to respectively derive new and strict expressions for the Drazin inverse of N and the Drazin inverse of M under certain conditions. In this way, we generalize and unify a series of results in the literature.

The next symbol description will be used throughout the paper. $\mathbb{C}^{m\times n}$ is the set of $m\times n$ complex matrices and I is the identity matrix of proper size. We always define the sum to be 0, if the lower limit of a sum is greater than its upper limit. For example, the sum $\sum_{k=0}^{-1} * = 0$. We adopt the convention that $[x]$ stands for the truncates integer of x, and $A^0 = I$. Since $(A^d)^n = (A^n)^d$ for any $A \in \mathbb{C}^{n\times n}$, we adopt the convention that $A^{dn} = A^{nd} = (A^d)^n$.

2. Key lemmas

In this section, we state key lemmas for proving the results of this paper.
We begin with the well-known Cline's Formula.

Lemma 2.1. [10] (Cline's Formula) *For* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times m}$, $(BA)^d = B[(AB)^{2d}]A$.

The Drazin inverse of triangle matrices are shown as the following auxiliary result.

Lemma 2.2. [21, 28] *Let* $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ *and* $N = \begin{bmatrix} D & 0 \\ B & A \end{bmatrix} \in \mathbb{C}^{n\times n}$, *where A and D are square matrices. Then*

$$M^d = \begin{bmatrix} A^d & X \\ 0 & D^d \end{bmatrix} \ \ and \ \ N^d = \begin{bmatrix} D^d & 0 \\ X & A^d \end{bmatrix},$$

where

$$X = \sum_{i=0}^{s-1}(A^d)^{i+2}BD^iD^\pi + A^\pi\sum_{i=0}^{r-1}A^iB(D^d)^{i+2} - A^dBD^d$$

such that $\mathrm{ind}(A) = r$ *and* $\mathrm{ind}(D) = s$.

To prove the main results, a needed formula for the Drazin inverse of a sum is taken into consideration.

Lemma 2.3. [35, Theorem 2.1] *Let* $PQP = 0$ *and* $PQ^2 = 0$, *where* $P, Q \in \mathbb{C}^{n\times n}$ *such that* $\mathrm{ind}(P) = r$ *and* $\mathrm{ind}(Q) = s$. *Then*

$$\begin{aligned}(P+Q)^d \ = \ & Q^\pi\sum_{i=0}^{s-1}Q^i(P^d)^{i+1} + \sum_{i=0}^{r-1}(Q^d)^{i+1}P^iP^\pi \\ & + Q^\pi\sum_{i=0}^{s-1}Q^i(P^d)^{i+2}Q + \sum_{i=0}^{r-2}(Q^d)^{i+3}P^{i+1}P^\pi Q \\ & - Q^dP^dQ - (Q^d)^2PP^dQ. \end{aligned} \tag{4}$$

3. Main results

Under the new assumptions, we develop expressions for the Drazin inverse of $\bar{N}$ and N given by (2) and (1), respectively.

Recall that $A^e = AA^d$ and $A^{id} = (A^d)^i$, where i is nonnegative integer. We now consider the Drazin inverse of a anti-triangle matrix $\bar{N}$ as the main result of this paper.

Theorem 3.1. *Let* $\bar{N}$ *be a matrix of the form* (2), *where A and B are square matrices of the same size. If*

$$AB^2 = 0, \quad A^2BA = 0, \quad ABA^2 = 0 \quad and \quad (AB)^2 = 0,$$

then

$$\bar{N}^d = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix},$$

where

$$\begin{aligned} E_1 &= -B^dA^dB+\sum_{i=0}^{s-1}B^\pi B^iA^{(2i+3)d}B+\sum_{i=0}^{s-1}B^\pi B^iA^{(2i+1)d}+\sum_{i=0}^{[\frac{r}{2}]-1}B^{(i+2)d}A^{2i+1}A^\pi B+\sum_{i=0}^{[\frac{r}{2}]-1}B^{(i+1)d}A^{2i+1}A^\pi,\\ E_2 &= \sum_{i=0}^{s-1}B^\pi B^iA^{(2i+2)d}B+\sum_{i=0}^{[\frac{r}{2}]}B^{(i+1)d}A^{2i}A^\pi B,\\ E_3 &= B^{3d}ABA-B^dA^{2d}B-B^d+\sum_{i=0}^{s-1}B^\pi B^iA^{(2i+2)d}+\sum_{i=0}^{[\frac{r}{2}]}B^{(i+1)d}A^{2i}A^\pi+\sum_{i=0}^{s-1}B^\pi B^iA^{(2i+4)d}B+\sum_{i=0}^{[\frac{r}{2}]}B^{(i+2)d}A^{2i}A^\pi B,\\ E_4 &= -B^dA^dB+\sum_{i=0}^{s-1}B^\pi B^iA^{(2i+3)d}B+\sum_{i=0}^{[\frac{r}{2}]}B^{(i+2)d}A^{2i+1}A^\pi B \end{aligned}$$

such that $\mathrm{ind}(A)=r$ *and* $\mathrm{ind}(B)=s$.

