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Composition Operators on Normal Weight Dirichlet Space

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Abstract. By using Bergman ball and Carleson domain, the authors give several equivalent characterizations for which composition operator is bounded or compact on the normal weight Dirichlet type spaces in this paper.

1. Introduction

Let $\mathbb{C}$ be the complex plane. Throughout this paper we fix a positive integer n and let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ denote the Euclidean space of complex dimension n. For $w = (w_1, \cdots, w_n)$ and $z = (z_1, \cdots, z_n)$ in $\mathbb{C}^n$, define $\langle w, z\rangle = w_1\overline{z_1} + \cdots + w_n\overline{z_n}$. The unit ball in $\mathbb{C}^n$ is the set $B_n = \{w \in \mathbb{C}^n : |w| = \sqrt{\langle w, w\rangle} < 1\}$. The space of holomorphic functions in B_n is denoted by $H(B_n)$. For $h \in H(B_n)$ and $w \in B_n$, let

$$\nabla h(w) = \left(\frac{\partial h}{\partial w_1}(w), \cdots, \frac{\partial h}{\partial w_n}(w)\right) \text{ and } Rh(w) = \sum_{k=1}^{n} w_k \frac{\partial h}{\partial w_k}(w).$$

Let dv be the Lebesgue measure on B_n. Suppose S_n is the boundary of B_n. For $a \in B_n$ and $r > 0$, let φ_a be the involutive automorphism of B_n with $\varphi_a(0) = a$ and $\varphi_a(a) = 0$. Let Bergman ball $D(a, r) = \{z : z \in B_n \text{ and } \beta(z, a) < r\}$, where

$$\beta(z, a) = \frac{1}{2} \log \frac{1 + |\varphi_a(z)|}{1 - |\varphi_a(z)|}.$$

For $\eta \in S_n$ and $t > 0$, let Carleson domain $S(\eta, t) = \{z \in B_n : |1 - \langle z, \eta\rangle| < t\}$.

If there exists constant $c > 0$ such that $A_1 \geq cA_2$ (or $A_1 \leq cA_2$), then we write "$A_1 \gtrsim A_2$" (or "$A_1 \lesssim A_2$"). If "$A_1 \gtrsim A_2$" and "$A_1 \lesssim A_2$", then we call "$A_1 \asymp A_2$".

A positive continuous function ν on $[0, 1)$ is called a normal function if there exist constants $0 < a \leq b < \infty$ and $0 \leq s_0 < 1$ such that $\dfrac{\nu(s)}{(1 - s^2)^a}$ is decreasing, and $\dfrac{\nu(s)}{(1 - s^2)^b}$ is increasing on $[s_0, 1)$. For example

$$\nu(s) = (1 - s^2)^p \log^\beta \frac{e}{1 - s^2} \left\{\log \log \frac{e^2}{1 - s^2}\right\}^\alpha \quad (p > 0,\ \beta \text{ and } \alpha \text{ are real}).$$

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In order to simplify the proof, let $s_0 = 0$ in this paper.

Let ν be a normal function on $[0,1)$. For $p > 0$, the normal weight Dirichlet space $D^p_\nu(B_n)$ consists of all holomorphic functions h on B_n such that

$$\|h\|^p_{D^p_\nu} = |h(0)|^p + \int_{B_n} |\nabla h(w)|^p \frac{\nu^p(|w|)}{1-|w|^2}\, dv(w) < \infty,$$

In particular, $D^p_\nu(B_n)$ is the Dirichlet type space $D^p_\alpha(B_n)$ when $\nu(s) = (1-s^2)^{\frac{\alpha+1}{p}}$ $(\alpha > -1)$. Moreover, $D^p_\alpha(B_n)$ is the Dirichlet space when $\alpha = 0$. By similar treatment of Theorem 3.2 in [27], we may obtain that

$$\|h\|^p_{D^p_\nu} \asymp |h(0)|^p + \int_{B_n} |Rh(w)|^p \frac{\nu^p(|w|)}{1-|w|^2}\, dv(w) \ \text{ for } h \in D^p_\nu(B_n).$$

