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Characterizations of Unconditionally Convergent and Weakly Unconditionally Cauchy Series via w_p^R-Summability, Orlicz-Pettis Type Theorems and Compact Summing Operator

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Abstract. In the present paper, we give a new characterization of unconditional convergent series and give some new versions of the Orlicz-Pettis theorem via FQ σ-family and a natural family $\mathcal{F}$ with the separation property S_1 through w_p^R-summability which may be considered as a generalization of the well-known strong p-Cesàro summability. Among other results, we obtain a new (weak) compactness criteria for the summing operator.

1. Introduction

In the study on analysis, the Orlicz-Pettis theorem is one of the important and useful results which is closely related to the sequence spaces, series and summability theory. In recent times, there is many versions of the classical Orlicz-Pettis theorem which are obtained from different types of summability methods. But, the first versions and the results of this theorem may be found in the famous monograph of Diestel [11], on the theory of sequences and series in Banach spaces.

The classical Orlicz-Pettis theorem says that weak subseries convergence of a series implies its (norm) subseries convergence in Banach spaces. Let us recall that the series $\sum_k x_k$ is subseries convergent if and only if $\sum_k x_{n_k}$ is convergent for all subsequences (n_k) of $\mathbb{N}$. For simplicity in notation, here and in what follows, the summation without limits runs from 1 to ∞ and $\mathbb{N}$ denotes the set of positive integers. Another well-known statement of Orlicz-Pettis theorem can be given as: *In a Banach space X, the series $\sum_k x_k$ is unconditionally convergent, that is, ℓ_∞-multiplier convergent, if the weakly sum $\sum_{k\in M} x_k$ exists for every $M \subset \mathbb{N}$* which exactly states that the equivalence of subseries convergence of the series in weak and strong topologies. This is why many authors study the new versions of Orlicz-Pettis theorem.

In a normed space X, a subseries convergent series is ℓ_∞-multiplier Cauchy, and if X is also sequentially complete, then the series is ℓ_∞-multiplier convergent. This result also coincides with unconditionally convergence of the series $\sum_k x_k$.

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A series $\sum_k x_k$ in a normed space X is unconditionally convergent -in short *uc*- (or unconditionally Cauchy -in short *uC*-) if the series $\sum_k x_{\pi(k)}$ converges (or a Cauchy series) for every permutation π of $\mathbb{N}$. It is called weakly unconditionally Cauchy -in short *wuC*- if for every permutation π of $\mathbb{N}$, the sequence $\left(\sum_{k=1}^n x_{\pi(k)}\right)$ is a weakly Cauchy sequence or as a useful result, $\sum_k x_k$ is *wuC* if and only if $\sum_k |x^*(x_k)| < \infty$ for all $x^* \in X^*$, the space of all linear and bounded (continuous) functionals defined on X. It is known that every *wuC* series in a Banach space X is *uc* if and only if X contains no copy of c_0; (cf. [6, pp. 42 and 44], [11, p. 44] and [26, p. 18]). The reader can also refer to the study of Albiac and Kalton [6] for specific investigations of Banach spaces.

In this study, we first establish a version of the well-known Orlicz-Pettis theorem by means of w_p^R-summability method and some natural families. Later, we employee (weakly) compact summing operator for characterization of subseries w_p^R- and w_{wp}^R-convergence of a series in a Banach space.

2. Background, Preliminaries and Notations

Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex fields, as usual, respectively. By ω, we denote the space of all real or complex valued sequences. Any vector subspace of ω is called a *sequence space*. We use the usual notations ℓ_∞, c and c_0 for the spaces of bounded, convergent and null sequences, respectively. A K-space is a locally convex sequence space X containing ϕ on which the coordinate functionals $\pi_k(x) = x_k$ are continuous for all $k \in \mathbb{N}$. Here and after, by ϕ we denote the space of finitely non-zero sequences spanned by the set $\{e^k : k \in \mathbb{N}\}$, where e^k denotes the sequence whose only non-zero term is 1 in the k^{th} place for all $k \in \mathbb{N}$. A complete linear metric (or complete normed linear) K-space is called an FK-space (or a BK-space).

Let X and Y be any given two subsets of ω, and $A = (a_{nk})_{n,k\in\mathbb{N}}$ be given an infinite matrix over the field $\mathbb{R}$ or $\mathbb{C}$. Then, we say that A defines a *matrix transformation* from X into Y and we denote it by writing $A : X \to Y$, if for every sequence $x = (x_k) \in X$ the A-transform $Ax = \{(Ax)_n\}$ of x exists and is in Y; where

$$(Ax)_n = \sum_k a_{nk}x_k, \quad (n \in \mathbb{N}). \tag{1}$$

By $(X : Y)$, we denote the class of all matrices A such that $A : X \to Y$. Thus, the A-transform of a sequence x exists if and only if the series on the right-hand side of (1) converges for each $n \in \mathbb{N}$. Furthermore, the sequence x is said to be A-summable to $a \in \mathbb{C}$ if Ax converges to a which is called the A-limit of x. The reader can refer to [8, 9, 25, 27] and [34] for recent results and related topics in summability.

