Filomat 36:18 (2022), 6375–6392
https://doi.org/10.2298/FIL2218375B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Quasi-Menger and Weakly Menger Frames

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Abstract. We study the quasi-Menger and weakly Menger properties in locales. Our definitions, which are adapted from topological spaces by replacing subsets with sublocales, are conservative in the sense that a topological space is quasi-Menger (resp. weakly Menger) if and only if the locale it determines is quasi-Menger (resp. weakly Menger). We characterize each of these types of locales in a language that does not involve sublocales. Regarding localic results that have no topological counterparts, we show that an infinitely extremally disconnected locale (in the sense of Arietta [1]) is weakly Menger if and only if its smallest dense sublocale is weakly Menger. We show that if the product of locales is quasi-Menger (or weakly Menger) then so is each factor. Even though the localic product $\prod_{j\in J}\Omega(X_j)$ is not necessarily isomorphic to the locale $\Omega\big(\prod_{j\in J}X_j\big)$, we are able to deduce as a corollary of the localic result that if the product of topological spaces is weakly Menger, then so is each factor.

1. Introduction and motivation

Recently there has been an interest in studying selection principles in a context that does not argue using points. In [3], we studied the Menger and almost Menger properties in frames. The latter property is a weaker form of the former introduced by Kočinac [14], and has since been studied by several authors in topological spaces. In [17], Mezabarba reprises a theorem of Hurewicz and one of Pawlikowski, each concerning topological games, within lattices that have "enough points" in the usual usage of this phrase in point-free topology. Notwithstanding the sufficiency of points (in the localic sense) in the lattices he considers, his arguments do not use points, in the topological sense.

There are several variations of the Menger property in spaces. For a thorough survey, we recommend the reader consult [15]. In this paper we concentrate on the localic versions of the quasi-Menger and the weakly Menger variations. Since non-spatial quasi-Menger and weakly Menger locales do exist, our study covers a wider scope than locales induced by quasi-Menger or weakly Menger spaces.

When working with locales, some questions come up that do not arise in topological spaces. The existence of the smallest dense sublocale in every locale frequently leads to questions that are not motivated by topological considerations, and indeed that do not have topological counterparts. This is a case in this paper too. For instance, we show that a locale that Arietta [1] calls infinitely extremally disconnected is weakly Menger if and only if its smallest dense sublocale is weakly Menger.

2020 *Mathematics Subject Classification*. Primary 06D22; Secondary 54A35, 54B05, 54C10, 54D20
Keywords. Frame, locale, Menger, quasi-Menger, weakly Menger, sublocale
Received: 06 January 2022; Revised: 10 March 2022; Accepted: 11 March 2022
Communicated by Ljubiša D.R. Kočinac
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Here is a brief outline of the paper. In Section 2 we recall some pertinent definitions. Quasi-Menger frames are defined and studied in Section 3. As in spaces, the definition requires that we start with a closed sublocale and a sequence of families of open sublocales interacting with the closed sublocale in a certain way. Starting with a regular-closed sublocale, instead, leads to a weaker variant, and we call the resulting locales regularly quasi-Menger. The topological counterpart (replacing "closed subspace" with "regular-closed subspace" in the definition of quasi-Menger spaces) has not been considered before in spaces defined in the form just mentioned. However, a closer look shows that the ensuing spaces (we call them regularly quasi-Menger spaces) actually do appear in a different guise as part of what Kočinac and Konca [16] call set-Menger spaces.

In Section 5 we find a sufficient condition for the spectrum of a locale to be regularly quasi-Menger if the locale is regularly quasi-Menger. The condition is that the points of the locale meet at the bottom – a condition which does not force the locale to be spatial, we hasten to add. We first prove this result topologically because the topological proof also brings to light some properties of the spectra of frames that do not seem to be recorded anywhere, so far as we have been able to determine. We then show how the same result appears as a corollary of a localic one proved completely differently.

In the last section we define and study weakly Menger locales. We show that a locale whose smallest dense sublocale is weakly Menger is itself weakly Menger. We actually have a characterization of when the smallest dense sublocale of any locale is weakly Menger.