Proof. We consider the splitting of $\bar{N}^2$ as follows

$$\bar{N}^2=\begin{bmatrix}A^2+B & AB\\ A & B\end{bmatrix}=\begin{bmatrix}A^2 & 0\\ A & A^eB\end{bmatrix}+\begin{bmatrix}B & AB\\ 0 & A^\pi B\end{bmatrix}, \tag{5}$$

and denote by P and Q the left matrix and the right matrix of the right-hand side in (5), respectively. We obtain the following relations derived directly from the condition

$$(A^eB)^d=0,\quad (A^\pi B)^n=B^{n-1}A^\pi B,\quad n\geq 1.$$

Note that

$$\begin{aligned}(A^\pi B)^d &= A^\pi B[(A^\pi B)^d]^2=A^\pi B[(A^\pi B)^2]^d=A^\pi B[B(A^\pi B)]^d=A^\pi B^2[(A^\pi B)B]^{2d}A^\pi B\\ &= A^\pi B^2(B^2)^{2d}A^\pi B=A^\pi(B^2)^dA^\pi B=B^{2d}A^\pi B.\end{aligned}$$

Then we utilize Lemma 2.2 to obtain

$$P^d=\begin{bmatrix}A^{2d} & 0\\ A^{3d} & 0\end{bmatrix},\quad Q^d=\begin{bmatrix}B^d & B^{2d}AB\\ 0 & B^{2d}A^\pi B\end{bmatrix},$$

and so

$$P^\pi=\begin{bmatrix}A^\pi & 0\\ -A^d & I\end{bmatrix},\quad Q^\pi=\begin{bmatrix}B^\pi & -B^dAB\\ 0 & I-B^dA^\pi B\end{bmatrix}.$$

Furthermore, we prove, for any $n\geq 2$,

$$P^n=\begin{bmatrix}A^{2n} & 0\\ A^{2n-1} & 0\end{bmatrix},$$

and for any $n\geq 1$,

$$Q^n=\begin{bmatrix}B^n & B^{n-1}AB\\ 0 & B^{n-1}A^\pi B\end{bmatrix},\quad P^{nd}=\begin{bmatrix}A^{(2n)d} & 0\\ A^{(2n+1)d} & 0\end{bmatrix},\ \text{and}\ Q^{nd}=\begin{bmatrix}B^{nd} & B^{(n+1)d}AB\\ 0 & B^{(n+1)d}A^\pi B\end{bmatrix}.$$

Let $\mathrm{ind}(A)=r$ and $\mathrm{ind}(B)=s$. We combine the computations of P^iP^π such that $i\geq 2$ and Q^iQ^π such that $i\geq 1$ to give $\mathrm{ind}(P)=[\frac{r}{2}]+1$ and $\mathrm{ind}(Q)=s+1$ as follows

$$P^iP^\pi=\begin{bmatrix}A^{2i} & 0\\ A^{2i-1} & 0\end{bmatrix}\begin{bmatrix}A^\pi & 0\\ -A^d & I\end{bmatrix}=\begin{bmatrix}A^{2i}A^\pi & 0\\ A^{2i-1}A^\pi & 0\end{bmatrix},$$

and

$$Q^iQ^\pi = \begin{bmatrix} B^i & B^{i-1}AB \\ 0 & B^{i-1}A^\pi B \end{bmatrix}\begin{bmatrix} B^\pi & -B^dAB \\ 0 & I-B^dA^\pi B \end{bmatrix} = \begin{bmatrix} B^iB^\pi & B^{i-1}B^\pi AB \\ 0 & B^{i-1}B^\pi A^\pi B \end{bmatrix}.$$

Easy computation gives $PQ^2 = 0$ and $PQP = 0$, which are the conditions in Lemma 2.3. Hence, we focus on obtaining the following relations as in (4):

$$Q^\pi \sum_{i=0}^{s} Q^i(P^d)^{i+1} = \begin{bmatrix} B^\pi \sum\limits_{i=0}^{s} B^iA^{(2i+2)d} & 0 \\ B^\pi \sum\limits_{i=0}^{s} B^iA^{(2i+3)d} & 0 \end{bmatrix},$$

$$\sum_{i=0}^{[\frac{r}{2}]} (Q^d)^{i+1}P^iP^\pi = \begin{bmatrix} \sum\limits_{i=0}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i}A^\pi + B^{3d}ABA & B^{2d}AB \\ \sum\limits_{i=1}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i-1}A^\pi - B^dA^d & B^d \end{bmatrix},$$