A sequence $x = (x_k) \in \omega$ is said to be statistically convergent to a complex number α, if

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |x_k - \alpha| \ge \epsilon\}| = 0$$

holds for each $\epsilon > 0$, where the vertical bars indicate the cardinality of the enclosed set and $x = (x_k) \in \omega$ is said to be strongly p-Cesàro summable (or w_p-convergent) to a complex number α, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n |x_k - \alpha|^p = 0$$

holds, where $0 < p < \infty$.

The spaces of all statistical convergent and strongly p-Cesàro summable sequences are denoted by w_0 and w_p, respectively. In 1988, Connor [10] proved that the inclusion $w_p \subset w_0$ strictly holds, and if a sequence $x = (x_k)$ is bounded statistical convergent, then it belongs to w_p. We should also note that statistical convergence can not be considered in terms of locally convex FK-spaces. One can find early results and a brief history of first studies on statistical and strong p-Cesàro convergence of scalar valued sequences in [10].

Before proceeding further, let us consider the Cesàro mean C_1 of order one given by

$$\begin{array}{rccl} C_1 : & \omega & \longrightarrow & \omega \\ & x = (x_k) & \longmapsto & C_1x = \left(\frac{1}{n}\sum_{k=1}^n x_k\right)_{n\in\mathbb{N}}. \end{array} \tag{2}$$

By using (2), one can define the strong p-Cesàro summability: A sequence $x = (x_k)$ in a normed space X is said to be strong p-Cesàro summable or w_p-summable to $x_0 \in X$ and is denoted by $w_p - \lim_{k\to\infty} x_k = x_0$, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|x_k - x_0\|^p = 0, \text{ where } 0 < p < \infty.$$

Similarly, a sequence $x = (x_k)$ in a normed space X is said to be weak w_p-summable to $x_0 \in X$ and is denoted by $w - w_p - \lim_{k\to\infty} x_k = x_0$, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x^*(x_k) - x^*(x_0)|^p = 0, \text{ where } 0 < p < \infty$$

for all $x^* \in X^*$, [23].

3. Multiplier spaces of w_p^R-Summability and wuC series

In this section, we focus on an improvement of the well-known w_p-summability method which will be used in the rest of the paper, and give the corresponding results of León-Saavedra et al. [23, 24]. Unless stated otherwise, we assume throughout that $p \in (0, \infty)$.

Definition 3.1. *Let X be a normed space, $r = (r_k)$ be a sequence of positive reals and $R_n = \sum_{k=1}^{n} r_k$ such that $\lim_{n\to\infty} R_n = \infty$. The sequence $x = (x_k)$ in a normed space X is said to be w_p^R-summable to $x_0 \in X$, and is denoted by $w_p^R - \lim_{k\to\infty} x_k = x_0$, if*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{r_k}{R_n} \|x_k - x_0\|^p = 0.$$

The point $x_0 \in X$ is also called the w_p^R-sum of $x = (x_k)$. Let $X^{\mathbb{N}}$ be the space of all X-valued sequences. Then, the space $w_p^R(X)$ of all w_p^R-summable sequences in the normed space X is given

$$w_p^R(X) := \left\{ x = (x_k) \in X^{\mathbb{N}} : \lim_{n\to\infty} \sum_{k=1}^{n} \frac{r_k}{R_n} \|x_k - x_0\|^p = 0 \text{ for some } x_0 \in X \right\}.$$

The space of all weakly w_p^R-summable sequences in a normed space X is denoted by $w_{wp}^R(X)$, i.e.,

$$w_{wp}^R(X) := \left\{ x = (x_k) \in X^{\mathbb{N}} : \exists x_0 \in X \ni \lim_{n\to\infty} \sum_{k=1}^{n} \frac{r_k}{R_n} \left| x^*(x_k) - x^*(x_0) \right|^p = 0 \text{ for all } x^* \in X^* \right\}.$$

Thus, we use the notation $w_{wp}^R - \lim_{k\to\infty} x_k = x_0$ if $x = (x_k)$ belongs to the space $w_{wp}^R(X)$.

Regarding a sequence $x^* = (x_k^*)$ of bounded linear functionals in the dual space X^* of a normed space X is said to be weak w_p^R-summable to $x_0^* \in X^*$, and is denoted by $w_{w^*p}^R - \lim_{k\to\infty} x_k^* = x_0^*$, if*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{r_k}{R_n} |x_k^*(x) - x_0^*(x)|^p = 0$$

holds for every $x \in X$. The point $x_0^* \in X^*$ is also called the $w_{w^*p}^R$-sum of $x^* = (x_k^*)$.