2. Preliminaries

We assume familiarity with frames and locales. Our references are [13] and [20]. In this section we recall just a few of the concepts that we shall need. Our notation is standard. The term "homomorphism" will always mean a frame homomorphism. The asterisk will appear as a subscript to denote the right adjoint of a homomorphism, and as a superscript to denote the pseudocomplement of an element.

2.1. Sublocales

Let L be a frame. The lattice of sublocales of L, ordered by inclusion, is a coframe denoted $\mathcal{S}(L)$. Meets in $\mathcal{S}(L)$ are intersections, and joins are given by

$$\bigvee_{i\in I} S_i = \Big\{\bigwedge M \mid M \subseteq \bigcup_{i\in I} S_i\Big\}.$$

The smallest element of $\mathcal{S}(L)$ is the sublocale $\mathsf{O} = \{1\}$, and is called the *void sublocale*. A sublocale is *complemented* in case it has a complement in $\mathcal{S}(L)$. Complemented sublocales are precisely the *linear* ones, meaning that a sublocale S is complemented if and only if

$$S \cap \bigvee_{i\in I} S_i = \bigvee_{i\in I} (S \cap S_i)$$

for every family $(S_i \mid i \in I)$ of sublocales of L.

The open sublocale associated with $a \in L$ is denoted by $\mathfrak{o}_L(a)$, and the closed one by $\mathfrak{c}_L(a)$. Recall that a is called *regular* in case $a = a^{**}$, and *complemented* in case $a \vee a^* = 1$. A sublocale of L is called *regular-closed* (resp. *regular-open*) in case it is of the form $\mathfrak{c}_L(a)$ (resp. $\mathfrak{o}_L(a)$) with a regular. The *clopen* sublocales of L are precisely the sublocales $\mathfrak{c}_L(a)$ for complemented elements a. For any family $(a_i \mid i \in I)$ of elements of L,

$$\mathfrak{c}_L\Big(\bigvee_{i\in I} a_i\Big) = \bigcap_{i\in I} \mathfrak{c}_L(a_i) \quad \text{and} \quad \mathfrak{o}_L\Big(\bigvee_{i\in I} a_i\Big) = \bigvee_{i\in I} \mathfrak{o}_L(a_i).$$

For any $a, b \in L$,

$$\mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(b) \quad \Longleftrightarrow \quad a \vee b = 1.$$

The *closure* of a sublocale S of L, denoted $\overline{S}$ or $\mathrm{cl}_L\, S$, is the sublocale

$$\overline{S} = \mathfrak{c}_L\Big(\bigwedge S\Big).$$

In particular, $\overline{\mathfrak{o}_L(a)} = \mathfrak{c}_L(a^*)$. If S and T are sublocales of L and $S \subseteq T$, then S is a sublocale of T. The closure of S in T will be denoted by $\mathrm{cl}_T\, S$, and $\overline{S}$ (unadorned) will be understood to be the closure in L. A sublocale S of L is *dense* if $\overline{S} = L$. We denote the smallest dense sublocale of L by $\mathfrak{B}L$, and recall that

$$\mathfrak{B}L = \{a \in L \mid a = a^{**}\} = \{b^* \mid b \in L\},$$

with meets as in L and joins given by

$$\bigvee^{\mathfrak{B}L}\{b_i \mid i \in I\} = \Big(\bigvee\{b_i \mid i \in I\}\Big)^{**},$$

for any family $(b_i \mid i \in I)$ of elements of $\mathfrak{B}L$.

2.2. Covers and coverings

By a *cover* of L we mean a set $C \subseteq L$ such that $\bigvee C = 1$. On the other hand, to avoid possible confusion, we say a collection $\mathscr{C}$ of sublocales of L is a *covering* of L if $\bigvee\{C \mid C \in \mathscr{C}\} = L$, where the join is calculated in $\mathcal{S}(L)$. This terminology is not standard. A cover consists of elements of L, whereas a covering consists of sublocales of L. If every sublocale in a covering $\mathscr{C}$ of L is open, then $\mathscr{C}$ is an *open covering* of L. There is a bijection between covers and open coverings given by

$$C \mapsto \mathscr{C}^C \overset{\text{def}}{=} \{\mathfrak{o}_L(c) \mid c \in C\} \qquad \text{and} \qquad \mathscr{C} \mapsto C^{\mathscr{C}} \overset{\text{def}}{=} \{x \in L \mid \mathfrak{o}_L(x) \in \mathscr{C}\}.$$

