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A Discrete Boundary Value Problem with Point Interaction

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Abstract. This paper is concerned with a boundary value problem (BVP) for discrete Sturm-Liouville equation with point interaction and boundary conditions depending on a hyperbolic eigenvalue parameter. This paper presents some spectral and scattering properties of this BVP in terms of Jost solution, scattering solutions, scattering function, continuous and discrete spectrum. In addition, the resolvent operator of the BVP is obtained to get the properties of eigenvalues. Furthermore, an example is considered as a special case of the main problem to demonstrate the effectiveness of our results.

1. Introduction

Boundary value problems in physics and applied mathematics have been intensively studied in many articles. Many papers such as $[1-3]$ consist boundary value problems on infinite intervals which frequently occur in mathematical modeling of various applied problems, for example, analysis of the mass transfer on a rotating disk in a non-Newtonian fluid [1] , discussion of electrostatic probe measurements [2], heat transfer in the radial flow between parallel circular disks [1] , and plasma physics [3] . Recently, many researchers have paid more attention to the boundary value problems on unbounded intervals with a point interaction in terms of spectral and scattering analysis, for instance see $[4-12]$. Boundary value problems with point interaction have discontinuities inside an interval and have great interest in mathematical physics and quantum mechanics. To solve these discontinuities, some extra conditions are necessary which are on the discontinuous point. These extra conditions are often called interface conditions, point interactions, transmission conditions, jump conditions or impulsive conditions. Since the theory of discrete Sturm-Liouville equations is related to theory of continuous Sturm-Liouville equations, for the mathematical theory of discrete Sturm-Liouville equations with point interaction, we refer to $[13-17]$. It is well-known that differential equations with point interaction and discrete analogs of them are widely used in mechanical engineer, medical science, life science and other fields. Many researchers have made a great contribution to this topic. There are also some recent works on spectral and scattering analysis of boundary value problems with point interactions or impulsive conditions $[4-12]$.

In this work, we are interested in a boundary value problem with a point interaction and it is generated by a discrete Sturm-Liouville equation. Differently from these recent works, our boundary value problem consists hyperbolic eigenparameter in boundary conditions that provides a new perspective in terms of applications.

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Let us introduce a discrete Sturm-Liouville boundary value problem with a point interaction (BVP)

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n \in \mathbb{N}\backslash\{k-1, k, k+1\} \tag{1}$$

$$(\alpha_0 + \alpha_1\lambda)\, y_1 + (\beta_0 + \beta_1\lambda)\, y_0 = 0,\ \alpha_0\beta_1 - \alpha_1\beta_0 \neq 0,\ \beta_1 \neq 0,\ \alpha_1 \neq \frac{\beta_0}{a_0} \tag{2}$$

$$\begin{aligned} y_{k+1} &= \gamma_1 y_{k-1} \\ y_{k+2} &= \gamma_2 y_{k-2}, \end{aligned} \tag{3}$$

where $\gamma_1\gamma_2 \neq 0$, $\gamma_1, \gamma_2 \in \mathbb{R}$, $\lambda = 2\cosh z$ and $\{a_n\}_{n\in\mathbb{N}\cup\{0\}}$, $\{b_n\}_{n\in\mathbb{N}}$ are real sequences in $\ell_2(\mathbb{N})$. Throughout the paper, we will assume that $a_n \neq 0$ for all $n \in \mathbb{N}\cup\{0\}$ and the sequences $\{a_n\}$ and $\{b_n\}$ satisfy

$$\sum_{n=1}^{\infty} n\left(|1-a_n| + |b_n|\right) < \infty. \tag{4}$$

Also, we introduce two semi-strips given by

$$T_- := \left\{z \in \mathbb{C} :\ Re(z) < 0,\ Im(z) \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}$$

and $T{:} = T_- \cup T_0$, where $T_0 := \left\{z \in \mathbb{C} :\ Re(z) = 0,\ Im(z) \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}$. Throughout the paper, we will show the set T_0 by $\left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right]$ shortly. The rest of the paper is organized as follows. In Section 2, we get the Jost solution, scattering solutions and scattering function of the BVP (1)-(3). Also, we investigate the properties of the scattering function in this section. In Section 3, we present the Green function, resolvent operator and continuous spectrum of the problem. Furthermore, we give an asymptotic equation to point the eigenvalues of the BVP (1)-(3). Section 4 consists an unperturbated form of BVP (1)-(3). We investigate our main results for this unperturbated BVP as an example.

2. Jost Solution and Scattering Function of BVP (1)-(3)

In this part, we present the solutions of BVP (1)-(3) including Jost and scattering solutions, and the relations between them. Moreover, we find the scattering function of the boundary value problem (1)-(3) and examine the properties of this scattering function.