The definition of w_p^R-, w_{wp}^R-summability of a series $\sum_j x_j$ in normed space X and $w_{w^*p}^R$-summability of the series $\sum_j x_j^*$ in the dual space X^* can easily be given, and are denoted by $w_p^R-\sum_j x_j$, $w_{wp}^R-\sum_j x_j$ and $w_{w^*p}^R-\sum_j x_j^*$, respectively.

w_p^R-summability can be considered as a generalization of strongly p-Cesàro summability, since it is reduced to strongly p-Cesàro summability whenever $r_k = 1$ for all $k \in \mathbb{N}$.

Now, we introduce the spaces of multiplier w_p^R-, weakly w_p^R- (w_{wp}^R-) and weakly* w_p^R-summability ($w_{w^*p}^R$-) associated to the series $\sum_j x_j$ in a normed space X and the series $\sum_j x_j^*$ in the dual space X^*, respectively.

Let $\sum_j x_j$ and $\sum_j x_j^*$ be the series in the normed space X and its dual X^*, respectively. We define the sets $S_{w_p^R}\left(\sum_j x_j\right)$, $S_{w_{wp}^R}\left(\sum_j x_j\right)$ and $S_{w_{w^*p}^R}\left(\sum_j x_j^*\right)$, as follows,

$$\begin{aligned} S_{w_p^R}\Big(\sum_j x_j\Big) &:= \left\{ y=(y_j)\in\ell_\infty : w_p^R - \sum_j y_j x_j \text{ exists} \right\}, \\ S_{w_{wp}^R}\Big(\sum_j x_j\Big) &:= \left\{ y=(y_j)\in\ell_\infty : w_{wp}^R - \sum_j y_j x_j \text{ exists} \right\}, \\ S_{w_{w^*p}^R}\Big(\sum_j x_j^*\Big) &:= \left\{ y=(y_j)\in\ell_\infty : w_{w^*p}^R - \sum_j y_j x_j^* \text{ exists} \right\} \end{aligned} \tag{3}$$

which are endowed with the sup norm and are subspaces of ℓ_∞.

We begin with the following proposition which gives a relationship between w_p^R-summability and w_p^R-boundedness of a sequence $x = (x_k)$ in a normed space X.

Proposition 3.2. *Let $x = (x_k)$ be a w_p^R-summable sequence to x_0 in a normed space X and $p \in (0, 1)$, as usual. Then, the sequence $\left(\sum_{k=1}^n \frac{r_k}{R_n}\|x_k\|^p\right)_{n\in\mathbb{N}}$ is bounded.*

Proof. Let $x = (x_k)$ be a w_p^R-summable sequence to $x_0 \in X$ with $p \in (0, 1)$, i.e.,

$$\lim_{n\to\infty} \sum_{k=1}^n \frac{r_k}{R_n} \|x_k - x_0\|^p = 0.$$

Therefore,

$$\begin{aligned} \sum_{k=1}^n \frac{r_k}{R_n}\|x_k\|^p &\leq \sum_{k=1}^n \frac{r_k}{R_n} (\|x_k - x_0\| + \|x_0\|)^p \\ &\leq \sum_{k=1}^n \frac{r_k}{R_n}\|x_k - x_0\|^p + \sum_{k=1}^n \frac{r_k}{R_n}\|x_0\|^p \\ &= \sum_{k=1}^n \frac{r_k}{R_n}\|x_k - x_0\|^p + \|x_0\|^p \end{aligned}$$

holds for every $n \in \mathbb{N}$.

This completes the proof. □

In the following proposition, we give a relationship between the w_1^R-summability and the classical Riesz convergence of a sequence $x = (x_k)$ in a normed space X.

Proposition 3.3. *If $x = (x_k)$ is a w_1^R-summable sequence to x_0 in a normed space X, then it is Riesz convergent to the same point.*

Proof. By a straightforward calculation, we have

$$\begin{aligned}\lim_{n\to\infty}\left\|\left(\sum_{k=1}^{n}\frac{r_k}{R_n}x_k\right)-x_0\right\| &= \lim_{n\to\infty}\frac{1}{R_n}\left\|\sum_{k=1}^{n}r_kx_k-\sum_{k=1}^{n}r_kx_0\right\| \\ &= \lim_{n\to\infty}\frac{1}{R_n}\left\|\sum_{k=1}^{n}r_k(x_k-x_0)\right\| \\ &\leq \lim_{n\to\infty}\sum_{k=1}^{n}\frac{r_k}{R_n}\|x_k-x_0\| = 0.\end{aligned}$$

This completes the proof. □

In the following, we have an analogue of Proposition 2.1 given in [23]. Since it can be easily proved, we omit details.

Proposition 3.4. *Let $r=(r_j)$ be a sequence of positive reals with $R_k=\sum_{j=1}^{k}r_j$ and $\lim_{k\to\infty}R_k=\infty$. If $x=(x_j)$ is a real valued sequence satisfying*

$$\lim_{k\to\infty}\sum_{j=1}^{n_k}\frac{r_j}{R_{n_k}}\left|x_j\right|^p=\infty$$

for an increasing subsequence (n_k) in $\mathbb{N}$. Then, there is no $x_0\in X$ such that

$$\lim_{k\to\infty}\sum_{j=1}^{k}\frac{r_j}{R_k}\left\|x_j-x_0\right\|^p=0,$$

that is, the sequence $x=(x_j)$ is not w_p^R-summable in X.