$$Q^\pi \sum_{i=0}^{s} Q^i(P^d)^{i+2}Q = \begin{bmatrix} B^\pi \sum\limits_{i=0}^{s} B^iA^{(2i+4)d}B & B^\pi \sum\limits_{i=0}^{s} B^iA^{(2i+3)d}B \\ B^\pi \sum\limits_{i=0}^{s} B^iA^{(2i+5)d}B & B^\pi \sum\limits_{i=0}^{s} B^iA^{(2i+4)d}B \end{bmatrix},$$

$$\sum_{i=0}^{[\frac{r}{2}]-1} (Q^d)^{i+3}P^{i+1}P^\pi Q = \begin{bmatrix} \sum\limits_{i=0}^{[\frac{r}{2}]-1} B^{(i+3)d}A^{2i+2}A^\pi B & \sum\limits_{i=0}^{[\frac{r}{2}]-1} B^{(i+3)d}A^{2i+3}A^\pi B \\ \sum\limits_{i=0}^{[\frac{r}{2}]-1} B^{(i+3)d}A^{2i+1}A^\pi B & \sum\limits_{i=0}^{[\frac{r}{2}]-1} B^{(i+3)d}A^{2i+2}A^\pi B \end{bmatrix},$$

$$Q^dP^dQ = \begin{bmatrix} B^dA^{2d}B & B^dA^dB \\ B^dA^{3d}B & B^dA^{2d}B \end{bmatrix},$$

and

$$(Q^d)^2PP^dQ = \begin{bmatrix} B^{2d}A^eB & B^{2d}A^eAB \\ B^{2d}A^dB & B^{2d}A^eB \end{bmatrix}.$$

Hence, we substitute the above expressions into (4) to conclude

$$\bar{N}^{2d} = (P+Q)^d = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

where

$$\begin{aligned} \alpha &= B^{3d}ABA - B^dA^{2d}B - B^{2d}A^eB + \sum_{i=0}^{s} B^\pi B^iA^{(2i+2)d} \\ &\quad + \sum_{i=0}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i}A^\pi + \sum_{i=0}^{s} B^\pi B^iA^{(2i+4)d}B + \sum_{i=0}^{[\frac{r}{2}]-1} B^{(i+3)d}A^{2i+2}A^\pi B, \\ \beta &= B^{2d}AB - B^dA^dB - B^{2d}A^eAB + \sum_{i=0}^{s} B^\pi B^iA^{(2i+3)d}B + \sum_{i=0}^{[\frac{r}{2}]-1} B^{(i+3)d}A^{2i+3}A^\pi B, \end{aligned}$$

$$\begin{aligned}
\gamma &= -B^dA^d - B^dA^{3d}B - B^{2d}A^dB + \sum_{i=0}^{s} B^\pi B^i A^{(2i+5)d}B \\
&\quad + \sum_{i=0}^{s} B^\pi B^i A^{(2i+3)d} + \sum_{i=0}^{[\frac{r}{2}]-1} B^{(i+3)d}A^{2i+1}A^\pi B + \sum_{i=1}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i-1}A^\pi, \\
\delta &= B^d - B^dA^{2d}B - B^{2d}A^eB + \sum_{i=0}^{s} B^\pi B^i A^{(2i+4)d}B + \sum_{i=0}^{[\frac{r}{2}]-1} B^{(i+3)d}A^{2i+2}A^\pi B
\end{aligned}$$

such that $\mathrm{ind}(A) = r$ and $\mathrm{ind}(B) = s$.

Next we compute $\bar{N}^d = \bar{N}\bar{N}^{2d}$ to get the following expression

$$\bar{N}^d = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix},$$

where

$$\begin{aligned}
E_1 &= B^\pi A^d + B^\pi A^{3d}B - B^dA^dB + \sum_{i=0}^{s} B^\pi B^{i+1} A^{(2i+5)d}B \\
&\quad + \sum_{i=0}^{s} B^\pi B^{i+1} A^{(2i+3)d} + \sum_{i=0}^{[\frac{r}{2}]-1} B^{(i+2)d}A^{2i+1}A^\pi B + \sum_{i=1}^{[\frac{r}{2}]} B^{id}A^{2i-1}A^\pi, \\
E_2 &= B^\pi A^{2d}B + B^dA^\pi B + \sum_{i=0}^{s} B^\pi B^{i+1} A^{(2i+4)d}B + \sum_{i=0}^{[\frac{r}{2}]-1} B^{(i+2)d}A^{2i+2}A^\pi B, \\
E_3 &= B^{3d}ABA - B^dA^{2d}B - B^{2d}A^eB + \sum_{i=0}^{s} B^\pi B^i A^{(2i+2)d} \\
&\quad + \sum_{i=0}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i}A^\pi + \sum_{i=0}^{s} B^\pi B^i A^{(2i+4)d}B + \sum_{i=0}^{[\frac{r}{2}]-1} B^{(i+3)d}A^{2i+2}A^\pi B, \\
E_4 &= B^{2d}A^\pi AB - B^dA^dB + \sum_{i=0}^{s} B^\pi B^i A^{(2i+3)d}B + \sum_{i=0}^{[\frac{r}{2}]-1} B^{(i+3)d}A^{2i+3}A^\pi B
\end{aligned}$$