Let $P(z) = \{P_n(z)\}$ and $Q(z) = \{Q_n(z)\}$ are the fundamental solutions of (1) for $z \in T$ and $n = 0, 1, \ldots, k-1$ satisfying the initial conditions

$$P_0(z) = 0,\ P_1(z) = 1$$

and

$$Q_0(z) = \frac{1}{a_0},\ Q_1(z) = 0,$$

respectively. It is well-known that $P_n(z)$ is the first kind, $Q_n(z)$ is the second kind polynomials. Because for each $n \geq 0$, $P_n(z)$ is a polynomial of degree $(n-1)$ and $Q_n(z)$ is a polynomial of degree $(n-2)$. Also, they are entire functions with respect to z. The Wronskian of these solutions is equal to -1 for all $z \in \mathbb{C}$. Note that the Wronskian for arbitrary two solutions $y = \{y_n(z)\}$ and $u = \{u_n(z)\}$ of (1) is given by

$$W[y, u] := a_n\left[y_n(z)\, u_{n+1}(z) - y_{n+1}(z)\, u_n(z)\right].$$

So, $P_n(z)$ and $Q_n(z)$ are linear independent solutions of (1). As a result of this, we can get the other solutions of (1) as

$$\psi_n(z) = -(\beta_0 + \lambda\beta_1) P_n(z) + a_0(\alpha_0 + \lambda\alpha_1) Q_n(z), \; n = 0, 1, \ldots, k-1 \tag{5}$$

by using the solutions $P_n(z)$, $Q_n(z)$ and the boundary condition (2). Note that, equation (1) also has the following bounded solution represented by $f(z) = \{f_n(z)\}$,

$$f_n(z) = \rho_n e^{nz}\left(1 + \sum_{m=1}^{\infty} A_{nm} e^{mz}\right), \; n = k+1, k+2, \ldots$$

with the condition $\lim_{n\to\infty} e^{-nz} f_n(z) = 1$, $z \in T$, where ρ_n and A_{nm} are given in terms of the sequences $\{a_n\}$ and $\{b_n\}$ as

$$\begin{aligned}
\rho_n &:= \prod_{k=n}^{\infty} a_k^{-1}, \\
A_{n1} &:= -\sum_{k=n+1}^{\infty} b_k, \\
A_{n2} &:= \sum_{k=n+1}^{\infty} \left\{1 - a_k^2 + b_k \sum_{p=k+1}^{\infty} b_p\right\}, \\
A_{n,m+2} &:= A_{n+1,m} + \sum_{k=n+1}^{\infty} \left\{\left(1 - a_k^2\right) A_{k+1,m} - b_k A_{k,m+1}\right\}
\end{aligned}$$

for $m \geq 1$. $f_n(z)$ is analytic according to z in $\mathbb{C}_{left} := \{z \in \mathbb{C} : Re(z) < 0\}$, continuous in $\overline{\mathbb{C}}_{left} := \{z \in \mathbb{C} : Re(z) \leq 0\}$ and $f_n(z) = f_n(z + 2\pi)$ for all $z \in \overline{\mathbb{C}}_{left}$. For $n = k+1, k+2, \ldots$ we also shall introduce the unbounded solution of (1) by $\widetilde{f}_n(z)$ with the condition $\lim_{n\to\infty} e^{nz}\widetilde{f}_n(z) = 1$, $z \in \overline{\mathbb{C}}_{left}$. By using the definition of Wronskian, it is clear that

$$W\left[f_n(z), \widetilde{f}_n(z)\right] = -2\sinh z$$

for $n = k+1, k+2, \ldots$ and $z \in T$ and they are independent solutions of (1) for $z \in T \setminus \{0, \pi i\}$. Now, we will define the following function for $z \in T$ by using the solutions $f(z)$, $P(z)$ and $Q(z)$

$$E_n(z) = \begin{cases} v(z) P_n(z) + \tau(z) Q_n(z) & ; \quad n = 1, 2, \ldots, k-1 \\ f_n(z) & ; \quad n = k+1, k+2, \ldots \end{cases}. \tag{6}$$

Using the condition (3), we get for $z \in T$

$$\begin{aligned}
\frac{1}{\gamma_1} f_{k+1}(z) &= v(z) P_{k-1}(z) + \tau(z) Q_{k-1}(z) \\
\frac{1}{\gamma_2} f_{k+2}(z) &= v(z) P_{k-2}(z) + \tau(z) Q_{k-2}(z),
\end{aligned}$$

and from these equations, we find the coefficients $v(z)$ and $\tau(z)$ as

$$v(z) = -\frac{a_{k-2}}{\gamma_1\gamma_2}\left[\gamma_1 f_{k+2}(z) Q_{k-1}(z) - \gamma_2 f_{k+1}(z) Q_{k-2}(z)\right] \tag{7}$$