We give the following remark for some useful results on weak* w_p^R-summability.

Remark 3.5. *Since Riesz method is regular, regarding a real sequence $a=(a_k)$ satisfying $\sum_k a_k=\alpha$ for some $\alpha\in\overline{\mathbb{R}}$, we have $\sum_{k=1}^{n}\frac{r_k}{R_n}s_k\to\alpha$, as $n\to\infty$, where $s_k=\sum_{j=1}^{k}a_j$ for all $k\in\mathbb{N}$ and $\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,+\infty\}$.*

4. A note on $w_{w^*p}^R$-summability

In this section, we focus on the multiplier space given with (3) which is the entire of the space ℓ_∞. Besides, if the normed space X is barrelled, we obtain a new characterization of the *wuC* series $\sum_k x_k^*$ in the dual space X^*. But first, in notation, we have

$$w_{w^*p}^R-\sum_k y_kx_k^*=x_0^*,\ x_0^*\in X^*\ \Leftrightarrow\ w_p^R-\sum_k y_kx_k^*(x)=x_0^*(x) \tag{4}$$

for all $x\in X$.

Theorem 4.1. *Let $\sum_k x_k^*$ be a series in the dual space X^* of a normed space X. Therefore, we list the following statements:*

(i) *$\sum_k x_k^*$ is a wuC series.*

(ii) *$\ell_\infty\subseteq S_{w_{w^*p}^R}\left(\sum_k x_k^*\right)$, i.e., $S_{w_{w^*p}^R}\left(\sum_k x_k^*\right)=\ell_\infty$.*

(iii) *$w_p^R-\sum_{k\in M}x_k^*(x)$ exists for each $x\in X$ with $M\subset\mathbb{N}$.*

Then, the implications (i) ⇒ (ii) ⇒ (iii) hold. Further, if X is a barrelled normed space then, the statements given by Parts (i), (ii) and (iii), above, are equivalent.

Proof. (i) ⇒ (ii): Suppose that $\sum_k x_k^*$ is a *wuC* series and take $y = (y_k) \in \ell_\infty$. It is easily seen that the series $\sum_k y_k x_k^*$ is weak* convergent in X^*, since the unit ball of X^* is weak* compact. So, (4) holds. Finally, we see that there exists $x_0^* \in X^*$ such that $w_{w^*p}^R - \sum_k y_k x_k^* = x_0^*$ holds. This means that $y = (y_k) \in S_{w_{w^*p}^R}\left(\sum_k x_k^*\right)$.

(ii) ⇒ (iii): This is clear.

(iii) ⇒ (i): Let X be a barrelled normed space. It is enough to show that the set G^* defined by

$$G^* := \left\{\sum_{k=1}^{n} y_k x_k^* : |y_i| \leq 1,\ k = 1, 2, \ldots, n,\ n \in \mathbb{N},\ x_k^* \in X^*\right\}$$

is pointwise bounded for all $x \in X$. Then, the boundedness of the set G^* follows from Banach-Steinhaus theorem. If G^* is not point-wise bounded, then there exists $x_0 \in X$ such that $\sum_k |x_k^*(x_0)|$ diverges. Let us define the sets M_1 and M_2 by

$$M_1 := \{k \in \mathbb{N} : x_k^*(x_0) \geq 0\} \text{ and } M_2 := \{k \in \mathbb{N} : x_k^*(x_0) < 0\}.$$

Under this hypothesis, one can see that $\sum_{k\in M_1} x_k^*(x_0)$ or $\sum_{k\in M_2}(-x^*)_k(x_0)$ diverges. Therefore, we conclude by Proposition 3.2 and Remark 3.5 that the series is not w_p^R-summable. This also contradicts our assumption for each $x \in X$ that $w_p^R - \sum_{k\in M} x_k^*(x)$ exists for $M \subset \mathbb{N}$. □

5. Unconditional Convergent Series Via w_p^R-summability

As a consequence of the classical Orlicz-Pettis theorem, in a Banach space X, the series $\sum_j x_j$ is *uc* (ℓ_∞-multiplier convergent) if and only if the weakly sum $\sum_{j\in M} x_j$ exists for every $M \subset \mathbb{N}$. By the following theorem, we extend this result to the w_p^R-summability of the series $\sum_j x_j$. Of course, we should also give the relationship between the subseries convergence and the characteristic function χ. Let us define the set $M_0 = \{\chi_\sigma | \sigma \subset \mathbb{N}\}$, where χ_σ is the characteristic function of σ. Then, M_0-multiplier convergence is just subseries convergence, and according to the classical Orlicz-Pettis theorem, given any weakly subseries convergent series $\sum_k x_k$ in a normed space X is actually norm (strong) subseries convergent, [31, p. vii].