such that $\mathrm{ind}(A) = r$ and $\mathrm{ind}(B) = s$. It is clearly that r and s are respectively the least nonnegative integers as follows

$$A^rA^\pi = 0, \ \ B^sB^\pi = 0,$$

and

$$r-2 \le 2[\frac{r}{2}] - 1 \le r-1, \ \ r-1 \le 2[\frac{r}{2}] \le r, \ \ r \le 2[\frac{r}{2}] + 1 \le r+1$$

for any nonnegative integer r. Therefore, we adjust appropriately the upper and lower limits of the corresponding sum to complete the proof. □

We next establish a relationship between $\bar{N}$ and N to derive the exact expression for the Drazin inverse of N under certain conditions as the second main result of this paper.

Theorem 3.2. *Let N be a matrix of the form* (1), *where A and BC are square matrices of the same size. If*

$$A(BC)^2 = 0, \quad A^2BCA = 0, \quad ABCA^2 = 0 \quad \text{and} \quad (ABC)^2 = 0,$$

then

$$N^d = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}, \tag{6}$$

where

$$\begin{aligned}
F_1 &= \sum_{i=0}^{s-1}(BC)^\pi(BC)^iA^{(2i+1)d} + \sum_{i=0}^{[\frac{r}{2}]-1}(BC)^{(i+1)d}A^{2i+1}A^\pi + \sum_{i=0}^{s-1}(BC)^\pi(BC)^iA^{(2i+3)d}BC \\
&+ \sum_{i=0}^{[\frac{r}{2}]-1}(BC)^{(i+2)d}A^{2i+1}A^\pi BC - (BC)^dA^dBC, \\
F_2 &= \sum_{i=0}^{s-1}(BC)^\pi(BC)^iA^{(2i+2)d}B + \sum_{i=0}^{s-1}(BC)^\pi(BC)^iA^{(2i+4)d}BCB + \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i}A^\pi B \\
&+ \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+2)d}A^{2i}A^\pi BCB + (BC)^{3d}ABCAB - (BC)^dA^{2d}BCB - (BC)^dB, \\
F_3 &= C\sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i}A^\pi + C\sum_{i=0}^{s-1}(BC)^\pi(BC)^iA^{(2i+2)d} + C\sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+2)d}A^{2i}A^\pi BC \\
&+ C\sum_{i=0}^{s-1}(BC)^\pi(BC)^iA^{(2i+4)d}BC - C(BC)^dA^{2d}BC - C(BC)^d, \\
F_4 &= C\sum_{i=0}^{[\frac{r}{2}]-1}(BC)^{(i+2)d}A^{2i+1}A^\pi B + C\sum_{i=0}^{[\frac{r}{2}]-1}(BC)^{(i+3)d}A^{2i+1}A^\pi BCB + C\sum_{i=0}^{s-1}(BC)^\pi(BC)^iA^{(2i+3)d}B \\
&+ C\sum_{i=0}^{s-1}(BC)^\pi(BC)^iA^{(2i+5)d}BCB - C(BC)^dA^dB - C(BC)^dA^{3d}BCB - C(BC)^{2d}A^dBCB
\end{aligned}$$

such that $\operatorname{ind}(A) = r$ *and* $\operatorname{ind}(BC) = s$.

Proof. We use the following splitting of N:

$$N = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}, \tag{7}$$

and denote by P and Q the left matrix and the right matrix of the right-hand side in (7), respectively. Then we switch P and Q to state