$$\tau(z) = \frac{a_{k-2}}{\gamma_1\gamma_2}\left[\gamma_1 f_{k+2}(z) P_{k-1}(z) - \gamma_2 f_{k+1}(z) P_{k-2}(z)\right] \tag{8}$$

for $z \in T$. The function $E(z) = \{E_n(z)\}$ is called the Jost solution of BVP (1)-(3). Since $P_n(z) = P_n(-z)$ and $Q_n(z) = Q_n(-z)$ for $z \in T$, then we find $v(-z) = \overline{v(z)}$ and $\tau(-z) = \overline{\tau(z)}$. Moreover, for all $z \in T \backslash \{0, i\pi\}$, we say that $f_n(z)$ and $f_n(-z)$ are independent solutions , because it is clear that

$$W[f_n(z), f_n(-z)] = -2\sinh z$$

for $z \in T \backslash \{0, i\pi\}$. Next, we will think other solution $F(z) = \{F_n(z)\}$ of BVP (1)-(3) by

$$F_n(z) := \begin{cases} \psi_n(z) & , \quad n = 1, 2, \dots, k-1 \\ c(z) f_n(z) + d(z) f_n(-z) & , \quad n = k+1, k+2, \dots \end{cases}$$

for $z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \backslash \{0, \pi i\}$. Since the solution $F(z)$ provides the condition (3), we get

$$c(z) = -\frac{a_{k+1}}{2\sinh z}\left[\gamma_1 f_{k+2}(-z)\psi_{k-1}(z) - \gamma_2 f_{k+1}(-z)\psi_{k-2}(z)\right] \tag{9}$$

and

$$d(z) = \frac{a_{k+1}}{2\sinh z}\left[\gamma_1 f_{k+2}(z)\psi_{k-1}(z) - \gamma_2 f_{k+1}(z)\psi_{k-2}(z)\right] \tag{10}$$

for $z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \backslash \{0, \pi i\}$. It follows from that $d(z) = c(-z) = \overline{c(z)}$ because of $\psi_n(z) = \psi_n(-z)$.

Lemma 2.1. *For all $z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \backslash \{0, \pi i\}$, the following equation holds*

$$W[E_n(z), F_n(z)] = \begin{cases} \dfrac{a_{k-2}}{a_{k+1}} \dfrac{2\sinh z}{\gamma_1\gamma_2} d(z) & , \quad n = 1, 2, \dots, k-1 \\[2ex] -2\sinh z d(z) & , \quad n = k+1, k+2, \dots \end{cases}.$$

Proof. *From the definition of wronskian for $n = 1, 2, \dots, k-1$, we write*

$$W[E_n(z), F_n(z)] = a_0\left[E_0(z)F_1(z) - F_0(z)E_1(z)\right].$$

Since $P_0(z) = 0$, $P_1(z) = 1$, $Q_0(z) = \frac{1}{a_0}$ and $Q_1(z) = 0$, the last equation gives

$$W[E_n(z), F_n(z)] = -a_0(\alpha_0 + \alpha_1\lambda)v(z) - (\beta_0 + \beta_1\lambda)\tau(z)$$

by using the definition of $\psi_1(z)$ and $\psi_0(z)$. Last equation is equal to $\frac{a_{k-2}}{a_{k+1}}\frac{2\sinh z}{\gamma_1\gamma_2}d(z)$ by (7), (8), (10) and the definition of $\psi(z)$. Similarly, for $n = k+1, k+2, \dots$, we write

$$W[E_n(z), F_n(z)] = a_{k+1}\left[E_{k+1}(z)F_{k+2}(z) - F_{k+1}(z)E_{k+2}(z)\right],$$

and the right side of this equation gives

$$a_{k+1}d(z)\left[f_{k+1}(z)f_{k+2}(-z) - f_{k+1}(-z)f_{k+2}(z)\right]$$

that equals to $d(z)W[f_n(z), f_n(-z)]_{n=k+1}$. Finally, for $n = k+1, k+2, \dots$ we get

$$W[E_n(z), F_n(z)] = d(z)W[f_n(z), f_n(-z)]_{n=k+1} = d(z)(-2\sinh z).$$

It completes the proof. □

Using the boundary condition (2) and the function $E_n(z)$, we define the Jost function of BVP (1)-(3) by

$$\begin{aligned} J(z) &:= (\alpha_0 + \lambda\alpha_1) E_1 + (\beta_0 + \lambda\beta_1) E_0 \\ &= (\alpha_0 + \lambda\alpha_1) v(z) + \frac{\tau(z)}{a_0}(\beta_0 + \lambda\beta_1). \end{aligned} \tag{11}$$

It can be easily written that J is analytic in $\mathbb{C}_{left}$ and continuous in $\overline{\mathbb{C}}_{left}$.