Theorem 5.1. *Let $\sum_j x_j$ be a series in a Banach space X. Then, the following statements are equivalent:*

(i) $\sum_j x_j$ is uc.

(ii) $\sum_j x_j$ is subseries weakly w_p^R-summable.

Proof. (i)⇒(ii): Let us suppose that $\sum_j x_j$ is an *uc* series in a Banach space X. Therefore, for every $M \subset \mathbb{N}$, there exists $x_0 \in X$ such that

$$\lim_{k\to\infty} \left| x^*\left(s_k^M\right) - x^*(x_0) \right| = 0 \tag{5}$$

for every $x^* \in X^*$, where $s_k^M = \sum_{j=1}^k \chi_M^{(j)} x_j$, the partial sum of the series obtained by multiplication of characteristic function determined by the set M and the sequence $x = (x_j)$.

We should prove that $\left(s_k^M\right)_k$ is in the space $w_{wp}^R(X)$, and also show that $\sum_j x_j$ is a subseries weakly w_p^R-summable. Now, for every $M \subset \mathbb{N}$, let us take

$$x_n := \sum_{k=1}^{n} r_k \left| x^*\left(s_k^M\right) - x^*(x_0) \right|^p \text{ for all } x^* \in X^*$$

and $y_n := R_n$. Therefore, we have

$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} \left| x^*\left(s_n^M\right) - x^*(x_0) \right|^p = 0 \tag{6}$$

for every $x^* \in X^*$, since (5) holds. By using Stolz-Cesàro theorem, we see from the existence of (6) that

$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{1}{R_n} \sum_{k=1}^{n} r_k \left| x^*\left(s_k^M\right) - x^*(x_0)\right|^p = 0$$

for every $x^* \in X^*$. This step proves that $\left(s_k^M\right)_k$ belongs to the space $w_{wp}^R(X)$.

(ii)⇒(i): Let us suppose that $\sum_j x_j$ is subseries weakly w_p^R-summable, that is, the series $\sum_j \chi_M^{(j)} x_j$ is weakly w_p^R-summable for every $M \subset \mathbb{N}$. We should show that $\sum_j x_j$ is weakly unconditionally convergent. Let us suppose that $\sum_j x_j$ is not a weakly unconditionally convergent series. Then, there must be some $x^* \in X^*$ such that $\sum_j |x^*(x_j)| = \infty$. Now, we choice $(\lambda_j) \subset \{-1, 1\}$ with

$$\lambda_j := \begin{cases} 1 & , \quad x^*(x_j) \geq 0, \\ -1 & , \quad x^*(x_j) < 0 \end{cases}$$

for all $j \in \mathbb{N}$. Therefore, $\sum_j \lambda_j x^*(x_j) = \infty$ and so the sequence $\left(\sum_{j=1}^k \lambda_j x_j\right)_k$ is not weakly w_p^R-summable from Proposition 3.4 and the Stolz-Cesàro theorem. Let us now define the sets M_3 and M_4 by $M_3 := \{j : \lambda_j = 1\}$ and $M_4 := \mathbb{N} - M_3$. Therefore, by our assumption, there exist $y_1, y_2 \in X$ such that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{r_k}{R_n} \left| x^*\left(s_k^{M_3}\right) - x^*(y_1)\right|^p = 0,$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{r_k}{R_n} \left| x^*\left(s_k^{M_4}\right) - x^*(y_2)\right|^p = 0$$

for every $x^* \in X^*$. This is a contradiction, since the following equality holds for every $x^* \in X^*$;

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{r_k}{R_n} \left| \sum_{j=1}^{k} \lambda_j x^*(x_j) - x^*(y_1 - y_2)\right|^p = 0.$$

Finally, we prove for every $M \subset \mathbb{N}$ that the series $\sum_j \chi_M^{(j)} x_j$ is weakly convergent and so, the series $\sum_j x_j$ will be *uc* by a result of classical Orlicz-Pettis theorem. For every $M \subset \mathbb{N}$ and $x^* \in X^*$, we have $y_0 \in X$ such that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{r_k}{R_n} \left| x^*\left(s_k^M\right) - x^*(y_0)\right|^p = 0 \tag{7}$$

and the series $\sum_j \chi_M^{(j)} x^*(x_j)$ is convergent for every $x^* \in X^*$, since it is weakly unconditionally convergent. Therefore, we obtain by Stolz-Cesàro theorem and (7) that

$$\lim_{k\to\infty} \left| \sum_{j=1}^{k} \chi_M^{(j)} x^*(x_j) - x^*(y_0)\right| = 0$$

for every $M \subset \mathbb{N}$ and $x^* \in X^*$.