Let $\varphi : B_n \to B_n$ be a holomorphic mapping. The composition operator C_φ with the symbol φ on $H(B_n)$ is defined by

$$C_\varphi(f) = f \circ \varphi \quad (f \in H(B_n)).$$

Composition type operators have been studied for a long time, and a lot of results have been obtained (such as, [1-18], [21-26]). For Dirichlet type spaces, there have been a lot of results involving composition operators or weighted composition operators, such as [1-18]. However, most of the above results were given on unit disc $\mathbb{D}$ and $\nu(s) = (1-s^2)^{\frac{\alpha+1}{p}}$ $(\alpha > -1)$. As for using Carleson domain to characterize composition operators on Dirichlet type spaces, there are the following results:

Theorem A ([1]) Let $\alpha > -1$. Suppose φ is an analytic self-map of $\mathbb{D}$ and $d\mu_\alpha = |\varphi'(w)|^2(1-|w|^2)^\alpha\, dv(w)$ $(w \in \mathbb{D})$.

(1) C_φ is a bounded operator on $D^2_\alpha(\mathbb{D})$ if and only if

$$\mu_\alpha \varphi^{-1} S(\xi, t) = O(t^{\alpha+2}) \quad \text{for all } \xi \in \partial\mathbb{D} \text{ and } t > 0.$$

(2) C_φ is a compact operator on $D^2_\alpha(\mathbb{D})$ if and only if

$$\sup_{\xi\in\partial\mathbb{D}} \mu_\alpha \varphi^{-1} S(\xi, t) = o(t^{\alpha+2}) \ \ (t \to 0^+).$$

In this paper, we generalize Theorem A from the concrete measure $(1-|z|^2)^\alpha\, dv(z)$ to the abstract measure $\dfrac{\nu^p(|z|)\, dv(z)}{1-|z|^2}$. At the same time, the dimension is extended from one dimension to n dimensions. Otherwise, we combine Carleson domain with Bergman ball to discuss the conditions for which the composition operator is bounded or compact, and we give four equivalent characterizations respectively.

2. Some Lemmas

Lemma 2.1. *Let $r > 0$ and ν be a normal function on $[0,1)$. Suppose a and b are the parameters in the definition of ν. Then*

(1) $\dfrac{\nu(|z|)}{\nu(|w|)} \le \left(\dfrac{1-|z|^2}{1-|w|^2}\right)^a + \left(\dfrac{1-|z|^2}{1-|w|^2}\right)^b$ *for all $z, w \in B_n$.*

(2) $\nu(|z|) \asymp \nu(|w|)$ *for any $z \in B_n$ and $w \in D(z,r)$.*

These results come from Lemma 2.2 in [28].

Lemma 2.2. *Let $p > 0$ and ν be a normal function on $[0, 1)$. Suppose $\varphi = (\varphi_1, \cdots, \varphi_n)$ is a holomorphic self-map of B_n and $\varphi_l \in D_\nu^p(B_n)$ for all $l \in \{1, 2, \cdots, n\}$. If g is nonnegative measurable on B_n, then*

$$\int_{B_n} g(w)\, dm_{p,\nu,\varphi}(w) = \int_{B_n} g[\varphi(w)] \frac{|R\varphi(w)|^p \nu^p(|w|)}{1 - |w|^2}\, dv(w), \text{ where}$$

$$m_{p,\nu,\varphi}(A) = \int_{\varphi^{-1}(A)} \frac{|R\varphi(w)|^p \nu^p(|w|)}{1 - |w|^2}\, dv(w), \ \ R\varphi = (R\varphi_1, R\varphi_2, \cdots, R\varphi_n),$$

and A is any Borel measurable set in B_n.

Proof. First, the condition $\varphi_l \in D_\nu^p(B_n)$ for all $l \in \{1, 2, \cdots, n\}$ means that $|R\varphi(w)|^p \nu^p(|w|)(1 - |w|^2)^{-1} dv(w)$ is a finite measure on B_n. The rest of proof is similar to that of Lemma 2.1 in [21]. □

Lemma 2.3. *There is a positive integer N such that for any $0 < r \le 1$ one can find a sequence $\{w^j\} \subset B_n$ with $B_n = \bigcup_{j=1}^{\infty} D(w^j, r)$, and for each point $z \in B_n$ belongs to at most N of the sets $D(w^j, 4r)$.*

This result comes from Lemma 2.23 in [20].

Lemma 2.4. *Let $c > 0$ and $\delta > -1$. Then the integral*

$$\int_{B_n} \frac{(1 - |z|^2)^\delta\, dv(z)}{|1 - \langle w, z\rangle|^{n+1+\delta+c}} \asymp \frac{1}{(1 - |w|^2)^c} \text{ for all } w \in B_n.$$

This result comes from Proposition 1.4.10 in [19].