Lemma 2.2. *For all* $z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \setminus \{0, \pi i\}$ *,we get*

$$J(z) = -\frac{1}{a_0}\frac{a_{k-2}}{a_{k+1}}\frac{2\sinh z}{\gamma_1\gamma_2} d(z).$$

Proof. We can write from (5), (10) and (11)

$$\begin{aligned} d(z) &= \frac{a_{k+1}}{2\sinh z}\left[\gamma_1 f_{k+2}(z)\psi_{k-1}(z) - \gamma_2 f_{k+2}(z)\psi_{k-2}(z)\right] \\ &= \frac{a_{k+1}}{2\sinh z}\left[-(\beta_0 + \lambda\beta_1)\tau(z)\frac{\gamma_1\gamma_2}{a_{k-2}} - a_0(\alpha_0 + \lambda\alpha_1) v(z)\frac{\gamma_1\gamma_2}{a_{k-2}}\right] \\ &= \frac{a_{k+1}}{2\sinh z}\frac{\gamma_1\gamma_2}{a_{k-2}}\left[-(\beta_0 + \lambda\beta_1)\tau(z) - a_0(\alpha_0 + \lambda\alpha_1) v(z)\right] \\ &= -a_0\frac{a_{k+1}}{a_{k-2}}\frac{\gamma_1\gamma_2}{2\sinh z}\left[\frac{1}{a_0}(\beta_0 + \lambda\beta_1)\tau(z) + (\alpha_0 + \lambda\alpha_1) v(z)\right] \\ &= -a_0\frac{a_{k+1}}{a_{k-2}}\frac{\gamma_1\gamma_2}{2\sinh z} J(z). \end{aligned}$$

It completes the proof of Lemma 2.2. □

Theorem 2.3. *For all* $z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \setminus \{0, \pi i\}$, $d(z) \neq 0$.

Proof. Let $d(z_0) = 0$ for at least $z_0 \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \setminus \{0, \pi i\}$. Since

$$d(z) = c(-z) = \overline{c(z)}$$

for all $z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \setminus \{0, \pi i\}$, we can write $d(z_0) = c(z_0) = 0$. It follows from that $F_n(z_0) = 0$ for all $\mathbb{N} \cup \{0\}$, but it gives a contradiction. It completes the proof of theorem. □

Definition 2.4. *The function* $S(z) := \frac{\overline{J(z)}}{J(z)}$, $z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \setminus \{0, \pi i\}$ *is called the scattering function of BVP (1)-(3).*

We know that there is a relation between $J(z)$ and $d(z)$, we give the scattering function by $d(z)$

$$S(z) = \frac{\overline{J(z)}}{J(z)} = \frac{J(-z)}{J(z)} = -\frac{\overline{d(z)}}{d(z)} = -\frac{d(-z)}{d(z)},\ z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \setminus \{0, \pi i\}. \tag{12}$$

Theorem 2.5. *For all* $z \in \left[-\frac{\pi}{2}i, \frac{3\pi}{2}i\right] \setminus \{0, \pi i\}$, *the scattering function satisfies*

$$S(-z) = S^{-1}(z) = \overline{S(z)}, \quad |S(z)| = 1.$$

Proof. Using the definition of function J and (12), we obtain

$$S(-z)=\frac{J(z)}{J(-z)}=S^{-1}(z)=\overline{S(z)}$$

for all $z\in\left[-\frac{\pi}{2}i,\frac{3\pi}{2}i\right]\setminus\{0,\pi i\}$. Furthermore,

$$|S(z)|^2=\overline{S(z)}S(z)=S^{-1}(z)S(z)=1$$

and it gives us $|S(z)|=1$ for all $z\in\left[-\frac{\pi}{2}i,\frac{3\pi}{2}i\right]\setminus\{0,\pi i\}$. $\square$

In the following, we will define another solution $G(z)=\{G_n(z)\}$ of the BVP (1)-(3) for all $z\in T$ to get the resolvent operator in next section

$$G_n(z):=\begin{cases}\psi_n(z) & ,\quad n=1,2,\ldots,k-1\\ q(z)f_n(z)+k(z)\widetilde{f_n}(z) & ,\quad n=k+1,k+2,\ldots\end{cases},$$

where

$$q(z)=-\frac{a_{k+1}}{2\sinh z}\left[\gamma_1\psi_{k-1}(z)\widetilde{f_{k+2}}(z)-\gamma_2\psi_{k-2}(z)\widetilde{f_{k+1}}(z)\right]$$

and

$$k(z)=\frac{a_{k+1}}{2\sinh z}\left[\gamma_1\psi_{k-1}(z)f_{k+2}(z)-\gamma_2\psi_{k-2}(z)f_{k+1}(z)\right].$$

Note that for all $z\in T$

$$W[E_n(z),G_n(z)]=\begin{cases}\frac{a_{k-2}}{a_{k+1}}\frac{2\sinh z}{\gamma_1\gamma_2}d(z) & ,\quad n=1,2,\ldots,k-1\\ -2\sinh z\,d(z) & ,\quad n=k+1,k+2,\ldots\end{cases}.$$

It is clear from Lemma 2.1 that for all $z\in\left[-\frac{\pi}{2}i,\frac{3\pi}{2}i\right]\setminus\{0,\pi i\}$, we can write $W[E_n(z),F_n(z)]=W[E_n(z),G_n(z)]$.