$$QP = \begin{bmatrix} A & BC \\ I & 0 \end{bmatrix}.$$

Utilizing Theorem 3.1, we rewrite the $(QP)^d$ as follows

$$(QP)^d = \begin{bmatrix} \lambda & \mu \\ \nu & \xi \end{bmatrix},$$

where

$$\begin{aligned}
\lambda &= \sum_{i=0}^{s-1}(BC)^{\pi}(BC)^{i}A^{(2i+3)d}BC + \sum_{i=0}^{s-1}(BC)^{\pi}(BC)^{i}A^{(2i+1)d} + \sum_{i=0}^{[\frac{r}{2}]-1}(BC)^{(i+2)d}A^{2i+1}A^{\pi}BC \\
&+ \sum_{i=0}^{[\frac{r}{2}]-1}(BC)^{(i+1)d}A^{2i+1}A^{\pi} - (BC)^{d}A^{d}BC, \\
\mu &= \sum_{i=0}^{s-1}(BC)^{\pi}(BC)^{i}A^{(2i+2)d}BC + \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i}A^{\pi}BC, \\
\nu &= \sum_{i=0}^{s-1}(BC)^{\pi}(BC)^{i}A^{(2i+2)d} + \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i}A^{\pi} + \sum_{i=0}^{s-1}(BC)^{\pi}(BC)^{i}A^{(2i+4)d}BC \\
&+ \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+2)d}A^{2i}A^{\pi}BC + (BC)^{3d}ABCA - (BC)^{d}A^{2d}BC - (BC)^{d}, \\
\xi &= \sum_{i=0}^{s-1}(BC)^{\pi}(BC)^{i}A^{(2i+3)d}BC + \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+2)d}A^{2i+1}A^{\pi}BC - (BC)^{d}A^{d}BC
\end{aligned}$$

such that $\mathrm{ind}(A) = r$ and $\mathrm{ind}(BC) = s$. We apply the Cline's Formula as in Lemma 2.1 to give

$$N^{d} = P(QP)^{2d}Q = \begin{bmatrix} \lambda^{2}A + \mu\nu A + \lambda\mu + \mu\xi & \lambda^{2}B + \mu\nu B \\ C\nu\lambda A + C\xi\nu A + C\nu\mu + C\xi^{2} & C\nu\lambda B + C\xi\nu B \end{bmatrix}. \tag{8}$$

Routine computations conclude the following main items $\xi^{2} = 0$, $\xi\nu = 0$, and

$$\begin{aligned}
\lambda^{2} &= \sum_{i=0}^{s-1}(BC)^{i}(BC)^{\pi}A^{(2i+2)d} + \sum_{i=0}^{s-1}(BC)^{i}(BC)^{\pi}A^{(2i+4)d}BC, \\
\mu\nu &= \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i}A^{\pi} + \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+2)d}A^{2i}A^{\pi}BC + (BC)^{3d}ABCA - (BC)^{d}A^{2d}BC - (BC)^{d}, \\
\lambda\mu &= \sum_{i=0}^{s-1}(BC)^{i}(BC)^{\pi}A^{(2i+3)d}BC, \\
\nu\lambda &= \sum_{i=0}^{[\frac{r}{2}]-1}(BC)^{(i+2)d}A^{2i+1}A^{\pi} + \sum_{i=0}^{[\frac{r}{2}]-1}(BC)^{(i+3)d}A^{2i+1}A^{\pi}BC + \sum_{i=0}^{s-1}(BC)^{i}(BC)^{\pi}A^{(2i+3)d} \\
&+ \sum_{i=0}^{s-1}(BC)^{i}(BC)^{\pi}A^{(2i+5)d}BC - (BC)^{d}A^{d} - (BC)^{d}A^{3d}BC - (BC)^{2d}A^{d}BC, \\
\nu\mu &= \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+2)d}A^{2i}A^{\pi}BC + \sum_{i=0}^{s-1}(BC)^{i}(BC)^{\pi}A^{(2i+4)d}BC - (BC)^{d}A^{2d}BC, \\
\mu\xi &= \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+2)d}A^{2i+1}A^{\pi}BC - (BC)^{d}A^{d}BC
\end{aligned}$$

such that $\mathrm{ind}(A) = r$ and $\mathrm{ind}(BC) = s$. Finally we substitute the above expressions into (8) to conclude the rest. □

Next we consider some specializations of our main result. Using Theorem 3.2 in the above, we both generalize [13, Theorem 3.1] and [13, Theorem 3.3] as follows.

Corollary 3.3. [13, Theorem 3.3] *Let N be a matrix of the form* (1)*, where A and BC are square matrices of the same size. If $ABC = 0$, then*

$$N^d = \begin{bmatrix} XA & XB \\ CX & C[XA^d + (BC)^d(XA - A^d)]B \end{bmatrix},$$

where

$$X = \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i} A^{\pi} + \sum_{i=0}^{s-1} (BC)^{\pi} (BC)^i A^{(2i+2)d} \tag{9}$$

such that $\mathrm{ind}(A) = r$ *and* $\mathrm{ind}(BC) = s$.

Proof. It is clear by Theorem 3.2 and equalities

$$\begin{aligned}
XA &= \sum_{i=0}^{s-1} (BC)^{\pi} (BC)^i A^{(2i+1)d} + \sum_{i=0}^{[\frac{r}{2}]-1} (BC)^{(i+1)d} A^{2i+1} A^{\pi}, \\
XB &= \sum_{i=0}^{s-1} (BC)^i (BC)^{\pi} A^{(2i+2)d} B + \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i} A^{\pi} B, \\
CX &= C \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i} A^{\pi} + C \sum_{i=0}^{s-1} (BC)^{\pi} (BC)^i A^{(2i+2)d}, \\
C[XA^d + (BC)^d(XA - A^d)]B &= C \sum_{i=0}^{[\frac{r}{2}]-1} (BC)^{(i+2)d} A^{2i+1} A^{\pi} B + C \sum_{i=0}^{s-1} (BC)^{\pi} (BC)^i A^{(2i+3)d} B - C(BC)^d A^d B
\end{aligned}$$

as desired. □

As a consequence of Corollary 3.3, we obtain the next result.