This completes the proof. □

In a Banach space X, since unconditionally convergence of a series $\sum_j x_j$ coincides with ℓ_∞-multiplier convergence of the series, as a consequence of Theorem 5.1, we have the following corollary which states that under certain hypothesis both of the spaces $S_{w_p^R}\left(\sum_j x_j\right)$ and $S_{w_{wp}^R}\left(\sum_j x_j\right)$ are the entire of the space ℓ_∞:

Corollary 5.2. *Let $\sum_j x_j$ be a series in a Banach space X. Then, the following statements are equivalent:*

(i) The series $\sum_j x_j$ is uc.

(ii) $\ell_\infty \subseteq S_{w_p^R}\left(\sum_j x_j\right)$, i.e., $\ell_\infty = S_{w_p^R}\left(\sum_j x_j\right)$.

(iii) $\ell_\infty \subseteq S_{w^R_{wp}}\Big(\sum_j x_j\Big)$, *i.e.*, $\ell_\infty = S_{w^R_{wp}}\Big(\sum_j x_j\Big)$.

6. Orlicz-Pettis Type Theorem with Natural Families

In this section, we obtain some new versions of Orlicz-Pettis theorem via FQ σ-family and a natural family $\mathcal{F}$ with the separation property S_1 through w^R_p-summability, respectively. Before this, we give the definitions of FQ σ-family and the separation property S_1 for a natural family $\mathcal{F}$. Recall that, for an infinite subset M of $\mathbb{N}$, $\mathcal{F}$ is called as a natural family on M, if the inclusions $\Phi_0(M) \subseteq \mathcal{F} \subseteq \mathcal{P}(M)$ hold, (cf. [1, Definition 1.1]). Here and in what follows, by $\Phi_0(X)$ and $\mathcal{P}(X)$, we denote the family of all finite subsets and the power set of the set X, respectively.

Definition 6.1. *A natural family $\mathcal{F}$ of subsets of $\mathbb{N}$ is said to be a finitely quasi σ-family (FQ σ-family), if for each disjoint sequence (A_j) of the sets belonging to the family $\Phi_0(\mathbb{N})$, there exists an infinite subset M of $\mathbb{N}$ such that $\Big(\bigcup_{j\in M} A_j\Big) \in \mathcal{F}$,* [29].

Regarding a natural family $\mathcal{F}$, the concept $\mathcal{F}$-convergence of a series $\sum_j x_j$ in a normed space X is closely related to the concept of subseries convergence (unconditional convergence). Let us remind that, for a given natural family $\mathcal{F}$, a series $\sum_j x_j$ in a normed space X is $\mathcal{F}$-convergent (or $\mathcal{F}$-weakly convergent) if $\sum_{j\in A} x_j$ is convergent (or weakly convergent) for every $A \in \mathcal{F}$, [1]. In [33], Swartz showed for an FQ σ-family $\mathcal{F}$ that every $\mathcal{F}$-convergent series in a Banach space X is also *uc*. Aizpuru et al. [2] also extended this result to the well-known Cesàro summability of series in Banach spaces. Quite recently, by using the concept FQ σ-family, León-Saavedra et al. [23] also obtained a new characterization of *uc* series in terms of weak w_p-summability.

Now, we can establish a new version of Orlicz-Pettis theorem via the concept FQ σ-family and w^R_p-summability method, as follows:

Theorem 6.2. *Suppose that $\sum_j x_j$ is a series in a Banach space X. Then, the following assertions are equivalent:*

(i) The series $\sum_j x_j$ is uc.

(ii) There is an FQ σ-family $\mathcal{F}$ such that the series $\sum_{j\in A} x_j$ is w^R_{wp}-summable for every $A \in \mathcal{F}$.

Proof. (i)⇒(ii): Let us suppose that the series $\sum_j x_j$ is an *uc* series in a Banach space X and $\mathcal{F}$ is an FQ σ-family. Then, the desired result follows from Theorem 5.1, since there exists $y_0 \in X$ such that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{r_k}{R_n} \left| \sum_{j=1}^{k} \chi_A^{(j)} x^*(x_j) - x^*(y_0) \right|^p = 0$$

for every $A \in \mathcal{F}$ and $x^* \in X^*$. That is to say that the series $\sum_{j\in A} x_j$ is w^R_{wp}-summable for every $A \in \mathcal{F}$.

(ii)⇒(i): Let $\mathcal{F}$ be an FQ σ-family and the series $\sum_{j\in A} x_j$ be w^R_{wp}-summable for every $A \in \mathcal{F}$. Hence, we have to prove that the series $\sum_j \chi_A^{(j)} x_j$ is weakly convergent for every $A \in \mathcal{F}$ and so, the series $\sum_j x_j$ is weakly *uc* series. Let us suppose that the series $\sum_j x_j$ is not weakly *uc*. Therefore, there exists $x^* \in X^*$ such that $\sum_j |x^*(x_j)| = \infty$. Let us define the sets $P := \{j : x^*(x_j) \geq 0\}$ and $Q := \{j : x^*(x_j) < 0\}$. Then, we can assume that

$$\sum_j \chi_P^{(j)} x^*(x_j) = \infty.$$

One can also inductively construct a pairwise disjoint sequence (A_n) which satisfies the following conditions for every $n \in \mathbb{N}$:

(i) $A_n \in (\Phi_0 \cap P)$.