3. Resolvent Operator, Eigenvalues and Continuous Spectrum

In this section, we find the resolvent operator, discrete spectrum and continuous spectrum of the BVP (1)-(3) by using scattering solutions and Jost solution. We also obtain an asymptotic equation to definite the set of eigenvalues and continuous spectrum of BVP (1)-(3).

Theorem 3.1. *The resolvent operator of BVP (1)-(3) has the representation*

$$R_\lambda g_n:=\sum_{m=1}^{\infty}I_{nm}(z)g_m,\ \ \{g_m\}\in l_2(\mathbb{N}),$$

where

$$I_{nm}(z)=\begin{cases}-\frac{G_m(z)E_n(z)}{W[E_m(z),G_m(z)]} & ;\quad m\le n\\ \\ -\frac{G_n(z)E_m(z)}{W[E_m(z),G_m(z)]} & ;\quad m>n\end{cases}$$

is the Green function of BVP (1)-(3) for $m,n\neq k$.

Proof. To get the Green function of BVP (1)-(3), we need to find the solution of the following equation

$$\nabla\left(a_n \triangle y_n\right)+h_n y_n-\lambda y_n=g_n, \tag{13}$$

where $h_n=a_{n-1}+a_n+b_n$, $\triangle$ and ∇ are respectively the forward and backward difference operators defined by $\triangle y_n=y_{n+1}-y_n$ and $\nabla y_n=y_n-y_{n-1}$. Since $E_n(z)$ and $G_n(z)$ are the fundamental solutions of BVP (1)-(3), we can write the general solution $y=\{y_n(z)\}$ of (13) as

$$y_n(z)=s_n E_n(z)+t_n G_n(z), \tag{14}$$

where s_n, t_n are nonzero coefficients. Using the method of variation of parameters, we find q_n and t_n by

$$s_n=-\sum_{m=1}^{n}\frac{G_m g_m}{W\left[E_m,G_m\right]},\ m\neq k \tag{15}$$

$$t_n=-\sum_{m=n+1}^{\infty}\frac{E_m g_m}{W\left[E_m,G_m\right]},\ m\neq k. \tag{16}$$

It follows from (14), (15) and (16) that the Green function of the BVP (1)-(3) is $I_{nm}(z)$ given in Theorem 3.1. As a result of this, the resolvent operator of BVP (1)-(3) is $R_\lambda g_n:=\sum\limits_{m=1}^{\infty} I_{nm}(z)\, g_m,\ \{g_m\}\in l_2(\mathbb{N})$. □

With the help of Theorem 3.1 and the definition of eigenvalues, we write the set of eigenvalues σ_d of BVP (1)-(3) as

$$\sigma_d:=\{\lambda=2\cosh z: z\in T_-,\ d(z)=0\}.$$

Theorem 3.2. *For all $z\in T_-$, $d(z)$ satisfies the following asymptotic equation under the condition (4).*

$$d(z)=e^{4z}\left(D+o(1)\right),\ |z|\to\infty, D\neq 0.$$

Proof. As we know that, the polynomial function $P_n(z)$ is $(n-1)$ degree polynomial function and $Q_n(z)$ is $(n-2)$ degree polynomial function of λ, we can get that

$$\lim_{|z|\to\infty}\{\psi_n(z)\, e^{nz}\}=-\frac{\beta_1}{a_1 a_2\ldots a_{n-1}},\ n=0,1,2,\ldots,k-1 \tag{17}$$

and

$$\lim_{|z|\to\infty}\{f_n(z)\, e^{-nz}\}=\rho_n,\ n=k+1,\ k+2,\ldots \tag{18}$$

by using (2), (5) and (10) for $z\in T_-$, where $\rho_i:=\left(\prod\limits_{k=i}^{\infty} a_k\right)^{-1}$. After that we write

$$d(z)=\frac{a_{k+1}}{2\sinh z}\left\{\begin{array}{l}\gamma_1\psi_{k-1}(z)\, e^{(k-1)z} f_{k+2}(z)\, e^{-(k+2)z} e^{3z}\\ -\gamma_2\psi_{k-2}(z)\, e^{(k-2)z} f_{k+1}(z)\, e^{-(k+1)z} e^{3z}\end{array}\right\}$$

by using (10), (17) and (18). Then it follows from that

$$\begin{aligned} d(z)\, e^{-4z} &= \frac{a_{k+1}}{2\sinh z}\left\{-\gamma_1\frac{\beta_1}{a_1 a_2\ldots a_{k-2}}\rho_{k+2}+\frac{\gamma_2\beta_1}{a_1 a_2\ldots a_{k-3}}\rho_{k+1}\right\}\\ &= \frac{-D}{e^{2z}-1}, \end{aligned} \tag{19}$$

where

$$D=-\frac{a_{k+1}\beta_1\gamma_1\rho_{k+1}}{(a_1 a_2\ldots a_{k-3})}\left(\frac{a_{k+1}}{a_{k-2}}-\frac{\gamma_2}{\gamma_1}\right).$$