3. Main Results and Proofs

Theorem 3.1. *Suppose ν is a normal function on $[0, 1)$. For $p > 0$, let φ be a holomorphic self-map of B_n and $\varphi_l \in D_\nu^p(B_n)$ for all $l \in \{1, 2, \cdots, n\}$. Define a measure $d\mu_{p,\nu,\varphi}(z) = dm_{p,\nu,\varphi}\varphi^{-1}(z)$, where $dm_{p,\nu,\varphi}(z) = \dfrac{\nu^p(|z|)|R\varphi(z)|^p}{1 - |z|^2}\, dv(z)$ $(z \in B_n)$. Given $0 < r \le 1$, then the following four conditions are equivalent:*

(1) $\mu_{p,\nu,\varphi}[S(\eta, t)] \lesssim t^n \nu^p(1 - t)$ for all $\eta \in S_n$ and $0 < t < 1/2$.

(2) $\mu_{p,\nu,\varphi}[D(w, r)] \lesssim (1 - |w|^2)^n \nu^p(|w|)$ for all $w \in B_n$.

(3) There exists a sufficiently large β such that

$$\sup_{w \in B_n} \frac{(1 - |w|^2)^\beta}{\nu^p(|w|)} \int_{B_n} \frac{|R\varphi(z)|^p\, \nu^p(|z|)\, dv(z)}{|1 - \langle \varphi(z), w\rangle|^{n+\beta}(1 - |z|^2)} < \infty.$$

(4) $\displaystyle\int_{B_n} |\nabla f(z)|^p\, d\mu_{p,\nu,\varphi}(z) \lesssim \|f\|_{D_\nu^p}^p$ for all $f \in D_\nu^p(B_n)$.

Proof. (1) ⇒ (2).

For any $w \in B_n$ and $z \in D(w, r)$, if $|w|^2 > (3 + \tanh r)/4$, then it is easy to prove

$$|1 - \langle z, \frac{w}{|w|}\rangle| \le |1 - \langle z, w\rangle| + |\langle z, w\rangle - \langle z, \frac{w}{|w|}\rangle|$$

$$\le \frac{1 + \tanh r}{1 - \tanh r}(1 - |w|^2) + (1 - |w|) < \frac{2(1 - |w|^2)}{1 - \tanh r} < \frac{1}{2}. \tag{3.1}$$

This means that $D(w,r) \subset S[w/|w|, 2(1-|w|^2)/(1-\tanh r)]$. Otherwise,

$$1 - \frac{2(1-|w|^2)}{1-\tanh r} < |w| \Rightarrow \nu\left[1 - \frac{2(1-|w|^2)}{1-\tanh r}\right] \leq \left(\frac{4}{1-\tanh r}\right)^b \nu(|w|),$$

where b is the parameter in the definition of ν. Therefore, it is clear that

$$\begin{aligned}
\mu_{p,\nu,\varphi}[D(w,r)] &\leq \mu_{p,\nu,\varphi}\left[S\left(\frac{w}{|w|}, \frac{2(1-|w|^2)}{1-\tanh r}\right)\right] \\
&\lesssim \left\{\frac{2(1-|w|^2)}{1-\tanh r}\right\}^n \nu^p\left[1 - \frac{2(1-|w|^2)}{1-\tanh r}\right] \\
&\leq \frac{2^{n+2pb}}{(1-\tanh r)^{n+pb}}(1-|w|^2)^n \nu^p(|w|).
\end{aligned}$$

If $|w|^2 \leq (3+\tanh r)/4$, then

$$(1-|w|^2)^n \nu^p(|w|) \geq \left(\frac{1-\tanh r}{4}\right)^n \nu^p\left(\sqrt{\frac{3+\tanh r}{4}}\right).$$

This implies that $\mu_{p,\nu,\varphi}[D(w,r)] \leq \mu_{p,\nu,\varphi}(B_n) \lesssim 1 \lesssim (1-|w|^2)^n \nu^p(|w|)$.

(2) $\Rightarrow$ (3).