(ii) $\sup_{n\in\mathbb{N}} A_n < \inf_{n\in\mathbb{N}} A_{n+1}$.

(iii) $\sum_j \chi_{A_n}^{(j)} x^*(x_j) > n$.

Now, let us consider the set $A := (\cup_{n\in M} A_n) \in \mathcal{F}$ for an infinite subset M of $\mathbb{N}$. It is easily seen that $\sum_{j\in A} x^*(x_j) = \infty$. By using Proposition 3.4 and the Stolz-Cesàro theorem, we see the fact that the series $\sum_{j\in A} x_j$ is not w_{wp}^R-summable for $A \in \mathcal{F}$, a contradiction. $\square$

In [1], Aizpuru and Gutiérrez-Dávila introduced the concept of a new natural family which includes the FQ σ-families. They called this family as the natural family with the property S_1. By using the families with the property S_1, they extended the corresponding result of Swartz [33] on Orlicz-Pettis theorem. In 2006, Aizpuru et al. [2] extended their study to the Cesàro convergence method under certain hypothesis. Recently, León-Saavedra et al. gave a new version of Orlicz-Pettis theorem and generalized the corresponding result of Aizpuru and Gutiérrez-Dávila via w_p-summability and the natural family $\mathcal{F}$ enjoying property S_1, [23].

Prior to giving our result concerning with a new version of Orlicz-Pettis theorem, we present the following definition of the families with the property S_1, and recall a lemma [2] which states an important result on the characterization of a weakly *uc* series in the Banach spaces.

Definition 6.3. *A natural family $\mathcal{F}$ is said to have the property S_1 if for each pair $[(A_j)_j, (B_j)_j]$ of disjoint sequences of mutually disjoint sets in $\Phi_0(\mathbb{N})$, there exist an infinite subset M of $\mathbb{N}$ and $B \in \mathcal{F}$ such that $A_j \subseteq B$ and $B_j \subseteq B^c$, the complement of the set B, for $j \in M$,* [1].

Lemma 6.4. *Regarding a series $\sum_j x_j$ in a Banach space X, the following assertions are equivalent,* [2]:

(i) The series $\sum_j x_j$ is weakly *uc*.
(ii) There is a natural family $\mathcal{F}$ with the property S_1 such that the partial sums of the series $\sum_{j\in A} x_j$ is bounded for every $A \in \mathcal{F}$.

By combining Lemma 6.4 and Theorems 5.1 and 6.2, we conclude the following result giving a new characterization of Orlicz-Pettis theorem by using w_p^R-summability through the natural families enjoying property S_1:

Corollary 6.5. *Let $\sum_j x_j$ be a series in a Banach space X. Then, the following assertions are equivalent:*

(i) $\sum_j x_j$ is a uc series.
(ii) There exists a natural family $\mathcal{F}$ with the property S_1 such that $\sum_{j\in A} x_j$ is w_{wp}^R-summable and the partial sums of the series are bounded for every $A \in \mathcal{F}$.

7. Compact Summing Operator

Let $\sum_k x_k$ be a series in a normed space X. Now, we define the summing operators from the spaces of multiplier w_p^R- and w_{wp}^R-summability to X by

$$\begin{array}{cccccc} \mathcal{S} & : & S_{w_p^R}\left(\sum_k x_k\right) & \longrightarrow & X & \\ & & y & \longmapsto & \mathcal{S}y = w_p^R - \sum_k x_k y_k, & \end{array} \tag{8}$$

$$\begin{array}{cccccc} \mathcal{S} & : & S_{w_{wp}^R}\left(\sum_k x_k\right) & \longrightarrow & X & \\ & & y & \longmapsto & \mathcal{S}y = w_{wp}^R - \sum_k x_k y_k, & \end{array} \tag{9}$$

respectively.

In the following, by using previous results on *uc* convergence and w_p^R- and w_{wp}^R-summability, we give a new compactness criteria for summing operator defined by (8) and (9), above.

Theorem 7.1. *Let $\sum_k x_k$ be a series in a normed space X. We list the following statements:*

(i) The summing operator $\mathcal{S}$ given by (8) is (weakly) compact.
(ii) The series $\sum_k x_k$ is subseries (weakly) w_p^R-convergent.

Then, Part (i) implies Part (ii).

Proof. Let us first suppose that $\mathcal{S}$ is compact and $M := \{\chi_\sigma | \sigma \text{ finite}\}$. Then, we have $M \subseteq B_{S_{w_p^R}(\sum_k x_k)}$, the closed unit ball of $S_{w_p^R}\left(\sum_k x_k\right)$. Therefore, the set

$$\mathcal{S}_M := \left\{ w_p^R - \sum_{k \in \sigma} x_k | \sigma \textit{ finite} \right\}$$

is relatively norm compact, and so, the series $\sum_k x_k$ is subseries norm w_p^R-convergent (cf. [31, Theorem 2.48]).