3. Quasi- and regularly quasi-Menger frames

Throughout, sequences will be indexed by the set $\mathbb{N}$ of positive integers. We recall from [5] that a topological space X is called quasi-Menger if for every closed set $F \subseteq X$ and every sequence $(\mathscr{V}_n)$ of covers of F by sets open in X, there exists, for each n, a finite $\mathscr{U}_n \subseteq \mathscr{V}_n$ such that $F \subseteq \overline{\bigcup_{n\in\mathbb{N}}\bigcup\mathscr{U}_n}$.

We aim to adapt this definition to frames and, simultaneously, consider a natural variant of the quasi-Menger property. Analogously to spaces, when we say a collection $\mathscr{C}$ of sublocales of a frame L *covers* a sublocale S we mean that $S \subseteq \bigvee\{T \mid T \in \mathscr{C}\}$. We shall at times abbreviate $\bigvee\{T \mid T \in \mathscr{C}\}$ as $\bigvee\mathscr{C}$. The variant in question appears in parenthesis in the following definition.

Definition 3.1. A frame L is *quasi-Menger* (resp. *regularly quasi-Menger*) if for every closed (resp. regular-closed) sublocale F of L and each sequence $(\mathscr{V}_n)$ with $\mathscr{V}_n$ consisting of open sublocales of L which cover F, there exists, for each n, a finite $\mathscr{U}_n \subseteq \mathscr{V}_n$ such that $F \subseteq \overline{\bigvee_{n\in\mathbb{N}}\bigvee\mathscr{U}_n}$. For spaces we define the weaker variant analogously, replacing sublocales with subsets, and joins of open sublocales with unions of open subsets.

As in [5], we abbreviate "quasi-Menger" as qM and "regularly quasi-Menger" as rqM. Before we plough ahead, let us show that although the rqM property (in spaces) does not appear in the literature as we have defined it, it actually does exist in a different guise.

In [16], Kočinac and Konca define, for a topological space X, Menger-type properties associated with collections of nonempty subsets of X. The relevant one for the present discussion is defined as follows. Let $\mathcal{P}$ be a collection of nonempty subsets of X. Then X is said to be *weakly $\mathcal{P}$-Menger* if for every $A \in \mathcal{P}$ and every sequence $(\mathscr{V}_n)$, where each $\mathscr{V}_n$ consists of sets open in X, such that $\overline{A} \subseteq \bigcup_{n\in\mathbb{N}}\mathscr{V}_n$, there exists, for each n, a finite $\mathscr{U}_n \subseteq \mathscr{V}_n$ such that $A \subseteq \overline{\bigcup_{n\in\mathbb{N}}\bigcup\mathscr{U}_n}$.

Now, since a subset of X is regular-closed if and only if it is the closure of some open set, reasoning exactly as in the proofs of [16, Theorem 4.8] and [21, Theorem 2.10], we see that if we let $\mathcal{O}$ be the collection of nonempty open subsets of X, then we have the following:

X is rqM *if and only if it is weakly O-Menger.*

It will be convenient to have characterizations of qM frames and rqM frames that are couched solely in terms of elements and do not mention sublocales. Among other things, such characterizations will enable us to show more easily that a space X is qM (resp. rqM) if and only if the frame $\Omega(X)$ is qM (resp. rqM).