Let a and b be the parameters in the definition of ν. For any $w \in B_n$ and $\beta > pb$, by Lemmas 2.1-2.4, Lemma 2.24 and Lemma 2.20 in [20], we have

$$\begin{aligned}
&\frac{(1-|w|^2)^\beta}{\nu^p(|w|)} \int_{B_n} \frac{|R\varphi(z)|^p \, \nu^p(|z|) \, dv(z)}{|1-\langle\varphi(z), w\rangle|^{n+\beta}(1-|z|^2)} \\
&= \frac{(1-|w|^2)^\beta}{\nu^p(|w|)} \int_{B_n} \frac{d\mu_{p,\nu,\varphi}(z)}{|1-\langle z, w\rangle|^{n+\beta}} \leq \frac{(1-|w|^2)^\beta}{\nu^p(|w|)} \sum_{k=1}^{\infty} \int_{D(w^k,r)} \frac{d\mu_{p,\nu,\varphi}(z)}{|1-\langle z, w\rangle|^{n+\beta}} \\
&\leq \frac{(1-|w|^2)^\beta}{\nu^p(|w|)} \sum_{k=1}^{\infty} \mu_{p,\nu,\varphi}[D(w^k,r)] \sup_{z\in D(w^k,r)} \frac{1}{|1-\langle z, w\rangle|^{n+\beta}} \\
&\lesssim \frac{(1-|w|^2)^\beta}{\nu^p(|w|)} \sum_{k=1}^{\infty} \frac{\nu^p(|w^k|)}{1-|w^k|^2} \sup_{z\in D(w^k,r)} \int_{D(z,r)} \frac{1}{|1-\langle u, w\rangle|^{n+\beta}} \, dv(u) \\
&\leq \frac{(1-|w|^2)^\beta}{\nu^p(|w|)} \sum_{k=1}^{\infty} \frac{\nu^p(|w^k|)}{1-|w^k|^2} \int_{D(w^k,2r)} \frac{1}{|1-\langle u, w\rangle|^{n+\beta}} \, dv(u) \\
&\lesssim \frac{(1-|w|^2)^\beta}{\nu^p(|w|)} \sum_{k=1}^{\infty} \int_{D(w^k,4r)} \frac{\nu^p(|u|) \, dv(u)}{|1-\langle u, w\rangle|^{n+\beta}(1-|u|^2)} \\
&\leq N \frac{(1-|w|^2)^\beta}{\nu^p(|w|)} \int_{B_n} \frac{\nu^p(|u|) \, dv(u)}{|1-\langle u, w\rangle|^{n+\beta}(1-|u|^2)} \\
&\lesssim \int_{B_n} \frac{(1-|w|^2)^{\beta-pb}(1-|u|^2)^{pb-1} \, dv(u)}{|1-\langle u, w\rangle|^{n+\beta}} + \int_{B_n} \frac{(1-|w|^2)^{\beta-pa}(1-|u|^2)^{pa-1} \, dv(u)}{|1-\langle u, w\rangle|^{n+\beta}} \\
&\lesssim 1.
\end{aligned}$$

(2) $\Rightarrow$ (4).

For any $f \in D^p_\nu(B_n)$, analogous to the proof of "(2) ⇒ (3)" we have

$$\begin{aligned}\int_{B_n} |\nabla f(z)|^p \, d\mu_{p,\nu,\varphi}(z) &\le \sum_{k=1}^{\infty} \int_{D(w^k,r)} |\nabla f(z)|^p \, d\mu_{p,\nu,\varphi}(z) \\ &\lesssim N \int_{B_n} \frac{|\nabla f(w)|^p \nu^p(|w|) \, dv(w)}{1-|w|^2} = N\|f\|^p_{D^p_\nu}.\end{aligned}$$

(3) ⇒ (2).

For any $w \in B_n$, it follows from Lemma 2.20 in [20] that

$$\begin{aligned} 1 &\gtrsim \frac{(1-|w|^2)^\beta}{\nu^p(|w|)} \int_{B_n} \frac{|R\varphi(z)|^p \, \nu^p(|z|) \, dv(z)}{|1-\langle \varphi(z), w\rangle|^{n+\beta}(1-|z|^2)} \\ &\ge \frac{(1-|w|^2)^\beta}{\nu^p(|w|)} \int_{D(w,r)} \frac{d\mu_{p,\nu,\varphi}(z)}{|1-\langle z, w\rangle|^{n+\beta}} \asymp \frac{\mu_{p,\nu,\varphi}[D(w,r)]}{(1-|w|^2)^n \nu^p(|w|)}.\end{aligned}$$

(4) ⇒ (1).