Corollary 3.4. [13, Theorem 3.1] *Let N be a matrix of the form* (1)*, where A and BC are square matrices of the same size. If $AB = 0$, then*

$$N^d = \begin{bmatrix} XA & (BC)^d B \\ CX & 0 \end{bmatrix},$$

where X is represented as in (9), $\mathrm{ind}(A) = r$ *and* $\mathrm{ind}(BC) = s$.

In order to illustrate our results, we present an example involving 4×4 matrices A, B and C which do not satisfy the assumptions of [13, Theorem 3.1 and Theorem 3.3], whereas the conditions of Theorem 3.2 are met, which allows us to compute N^d.

Example 3.5. *Consider* 4×4 *complex block matrices*

$$A = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $a \neq 0$, $b \neq 0$ and $c \neq 0$. We observe that

$$AB = \begin{bmatrix} 0 & 0 & ab & 0 \\ 0 & 0 & 0 & ab \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0 \quad \textit{and} \quad ABC = \begin{bmatrix} 0 & 0 & 0 & abc \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0.$$

Because the assumptions of [13, Theorem 3.1 and Theorem 3.3] *are not satisfied, we can not use these results. Since $ABCA = 0$ and $ABCB = 0$, we can apply Theorem 3.2 to get*

$$N^d = \begin{bmatrix} 0 & a & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^d = 0.$$

4. Application of main results

As application, we utilize the relationship between the matrices N and M given by (1) and (3), respectively, to establish representations for the Drazin inverse of M under certain restriction, which generalize and unify a series of results in the literature.

Theorem 4.1. *Let M be a matrix of the form* (3) *and N be a matrix of the form* (1), *where A, D and BC are square matrices such that A and BC are of the same size. If*

$$A(BC)^2 = 0, \quad A^2BCA = 0, \quad ABCA^2 = 0, \quad (ABC)^2 = 0, \quad BDC = 0 \quad \textit{and} \quad BD^2 = 0,$$

then

$$\begin{aligned} M^d \;=\; & \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i N^{(i+1)d} \begin{bmatrix} I & F_2D \\ 0 & I + F_4D \end{bmatrix} + \sum_{i=0}^{r-2} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+3)d} \end{bmatrix} N^{i+1} \begin{bmatrix} 0 & -F_1BD \\ 0 & (I - F_3B)D \end{bmatrix} \\ & + \sum_{i=0}^{r-1} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} N^i \begin{bmatrix} I - XA^2 - (BC)^{2d}ABCA - XBC & -F_1B \\ -CXA - C[XA^d + (BC)^d(XA - A^d)]BC & I - F_3B \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D^d(F_4 + D^dF_3B)D \end{bmatrix}, \end{aligned}$$

where N^d is given by (6) *and X is given by* (9) *such that* $\mathrm{ind}(N) = r$ *and* $\mathrm{ind}(D) = s$.

Proof. Let

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} := N + Q.$$

Then $NQN = 0$ and $NQ^2 = 0$. By Theorem 3.2, N^d is given by (6) and

$$N^\pi = \begin{bmatrix} I - F_1A - F_2C & -F_1B \\ -F_3A - F_4C & I - F_3B \end{bmatrix}.$$

Routine calculation gives as follows

$$\begin{aligned} F_1A &= XA^2 + (BC)^{2d}ABCA, \\ F_2C &= XBC, \\ F_3A &= CXA, \\ F_4C &= C[XA^d + (BC)^d(XA - A^d)]BC. \end{aligned}$$

So,

$$N^{\pi}=\begin{bmatrix} I-XA^2-(BC)^{2d}ABCA-XBC & -F_1B \\ -CXA-C[XA^d+(BC)^d(XA-A^d)]BC & I-F_3B \end{bmatrix}.$$

Also, notice that

$$Q^d=\begin{bmatrix} 0 & 0 \\ 0 & D^d \end{bmatrix} \quad \text{and} \quad Q^{\pi}=\begin{bmatrix} I & 0 \\ 0 & D^{\pi} \end{bmatrix}.$$

Applying Lemma 2.3, we finish this proof. □

Remark 4.2. *Theorem 4.1 can generalize and unify the following conditions about the expression for M^d.*

1. $BC=0, BDC=0$ *and* $BD^2=0$ *(see* [18, Theorem 2.2]*);*
2. $BC=0, BD=0$ *and* $DC=0$ *(see* [16, Theorem 5.3]*);*
3. $BC=0, BD=0$ *and D is nilpotent (see* [20, Corollary 2.3]*);*
4. $A=0$ *and* $D=0$ *(see* [6, Theorem 2.1]*) ;*
5. $ABC=0, DC=0$ *and* $BD=0$ *(see* [11, Theorem 1]*).*

Moreover, we give some specific corollaries as follows. It is worth mentioning that the following corollary of Theorem 4.1 also respectively generalizes all conditions above.