By using (19), we obtain $\lim\limits_{|z|\to\infty} d(z)\, e^{-4z}=D$ for all $z\in T_-$. □

We can say from Theorem 3.2 that the set of eigenvalues of BVP (1)-(3) is bounded under the condition (4).

Theorem 3.3. *Assume (4). Then the continuous spectrum of the BVP (1)-(3) is equal to the set* $[-2, 2]$.

Proof. First, we will show the operator generated by the BVP (1)-(3) by L. If we introduce the operators L_1 and L_2 generated by the following difference expressions in $l_2(\mathbb{N})$ together with (2) and (3)

$$(L_1 y)_n = y_{n-1} + y_{n+1}, \ \mathbb{N}\setminus\{k-1, \ k+1\}$$

$$(L_2 y)_n = (a_{n-1} - 1)\, y_{n-1} + b_n y_n + (a_n - 1)\, y_{n+1}, \ \mathbb{N}\setminus\{k-1, \ k, \ k+1\},$$

respectively. As you see $L = L_1 + L_2$ and we find that L_2 is a compact operator in $l_2(\mathbb{N})$ under the assumption (4) by using the compactness criteria in $l_2(\mathbb{N})$ (see [18]). On the other hand, we can write $L_1 = L_3 + L_4$, where L_3 is a selfadjoint operator with $\sigma_c(L_3) = [-2, 2]$, σ_c denotes the continuous spectrum and L_4 is a finite dimensional operator in $l_2(\mathbb{N})$. Since L_4 is a finite dimensional operator in $l_2(\mathbb{N})$, it is a compact operator in $l_2(\mathbb{N})$. It gives that the sum of two compact operators $L_2 + L_4$, is also a compact operator. Finally, we can write our main operator L as $L = L_3 + L_4 + L_2$ and by using Weyl theorem [19] of a compact perturbation, we find $\sigma_c(L_3) = \sigma_c(L) = [-2, 2]$. □

4. An Example

In this part, we will be interested in an example which is a special or simple case of our main problem. We will discuss our main results given in previous sections for this simple problem. In that way, readers will have an opportunity to understand the main results easily. Let us consider the following discrete boundary value problem with point interaction

$$y_{n-1} + y_{n+1} = 2\cosh z y_n, \ n \in \mathbb{N}\setminus\{3, 4, 5\} \tag{20}$$

$$(\alpha_0 + \alpha_1\lambda)\, y_1 + (\beta_0 + \beta_1\lambda)\, y_0 = 0, \ \alpha_0\beta_1 - \alpha_1\beta_0 \neq 0, \ \beta_1 \neq 0, \ \alpha_1 \neq \beta_0 \tag{21}$$

$$\begin{aligned} y_5 &= \gamma_1 y_3 \\ y_6 &= \gamma_2 y_1, \end{aligned} \tag{22}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_1, \gamma_2 \in \mathbb{R}$ and $\gamma_1\gamma_2 \neq 0$. It is evident that we get this problem taking $a_n \equiv 1$, $b_n \equiv 0$ for all $n \in \mathbb{N}$ and $k = 4$ in problem (1)-(3). Then, the solution $f_n(z)$ turns into e^{nz} and the fundamental solutions $P_n(z)$ and $Q_n(z)$ of (3) have the following values for the problem (20)-(22) for $n = 0, 1, 2, 3$

$$\begin{aligned} &P_0(z) = 0, \quad P_1(z) = 1, \quad P_2(z) = \lambda, \quad P_3(z) = \lambda^2 - 1 \\ &Q_0(z) = 1, \quad Q_1(z) = 0, \quad Q_2(z) = -1, \quad Q_3(z) = -\lambda. \end{aligned}$$

Thus, by using (6) and (10), we find $d(z)$ and Jost solution of this problem

$$d(z) = \frac{1}{2\sinh z}\left[\gamma_1 f_6(z)\, \psi_3(z) - \gamma_2 f_5(z)\, \psi_2(z)\right] \tag{23}$$

$$E_n(z) = \begin{cases} v(z)\, P_n(z) + \tau(z)\, Q_n(z) & ; \quad n = 0, 1, 2, 3 \\ f_n(z) & ; \quad n = 5, 6, 7, \ldots \end{cases}.$$