For any $\eta \in S_n$ and $0 < t < 1/2$, we take

$$f_{t,\eta}(z) = \frac{t^{2b+1}}{\nu(1-t)[1-(1-t)\langle z,\eta\rangle]^{\frac{n}{p}+2b}} \quad (z \in B_n).$$

It follows from Lemma 2.1 and Lemma 2.4 that

$$\begin{aligned} &\frac{(n+2pb)^n \mu_{p,\nu,\varphi}[S(\eta,t)]}{p^p 2^{n+2pb+2p} t^n \nu^p(1-t)} \\ &\le \int_{S(\eta,t)} |\nabla f_{t,\eta}(z)|^p \, d\mu_{p,\nu,\varphi}(z) \\ &\lesssim \int_{B_n} \frac{t^{2pb+p}}{\nu^p(1-t)|1-\langle z,(1-t)\eta\rangle|^{n+2pb+p}} \frac{\nu^p(|z|)}{1-|z|^2} \, dv(z) \\ &\lesssim \int_{B_n} \frac{t^{2pb+p}(1-|z|^2)^{pa-1}[1-(1-t)^2]^{-pa}}{|1-\langle z,(1-t)\eta\rangle|^{n+2pb+p}} \, dv(z) + \int_{B_n} \frac{t^{2pb+p}(1-|z|^2)^{pb-1}[1-(1-t)^2]^{-pb}}{|1-\langle z,(1-t)\eta\rangle|^{n+2pb+p}} \, dv(z) \\ &\lesssim 1.\end{aligned}$$

This shows that $\mu_{p,\nu,\varphi}[S(\eta,t)] \lesssim t^n \nu^p(1-t)$ for all $\eta \in S_n$ and $0 < t < 1/2$.

The proof is completed. □

Remark 3.2. *We know that*

$$\begin{aligned} \|C_\varphi f\|^p_{D^p_\nu} &\asymp |f[\varphi(0)]|^p + \int_{B_n} |R[C_\varphi f](z)|^p \frac{\nu^p(z)}{1-|z|^2} \, dv(z) \\ &= |f[\varphi(0)]|^p + \int_{B_n} |\langle (\nabla f)[\varphi(z)], \overline{R\varphi(z)}\rangle|^p \frac{\nu^p(z)}{1-|z|^2} \, dv(z) \\ &\le |f[\varphi(0)]|^p + \int_{B_n} |\nabla f(z)|^p \, d\mu_{p,\nu,\varphi}(z).\end{aligned}$$

By Theorem 3.1, C_φ is a bounded operator on $D^p_\nu(B_n)$ when one of (1)-(3) holds.

Theorem 3.3. *Suppose ν is a normal function on $[0,1)$. For $p>0$, let φ be a holomorphic self-map of B_n and $\varphi_l \in D^p_\nu(B_n)$ for all $l\in\{1,2,\cdots,n\}$. Define a measure $d\mu_{p,\nu,\varphi}(z)=dm_{\varphi,p,\nu}\varphi^{-1}(z)$, where $dm_{\varphi,p,\nu}(z)=\dfrac{\nu^p(|z|)|R\varphi(z)|^p}{1-|z|^2}dv(z)$ $(z\in B_n)$. Given $0<r\le 1$, then the following four conditions are equivalent:*

(1) $\sup\limits_{\eta\in S_n}\mu_{p,\nu,\varphi}[S(\eta,t)]=o[t^n\nu^p(1-t)]\quad (t\to 0^+)$.

(2) $\mu_{p,\nu,\varphi}[D(w,r)]=o[(1-|w|^2)^n\nu^p(|w|)]\quad (|w|\to 1^-$ *for* $w\in B_n)$.

(3) There exists a sufficiently large β such that

$$\lim_{|w|\to 1^-}\frac{(1-|w|^2)^\beta}{\nu^p(|w|)}\int_{B_n}\frac{|R\varphi(z)|^p\,\nu^p(|z|)\,dv(z)}{|1-\langle\varphi(z),w\rangle|^{n+\beta}(1-|z|^2)}=0\ \textit{for}\ w\in B_n.$$

(4) $\lim\limits_{k\to\infty}\int_{B_n}|\nabla f_k(z)|^p\,d\mu_{p,\nu,\varphi}(z)=0$, *where $\{f_k\}$ is any sequence such that $\{f_k\}$ converges to 0 uniformly on any compact subset of B_n and* $\sup\limits_{k\ge 1}\|f_k\|_{D^p_\nu}\le 1$.