Corollary 4.3. *Let M be a matrix of the form* (3) *and N be a matrix of the form* (1)*, where A, D and BC are square matrices such that A and BC are of the same size. If*

$$ABC=0, \quad BDC=0 \quad \text{and} \quad BD^2=0,$$

then

$$\begin{aligned} M^d &= \begin{bmatrix} I & 0 \\ 0 & D^{\pi} \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i N^{(i+1)d} \begin{bmatrix} I & XBD \\ 0 & I+C[XA^d+(BC)^d(XA-A^d)]BD \end{bmatrix} \\ &+ \sum_{i=0}^{r-2} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+3)d} \end{bmatrix} N^{i+1} \begin{bmatrix} 0 & -XABD \\ 0 & (I-CXB)D \end{bmatrix} \\ &+ \sum_{i=0}^{r-1} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} N^i \begin{bmatrix} (BC)^{\pi}-XA^2 & -XAB \\ -CXA & I-CXB \end{bmatrix} \\ &- \begin{bmatrix} 0 & 0 \\ 0 & D^d[C(XA^d+(BC)^d(XA-A^d))+D^dCX]BD \end{bmatrix}, \end{aligned}$$

where N^d is given as in Corollary 3.3 and X is represented by (9) *such that* $\mathrm{ind}(N)=r$ *and* $\mathrm{ind}(D)=s$.

Utilizing Corollary 4.3, we obtain the expression for M^d as in [3, Theorem 2.3].

Corollary 4.4. [3, Theorem 2.3] *Let M be a matrix of the form* (3) *and N be a matrix of the form* (1)*, where A, D and BC are square matrices such that A and BC are of the same size. If*

$$ABC=0 \quad \text{and} \quad BD=0,$$

then

$$M^d=\begin{bmatrix} I & 0 \\ 0 & D^{\pi} \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i N^{(i+1)d} + \sum_{i=0}^{r-1} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} N^i \begin{bmatrix} (BC)^{\pi}-XA^2 & -XAB \\ -CXA & I-CXB \end{bmatrix},$$

where N^d is given as in Corollary 3.3 and X is represented by (9) *such that* $\mathrm{ind}(N)=r$ *and* $\mathrm{ind}(D)=s$.

We utilize Corollary 4.4 to get the next formula.

Corollary 4.5. *Let M be a matrix of the form* (3) *and N be a matrix of the form* (1), *where A, D and BC are square matrices such that A and BC are of the same size. If*

$$AB = 0 \quad \text{and} \quad BD = 0,$$

then

$$M^d = \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i N^{(i+1)d} + \sum_{i=0}^{r-1} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} N^i \begin{bmatrix} (BC)^\pi - XA^2 & 0 \\ -CXA & I - C(BC)^d B \end{bmatrix},$$

where N^d is given as in Corollary 3.4 and X is represented by (9) *such that* $\operatorname{ind}(N) = r$ *and* $\operatorname{ind}(D) = s$.

Similarly as Theorem 4.1, we can verify the following main result, which generalizes and unifies some more results than Theorem 4.1 in the literature.

Theorem 4.6. *Let M be a matrix of the form* (3) *and N be a matrix of the form* (1), *where A, D and BC are square matrices such that A and BC are of the same size. If*

$$A(BC)^2 = 0, \quad A^2BCA = 0, \quad ABCA^2 = 0, \quad (ABC)^2 = 0, \quad DCA = 0 \quad \text{and} \quad DCB = 0,$$

then

$$\begin{aligned} M^d \;=\; & \begin{bmatrix} I - XA^2 - (BC)^{2d}ABCA - XBC & -F_1B \\ -CXA - C[XA^d + (BC)^d(XA - A^d)]BC & I - F_3B \end{bmatrix} \sum_{i=0}^{r-1} N^i \begin{bmatrix} 0 & 0 \\ D^{(i+2)d}C & D^{(i+1)d} \end{bmatrix} \\ + & \sum_{i=0}^{s-1} N^{(i+1)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} + \sum_{i=0}^{s-2} N^{(i+3)d} \begin{bmatrix} 0 & 0 \\ D^{i+1}D^\pi C & 0 \end{bmatrix} \\ - & \begin{bmatrix} F_2D^dC & 0 \\ F_4D^dC & 0 \end{bmatrix} - N^{2d} \begin{bmatrix} 0 & 0 \\ D^eC & 0 \end{bmatrix}, \end{aligned}$$

where N^d is given by (6) *and X is given by* (9) *such that* $\operatorname{ind}(N) = r$ *and* $\operatorname{ind}(D) = s$.