From the equation (23), we obtain the scattering function of (20)-(22)

$$S(z) = -e^{-10z}\left[\frac{\gamma_1\psi_3(z)\, e^7 - \gamma_2\psi_1(z)}{\gamma_1\psi_3(z)\, e^7 - \gamma_2\psi_2(z)}\right].$$

Also, continuous spectrum of the problem (20)-(22) is $[-2,2]$ from Theorem 3.3. It is necessary to find the zeros of $d(z)$ for $z \in T_-$ to obtain the eigenvalues of the problem (20)-(22). Because the set of eigenvalues of the problem (20)-(22) is given by

$$\sigma_d = \{\lambda = 2\cosh z : z \in T_-,\ d(z) = 0\} \tag{24}$$

where $d(z)$ is defined by (23). By using the values of $P_i(z), Q_i(z); i = 0,1,2,3$, we obtain

$$\psi_2(z) = -\lambda(\beta_0 + \lambda\beta_1) - (\alpha_0 + \lambda\alpha_1)$$
$$\psi_3(z) = \left(1 - \lambda^2\right)(\beta_0 + \lambda\beta_1) - \lambda(\alpha_0 + \lambda\alpha_1).$$

It follows from last equations and (23) that

$$d(z) = \frac{1}{2\sinh z}\left\{ \begin{array}{c} \gamma_1[\left(1 - \lambda^2\right)(\beta_0 + \lambda\beta_1) - \lambda(\alpha_0 + \lambda\alpha_1)]e^{6z} \\ +\gamma_2[\lambda(\beta_0 + \lambda\beta_1) + (\alpha_0 + \lambda\alpha_1)]e^{5z} \end{array} \right\}. \tag{25}$$

Equation (25) implies that $d(z) = 0$ if and only if

$$\frac{\gamma_2}{\gamma_1} = \frac{\left(\lambda^2 - 1\right)(\beta_0 + \lambda\beta_1) + \lambda(\alpha_0 + \lambda\alpha_1)}{\lambda(\beta_0 + \lambda\beta_1) + (\alpha_0 + \lambda\alpha_1)} e^z. \tag{26}$$

For the simplicity on calculations, if we choose $\alpha_0 = 1, \alpha_1 = -2, \beta_1 = 0$ and $\beta_0 = 2$ in (26), we find

$$(e^z - 1)^2 = \frac{\gamma_2}{\gamma_1}.$$

Let $\gamma_2 = 4a^2\gamma_1$, $a \in \mathbb{R}$. By using last equation, we find $e^z = 2a + 1$ or $e^z = 1 - 2a$.

Case 1: If $e^z = 2a + 1$, then it gives

$$z_m = \ln|2a + 1| + iArg(2a + 1) + 2im\pi,\ m \in \mathbb{Z}. \tag{27}$$

It is clear from (24) and (27) that the boundary value problem (20)-(22) has eigenvalues if and only if $\ln|2a + 1| < 0$. It implies that $-1 < 2a + 1 < 1$. Consequently, the necessary condition for the boundary value problem (20)-(22) to have an eigenvalue is that $-1 < a < 0$. These eigenvalues are real and lie on $(-\infty, -2) \cup (2, \infty)$. Note that, $a \neq 0$. Because the impulsive conditions do not work when $a = 0$. Moreover, $a \neq -1$. Because if $a = -1$, then $z_m = i(2m + 1)\pi$, $m \in \mathbb{Z}$. But only for $m = 0$, *i.e.*, $z_0 = i\pi \in \left[-\frac{\pi i}{2}, \frac{3\pi i}{2}\right]$. For $z_0 = i\pi$, we obtain $\lambda_0 = 2\cosh z_0 = -2$. Since $\lambda = -2$ is in continuous spectrum, it is not an eigenvalue of the boundary value problem (20)-(22).

Case 2: If $e^z = 1 - 2a$, then it gives

$$z_m = \ln|1 - 2a| + iArg(1 - 2a) + 2im\pi,\ m \in \mathbb{Z}. \tag{28}$$

It is clear from (24) and (28) that the boundary value problem (20)-(22) has eigenvalues if and only if $\ln|1 - 2a| < 0$. It implies that $-1 < 1 - 2a < 1$. Consequently, the necessary condition for the boundary value problem (20)-(22) to have an eigenvalue is that $0 < a < 1$. These eigenvalues are real and lie on $(-\infty, -2) \cup (2, \infty)$. Note that, $a \neq 0$ and $a \neq 1$ because of the same reasons with case 1.

Therefore, the necessary condition or the boundary value problem (20)-(22) to have an eigenvalue is that $a \in (-1, 0) \cup (0, 1)$.