Theorem 3.2. *A frame L is* qM *(resp.* rqM*) if and only if for every $a \in L$ (resp. regular $a \in L$) and every sequence (V_n) of subsets of L with $a \vee \bigvee V_n = 1$ for each n, there is a finite $U_n \subseteq V_n$ such that $\left(\bigvee_{n\in\mathbb{N}} u_n\right)^* \leq a$, where $u_n = \bigvee U_n$.*

Proof. ($\Rightarrow$) Suppose that L is qM and let $a \in L$. Let (V_n) be a sequence of subsets of L such that $a \vee \bigvee V_n = 1$ for every n. Then, for each n,

$$\mathfrak{c}_L(a) \subseteq \mathfrak{o}_L\Big(\bigvee V_n\Big) = \bigvee\{\mathfrak{o}_L(x) \mid x \in V_n\}.$$

Thus, the family

$$\mathscr{V}_n = \{\mathfrak{o}_L(x) \mid x \in V_n\}$$

covers the closed sublocale $\mathfrak{c}_L(a)$, and, of course, consists of open sublocales of L. Since L is qM, for each n, there is a finite set $U_n \subseteq V_n$ such that for the collection

$$\mathscr{U}_n = \{\mathfrak{o}_L(u) \mid u \in U_n\}$$

we have $\mathfrak{c}_L(a) \subseteq \overline{\bigvee_{n\in\mathbb{N}} \bigvee \mathscr{U}_n}$. Putting $u_n = \bigvee U_n$, we have

$$\bigvee \mathscr{U}_n = \bigvee\{\mathfrak{o}_L(u) \mid u \in U_n\} = \mathfrak{o}_L\Big(\bigvee U_n\Big) = \mathfrak{o}_L(u_n),$$

so that

$$\mathfrak{c}_L(a) \subseteq \overline{\bigvee_{n\in\mathbb{N}} \mathfrak{o}_L(u_n)} = \overline{\mathfrak{o}_L\Big(\bigvee_{n\in\mathbb{N}} u_n\Big)} = \mathfrak{c}_L\Big(\Big(\bigvee_{n\in\mathbb{N}} u_n\Big)^*\Big),$$

which implies $\left(\bigvee_{n\in\mathbb{N}} u_n\right)^* \leq a$, as required.

($\Leftarrow$) Suppose that the condition in the statement of the theorem holds, and let F be a closed sublocale of L. Pick $a \in L$ with $F = \mathfrak{c}_L(a)$. Suppose that $(\mathscr{V}_n)$ is a sequence where each $\mathscr{V}_n$ consists of open sublocales of L and $\mathscr{V}_n$ covers F. So, for each n, there exists a set $V_n \subseteq L$ such that $\mathscr{V}_n = \{\mathfrak{o}_L(v) \mid v \in V_n\}$. The containment $\mathfrak{c}_L(a) \subseteq \bigvee \mathscr{V}_n$ implies

$$\mathfrak{c}_L(a) \subseteq \bigvee\{\mathfrak{o}_L(v) \mid v \in V_n\} = \mathfrak{o}_L\Big(\bigvee V_n\Big),$$

which, in turn, implies $a \vee \bigvee V_n = 1$. By hypothesis, for each n, there is a finite $U_n \subseteq V_n$ such that $\left(\bigvee_{n\in\mathbb{N}} u_n\right)^* \leq a$, where $u_n = \bigvee U_n$. For each n, put

$$\mathscr{U}_n = \{\mathfrak{o}_L(x) \mid x \in U_n\}.$$

Then $\mathscr{U}_n$ is a finite subset of $\mathscr{V}_n$ and

$$\begin{aligned}\mathfrak{c}_L(a) \subseteq \mathfrak{c}_L\Big(\Big(\bigvee_{n\in\mathbb{N}} u_n\Big)^*\Big) = \overline{\mathfrak{o}_L\Big(\bigvee_{n\in\mathbb{N}} u_n\Big)} &= \overline{\bigvee_{n\in\mathbb{N}} \mathfrak{o}_L(u_n)}\\ &= \overline{\bigvee_{n\in\mathbb{N}} \mathfrak{o}_L\Big(\bigvee U_n\Big)}\\ &= \overline{\bigvee_{n\in\mathbb{N}} \bigvee\{\mathfrak{o}_L(x) \mid x \in U_n\}} = \overline{\bigvee_{n\in\mathbb{N}} \bigvee \mathscr{U}_n}.\end{aligned}$$

Therefore L is qM.