Proof. Using the same notations as in the proof of Theorem 4.1, we observe that $QNQ = 0$ and $QN^2 = 0$. We utilize Theorem 3.2 and Lemma 2.3 to prove the rest as in the proof of Theorem 4.1. □

Remark 4.7. *Using Theorem 4.6, we can obtain some results in Remark 4.2 such as [16, Theorem 5.3], [6, Theorem 2.1] and [11, Theorem 1], and some others as follows*

1. *$BC = 0, DC = 0$ and D is nilpotent (see* [20, Lemma 2.2]*);*
2. *$ABC = 0, DC = 0$ and BC is nilpotent (or D is nilpotent) (see* [11, Theorem 2 and Theorem 3]*);*
3. *$BCB = 0$, $BCA = 0$, $DCA = 0$ and $DCB = 0$ (see* [35, Theorem 3.1]*).*

In particular, we next give some extra and specific corollaries as follows. Applying Theorem 4.6, we develop the formula for the Drazin inverse of M under the assumptions $ABC = 0$, $DCB = 0$ and $DCA = 0$.

Corollary 4.8. *Let M be a matrix of the form* (3) *and N be a matrix of the form* (1), *where A, D and BC are square matrices such that A and BC are of the same size. If*

$$ABC = 0, \quad DCB = 0 \quad \text{and} \quad DCA = 0,$$

then

$$
\begin{aligned}
M^d &= \begin{bmatrix} (BC)^\pi - XA^2 & -XAB \\ -CXA & I-CXB \end{bmatrix} \sum_{i=0}^{r-1} N^i \begin{bmatrix} 0 & 0 \\ D^{(i+2)d}C & D^{(i+1)d} \end{bmatrix} \\
&+ \sum_{i=0}^{s-1} N^{(i+1)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} + \sum_{i=0}^{s-2} N^{(i+3)d} \begin{bmatrix} 0 & 0 \\ D^{i+1}D^\pi C & 0 \end{bmatrix} \\
&- \begin{bmatrix} XBD^dC & 0 \\ C[XA^d + (BC)^d(XA - A^d)]BD^dC & 0 \end{bmatrix} - N^{2d} \begin{bmatrix} 0 & 0 \\ D^eC & 0 \end{bmatrix},
\end{aligned}
$$

where N^d is given as in Corollary 3.3 and X is represented by (9) *such that* $\operatorname{ind}(N) = r$ *and* $\operatorname{ind}(D) = s$.

We utilize Corollary 4.8 to develop the formula for the Drazin inverse of M in the case of [3, Theorem 2.2].

Corollary 4.9. [3, Theorem 2.2] *Let M be a matrix of the form* (3) *and N be a matrix of the form* (1), *where A, D and BC are square matrices such that A and BC are of the same size. If*

$$ABC = 0 \quad and \quad DC = 0,$$

then

$$M^d = \begin{bmatrix} I - XA^2 - (BC)^dBC & -XAB \\ -CXA & I-CXB \end{bmatrix} \sum_{i=0}^{r-1} N^i \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} + \sum_{i=0}^{s-1} N^{(i+1)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix},$$

where N^d is given as in Corollary 3.3 and X is represented by (9) *such that* $\operatorname{ind}(N) = r$ *and* $\operatorname{ind}(D) = s$.

Corollary 4.9 gives the next expression for M^d.

Corollary 4.10. *Let M be a matrix of the form* (3) *and N be a matrix of the form* (1), *where A, D and BC are square matrices such that A and BC are of the same size. If*

$$AB = 0 \quad and \quad DC = 0,$$

then

$$M^d = \begin{bmatrix} (BC)^\pi - XA^2 & 0 \\ -CXA & I-CXB \end{bmatrix} \sum_{i=0}^{r-1} N^i \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} + \sum_{i=0}^{s-1} N^{(i+1)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix},$$

where N^d is given as in Corollary 3.4 and X is represented by (9) *such that* $\operatorname{ind}(N) = r$ *and* $\operatorname{ind}(D) = s$.

In the end, we give an example with 4×4 matrices A, B, C and D and apply Theorem 4.1 to calculate M^d.

Example 4.11. *Let A, B and C be as in Example 3.5 and*

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that $D = D^2 = D^{\#}$, $BD = 0$ and $N^d = 0$ by Example 3.5. Applying Theorem 4.1, we calculate

$$M^d = \begin{bmatrix} 0 & a & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^d$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & c+ca & c+ca+ca^2+cbc & 1 & 1 & 1+cb & 1+cb+cab \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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