Proof. (1) $\Rightarrow$ (2).

Let $\sup\limits_{\eta\in S_n}\mu_{p,\nu,\varphi}[S(\eta,t)]=o[t^n\nu^p(1-t)]\quad(t\to 0^+)$. For any $w\in B_n$ and $|w|^2>(3+\tanh r)/4$, we write $\eta=w/|w|$. It follows from (3.1) that

$$\frac{\mu_{p,\nu,\varphi}[D(w,r)]}{(1-|w|^2)^n\nu^p(|w|)}\le \mu_{p,\nu,\varphi}[S(\eta,\frac{2(1-|w|^2)}{1-\tanh r})]/(1-|w|^2)^n\nu^p(|w|).$$

It follows from $\dfrac{2(1-|w|^2)}{1-\tanh r}\to 0^+$ $(|w|\to 1^-)$ that

$$\lim_{|w|\to 1^-}\frac{\mu_{p,\nu,\varphi}[D(w,r)]}{(1-|w|^2)^n\nu^p(|w|)}=0.$$

(2) $\Rightarrow$ (3).

Let $\mu_{p,\nu,\varphi}[D(w,r)]=o[(1-|w|^2)^n\nu^p(|w|)]\quad(|w|\to 1^-$ for $w\in B_n)$. Then for any $\varepsilon>0$, there exists a $0<\delta_0<1$ such that

$$\frac{\mu_{p,\nu,\varphi}[D(w,r)]}{(1-|w|^2)^n\nu^p(|w|)}<\varepsilon\ \text{ when } |w|>\delta_0 \text{ and } w\in B_n. \tag{3.2}$$

Take the sequence $\{w^j\}$ in Lemma 2.3 and let $|w^j|\to 1^-$ when $j\to\infty$. Therefore, there exists a positive integer J_0 such that $|w^j|>\delta_0$ when $j>J_0$. Let $A=\bigcup_{j=1}^{J_0}D(w^j,r)$. Let b be the parameter in the definition of ν. By Lemmas 2.1-2.4, Lemma 2.24 and Lemma 2.20 in [20], (3.2), we have

$$\begin{aligned}
&\frac{(1-|w|^2)^\beta}{\nu^p(|w|)}\int_{B_n}\frac{|R\varphi(z)|^p\,\nu^p(|z|)\,dv(z)}{|1-\langle\varphi(z),w\rangle|^{n+\beta}(1-|z|^2)}\\
&=\frac{(1-|w|^2)^\beta}{\nu^p(|w|)}\int_{B_n}\frac{d\mu_{p,\nu,\varphi}(z)}{|1-\langle z,w\rangle|^{n+\beta}}\\
&\le\frac{(1-|w|^2)^\beta}{\nu^p(|w|)}\sum_{j=1}^{J_0}\mu_{p,\nu,\varphi}(B_n)\sup_{z\in\overline{A}}\frac{1}{|1-\langle z,w\rangle|^{n+\beta}}+\frac{(1-|w|^2)^\beta}{\nu^p(|w|)}\sum_{j=J_0+1}^{\infty}\mu_{p,\nu,\varphi}[D(w^j,r)]\sup_{z\in D(w^j,r)}\frac{1}{|1-\langle z,w\rangle|^{n+\beta}}\\
&\lesssim(1-|w|^2)^{\beta-pb}+N\varepsilon\frac{(1-|w|^2)^\beta}{\nu^p(|w|)}\int_{B_n}\frac{\nu^p(|u|)\,dv(u)}{|1-\langle u,w\rangle|^{n+\beta}(1-|u|^2)}\\
&\lesssim(1-|w|^2)^{\beta-pb}+\varepsilon\to\varepsilon\ \ (|w|\to 1^-).
\end{aligned}$$

It follows from the arbitrariness of ε that

$$\lim_{|w|\to 1^-} \frac{(1-|w|^2)^\beta}{\nu^p(|w|)} \int_{B_n} \frac{|R\varphi(z)|^p \ \nu^p(|z|)\ dv(z)}{|1-\langle \varphi(z), w\rangle|^{n+\beta}(1-|z|^2)} = 0.$$

(2) $\Rightarrow$ (4).