Since the second proof can be given by the similar way, we omit details. So, weak compactness of $\mathcal{S}$ implies subseries weakly w_p^R-convergence of the series $\sum_k x_k$. □

It is well-known that in a Banach space X, norm subseries convergence (also weak subseries convergence from Orlicz-Pettis theorem) of a series $\sum_k x_k$ coincides with *uc* convergence of the series. Therefore, we give the following theorem without proof, since it is clear by using Theorem 5.1 and Corollary 5.2 with previous result.

Theorem 7.2. *Let $\sum_k x_k$ be a series in a Banach space X. Then, the following statements are equivalent:*

(i) The summing operator $\mathcal{S}$ given with (8) or (9) is compact.
(ii) The summing operator $\mathcal{S}$ given with (8) or (9) is weakly compact.
(iii) The series $\sum_k x_k$ is subseries w_p^R-convergent.
(iv) The series $\sum_k x_k$ is subseries weakly w_p^R-convergent.
(v) The series $\sum_k x_k$ is an uc series.

8. Conclusion

Multiplication of the series $\sum_k x_k$ in a Banach space X and an arbitrary real or complex sequence $a = (a_k)$ has the form $\sum_k a_k x_k$ and is important to characterize the behaviors of the series in X. A series $\sum_k x_k$ in a normed space X is E-multiplier convergent if the series $\sum_k a_k x_k$ converges in X for every $a = (a_k) \in E$. The series is multiplier Cauchy in X if the partial sums of the series $\sum_k a_k x_k$ form a norm Cauchy sequence in X for all $a = (a_k) \in E$, [31]. The series $\sum_k x_k$ in X is *wuC* (or *uc*) series if and only if $\sum_k a_k x_k$ is convergent for every null (or bounded) sequence $a = (a_k)$, that is, $\sum_k x_k$ is a c_0-(or an ℓ_∞-) multiplier convergent series, [11].

For the researchers of series and summability theory, the studies of characterizations of completeness and barrelledness of normed spaces through *wuC* series, and obtaining new multiplier spaces by means of summability methods are the interesting field of recent times in the theory of multiplier convergent series. One can refer to Aizpuru et al. [2, 4, 5, 28], Swartz [32], Kama [16, 17], Kama and Altay [13, 15], Kama et al. [14], Karakuş [18], Karakuş and Başar [19–21] and León-Saavedra et al. [24] for recent studies on multiplier convergent series. In [22], the authors also characterize some classical properties of normed spaces by using generalized almost summability methods given for vector valued sequences, (see also [21]).

Orlicz-Pettis theorem is one of the most important result in summability theory. Kalton's survey [12] and Diestel's monograph [11] are worth to see for first studies on this theorem. By using some different types of summability methods, this theorem has been employed and improved by many authors; [1–5, 7, 12, 19, 23, 30, 31] and [33]. Especially, some new versions of Orlicz-Pettis theorem which are used in this study can be listed as: Aizpuru et al. [2–5], Altay and Kama [7], Kama and Altay [15], Karakuş and Başar [19], Pérez-Fernández et al. [28] and Swartz [30, 31]. Recently, León-Saavedra et al. [23] obtained a new version of the Orlic-Pettis theorem by using strong p-Cesàro summability method. For more details and recent results in Orlicz-Pettis theorem, the reader may refer to Swartz's book [31, ch. 4-5-6] on the multiplier convergent series.

The compact summing operator, we placed at the end of this paper is also a useful tool in our research area. For instance, one can characterize a bounded multiplier convergent series or *uc* series with compact summing operator.

In this study, we aimed to give some new results related to the Orlicz-Pettis theorem via w_p^R-summability method which is more general form of w_p-summability. We obtained a new characterization of *uc* series in Banach spaces by using finitely quasi σ-family (FQ σ-family) of $\mathbb{N}$ which is firstly used in [29] and w_p^R-summability. Therefore, we also gave another result concerning characterization of weakly uc by the family carrying property S_1 through w_p^R-summability. Finally, we characterized compactness of summing operator which is defined from certain multiplier convergent spaces to a normed space X.

One can also improve the concept of w_p^R-summability and give a new version of Orlicz-Pettis theorem and corresponding results of this paper by using more general version of w_p^R-summability method via a regular matrix $A = (a_{nk})$, as follows: *Let $A = (a_{nk})$ be a regular matrix and $p \in (0, \infty)$. A sequence $x = (x_k)$ in a normed space X is said to be w_p^A-summable to $x_0 \in X$ if the relation*

$$\lim_{n\to\infty} \sum_k a_{nk} \|x_k - x_0\|^p = 0$$

holds.

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