5. Conclusion

The investigation of boundary value problems with point interaction on scattering analysis, specifically questions about the scattering solutions and scattering function, as well as its spectral analysis in terms of finding resolvent operator, spectrum, eigenvalues, is a recent research topic; see [4-12] and references therein. In this paper, we have concerned a discrete boundary value problem with point interaction and with boundary condition depending on spectral parameter. We have presented the main properties of scattering function of this BVP by using the scattering solutions and Jost solution. Furthermore, we have examined the resolvent operator, eigenvalues and continuous spectrum of the BVP. We also have looked on the effects of spectral parameter on the problem both when it is hyperbolic and when it is found in the boundary condition. We hope that the results obtained in this article will initiate new and interesting developments. The results can be generalized by taking $n \in \mathbb{Z}$ or by taking more complicated impulsive conditions.

References

[1] T. Y. Na, Computational Methods in Engineering Boundary Value Problems, Academic Press, New York, 1979.
[2] G. Maise, A. J. Sabadell, Electrostatic probe measurement in solid-propellant rocket exhausts, AIAA J. 8(5) (1970) 895-901. DOI: 10.2514/3.5784.
[3] R. P. Agarwal, D. O'Regan, Infinite interval problems modeling phenomena which arise in the theory of plasma and electrical potential theory, Stud. Appl. Math. 111 (2003) 339-358. DOI: 10.1111/1467-9590.t01-1-00237.
[4] E. Bairamov, Y. Aygar, D. Karslioglu, Scattering analysis and spectrum of discrete Schrödinger equations with transmission conditions, Filomat 31(17) (2017) 5391-5399. DOI: 10.2298/FIL1717391B.
[5] E. Bairamov, Y. Aygar, B. Eren, Scattering theory of impulsive Sturm-Liouville equations, Filomat 31(17) (2017) 5401-5409. DOI: 10.2298/FIL1717401B.
[6] Y. Aygar, E. Bairamov, Scattering theory of impulsive Sturm-Liouville Equation in quantum calculus, Bull. Malys. Math. Sci. Soc. 42(6) (2019) 3247-3259. DOI: 10.1007/s40840-018-0657-2.
[7] E. Bairamov, Y. Aygar, G. B. Oznur, Scattering properties of eigenparameter-dependent impulsive Sturm-Liouville equations, Bull. Malys. Math. Sci. Soc. 43(3) (2020) 2769-2781. DOI: 10.1007/s40840-019-00834-5.
[8] E. Bairamov, Y. Aygar, S. Cebesoy, Investigation of spectrum and scattering function of impulsive matrix difference operators, Filomat 33(5) (2019) 1301-1312. DOI: 10.2298/FIL1905301B.
[9] Y. Aygar Küçükevcilioğlu, E. Bairamov, G. G. Özbey, On the spectral and scattering properties of eigenparameter dependent discrete impulsive Sturm-Liouville equations, Turkish J. of Math. 45(2) (2021) 988-1000. DOI: 10.3906/mat-2101-45.
[10] E. Bairamov, S. Cebesoy, I. Erdal, Properties of eigenvalues and spectral singularities for impulsive quadratic pencil of difference operators, J. Appl. Anal. Comput. 9(4) (2019) 1454-1469. DOI: 10.11948/2156-907X.20180280.
[11] E. Bairamov, I. Erdal, S. Yardımcı Spectral properties of an impulsive Sturm-Liouville operator, J. Inequal. Appl. Paper No. 191 (2018) 16 ppNo. 191, 16 pp. DOI: 10.1186/s13660-018-1781-0.
[12] B. P. Allahverdiev, E. Bairamov, E. Ugurlu, Eigenparameter dependent Sturm-Liouville problems in boundary conditions with transmission conditions, J. Math. Anal. Appl. 401(1) (2013) 388-396. DOI: 10.1016/j.jmaa.2012.12.020.
[13] A. M. Samoilenko, N. A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995. DOI: 10.1142/2892.
[14] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, Theory of Impulsive Differential Equations, Series in Modern Applied Mathematics 6, World Scientific, Singapore, 1989. DOI: 10.1142/0906.
[15] D. D. Bainov, P. S. Simeonov, Impulsive Differential Equations: Asymptotic Properties of the Solutions, World Scientific, Singapore, 1995. DOI: 10.1142/2413.
[16] D. D. Bainov, P. S. Simeonov, Oscillation Theory of Impulsive Differential Equations, International Publications, Orlando, FL, USA, 1998.
[17] D. D. Bainov, P. S. Simeonov, Systems with Impulse Effect Stability Theory and Applications, Ellis Horwood Limited, Chichester, 1989.
[18] L. A. Lusternik, V. I. Sobolev, Elements of Functional Analysis, Halsted Press, New York, 1974.
[19] I. A. Glazman, Direct Methods of Qualitative Spectral Analysis of Singular Diferential Operators, Israel Program for Scientific Translations, Jerusalem, 1965.
[20] M. A. Naimark, Linear Differential Operators. I-II, Ungar, New York, 1968.