Let $\{f_k\}$ be any sequence which converges to 0 uniformly on any compact set of B_n and $\sup_{k\geq 1}\|f_k\|_{D^p_\nu} \leq 1$. It is similar to the proof of "(2) $\Rightarrow$ (3)". We have

$$\begin{aligned}\int_{B_n} |\nabla f_k(z)|^p \ d\mu_{p,\nu,\varphi}(z) &\leq \sum_{j=1}^{\infty} \int_{D(w^j,r)} |\nabla f_k(z)|^p \ d\mu_{p,\nu,\varphi}(z)\\ &\lesssim \sum_{j=1}^{J_0} \mu_{p,\nu,\varphi}(B_n) \sup_{z\in \overline{A}} |f_k(z)|^p + N\varepsilon \int_{B_n} \frac{|\nabla f_k(w)|^p \nu^p(|w|)\ dv(w)}{1-|w|^2}\\ &\leq \sum_{j=1}^{J_0} \mu_{p,\nu,\varphi}(B_n) \sup_{z\in \overline{A}} |f_k(z)|^p + N\varepsilon \to N\varepsilon \ \ (k\to\infty).\end{aligned}$$

It follows from the arbitrariness of ε that $\lim_{k\to\infty} \int_{B_n} |\nabla f_k(z)|^p \ d\mu_{p,\nu,\varphi}(z) = 0$.

(3) $\Rightarrow$ (2).

For any $w \in B_n$, if $|w| \to 1^-$, then it follows from Lemma 2.20 in [20] that

$$\begin{aligned}0 \leftarrow & \frac{(1-|w|^2)^\beta}{\nu^p(|w|)} \int_{B_n} \frac{|R\varphi(z)|^p \ \nu^p(|z|)\ dv(z)}{|1-\langle \varphi(z), w\rangle|^{n+\beta}(1-|z|^2)}\\ \geq & \frac{(1-|w|^2)^\beta}{\nu^p(|w|)} \int_{D(w,r)} \frac{d\mu_{p,\nu,\varphi}(z)}{|1-\langle z, w\rangle|^{n+\beta}} \asymp \frac{\mu_{p,\nu,\varphi}[D(w,r)]}{(1-|w|^2)^n \nu^p(|w|)}.\end{aligned}$$

(4) $\Rightarrow$ (1).

Assume $\sup_{\eta\in S_n} \mu_{p,\nu,\varphi}[S(\eta,t)] \neq o[t^n \nu^p(1-t)]$ as $t\to 0^+$. Then there exist $\{\eta^k\}\subset S_n$, $c>0$ and $\{t_k\}\subset (0,1)$ with $t_k \to 0$ such that $\mu_{p,\nu,\varphi}[S(\eta^k,t_k] \geq c t_k^n \nu^p(1-t_k)$.

Let a and b be the parameters in the definition of ν. For $\delta > b$, we take

$$f_k(z) = \frac{t_k^{\delta+1}}{\nu(1-t_k)(1-\langle z, (1-t_k)\eta^k\rangle)^{\frac{n}{p}+\delta}} \ \ (z\in B_n).$$

It follows from Lemma 2.1 and Lemma 2.4 that $\|f_k\|_{D^p_\nu} \lesssim 1$, and $\{f_k\}$ converges to 0 uniformly on any compact set of B_n. If $k\to\infty$, then we have

$$\begin{aligned}0 \leftarrow & \int_{B_n} |\nabla f_k(z)|^p \ d\mu_{p,\nu,\varphi}(z) = \int_{B_n} \frac{t_k^{p\delta+p}(1-t_k)^p (n+p\delta)^p \ d\mu_{p,\nu,\varphi}(z)}{p^p \nu^p(1-t_k)|1-\langle z,(1-t_k)\eta^k\rangle|^{n+p\delta+p}}\\ \geq & \frac{(n+p\delta)^p \ \mu_{p,\nu,\varphi}[S(\eta^k,t_k)]}{p^p 2^{n+p\delta+2p} t_k^n \nu^p(1-t_k)} \geq \frac{(n+p\delta)^p \ c}{p^p 2^{n+p\delta+2p}}.\end{aligned}$$

This contradiction means that $\sup_{\eta\in S_n} \mu_{p,\nu,\varphi}[S(\eta,t)] = o[t^n \nu^p(1-t)]$ as $t\to 0^+$.

The proof is completed. □

Remark 3.4. *Theorem 3.3 shows that C_φ is a compact operator on $D^p_\nu(B_n)$ when one of (1)-(3) in Theorem 3.3 holds.*

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