The parenthetical claim follows similarly because a sublocale of L is regular-closed if and only if it is of the form $\mathfrak{c}_L(a)$ for some regular $a \in L$. $\square$

It is clear that qM implies rqM. Recall that a frame L is *normal* if whenever $a \vee b = 1$ in L, there exist elements c and d in L such that

$$c \wedge d = 0 \quad \text{and} \quad a \vee c = 1 = b \vee d.$$

Note that if L is normal and $a \vee b = 1$, there exists a $c \in L$ such that $c \prec a$ and $c \vee b = 1$, where $\prec$ denotes the familiar "rather below" relation. Also, if $c \prec a$, then $c^{**} \leq a$. We show that for normal frames qM and rqM coincide.

Proposition 3.3. *A normal frame is* qM *if and only if it is* rqM*.*

Proof. Only one implication needs to be proved. So suppose that L is a normal rqM frame. Let $a \in L$ and (V_n) be sequence of subsets of L such that $a \vee \bigvee V_n = 1$ for every n. Since L is normal, we can find a $c \in L$ such that $c \prec a$ and $c \vee \bigvee V_n = 1$ for each n. Then $c^{**} \vee \bigvee V_n = 1$ for each n. Since c^{**} is regular and L is rqM, there exists, for each n, a finite $U_n \subseteq V_n$ such that $\left(\bigvee_{n\in\mathbb{N}} u_n\right)^* \leq c^{**}$, where $u_n = \bigvee U_n$. It therefore follows from Theorem 3.2 that L is qM because $c^{**} \leq a$. $\square$

When we were writing [3] we came to realize that working with directed collections can at times be more convenient in selection principles. By a *directed* subset of a poset we mean an up-directed one. A proof similar to that of [3, Proposition 2.7] yields the following characterizations.

Corollary 3.4. *A frame L is* qM *(resp.* rqM*) if and only if for every $a \in L$ (resp. regular $a \in L$) and every sequence (V_n) of directed subsets of L with $a \vee \bigvee V_n = 1$ for each n, there exists an element $v_n \in V_n$ such that $(\bigvee_{n\in\mathbb{N}} v_n)^* \leq a$.*

We recall from [3] that a frame L is called *Menger* if for every sequence $(\mathscr{C}_n)$ of open coverings of L, there exists, for each n, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathscr{D}_n$ is a covering of L. It is easiest to see from the foregoing corollary and the characterization of Menger frames in terms of directed covers presented in [3, Proposition 2.7] that the Menger property is stronger than the qM property.

Corollary 3.5. *Every Menger frame is* qM*.*

Proof. Let L be a Menger frame. Let $a \in L$, and suppose that (V_n) is a sequence of directed subsets of L such that $a \vee \bigvee V_n = 1$ for each n. For each n, the set

$$V_n^{(a)} = \{a \vee x \mid x \in V_n\}$$

is a directed cover of L, and so, by [3, Proposition 2.2], for each n, there exists an element $v_n \in V_n$ such that

$$1 = \bigvee_{n\in\mathbb{N}} (a \vee v_n) = a \vee \bigvee_{n\in\mathbb{N}} v_n.$$

This implies $(\bigvee_{n\in\mathbb{N}} v_n)^* \leq a$, whence we deduce that L is qM. $\square$

We now use Theorem 3.2 to show that the qM property has been conservatively extended to frames.

Theorem 3.6. *A space X is* qM *if and only if $\Omega(X)$ is* qM*.*

Proof. Suppose that X is qM. Let $A \in \Omega(X)$ and $(\mathscr{V}_n)$ be a sequence of subsets of $\Omega(X)$ such that $A \vee \bigvee \mathscr{V}_n = 1_{\Omega(X)}$ for each n. In topological language this equality says $A \cup \bigcup \mathscr{V}_n = X$, hence, for the closed set $F = X \smallsetminus A$, we have $X \smallsetminus A \subseteq \bigcup \mathscr{V}_n$. Therefore $(\mathscr{V}_n)$ is a sequence of covers of F by open subsets of X, and so, since X is qM, we can select, for each n, a finite $\mathscr{U}_n \subseteq \mathscr{V}_n$ such that $F \subseteq \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathscr{U}_n}$. Taking complements yields $X \smallsetminus \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathscr{U}_n} \subseteq A$. In frame language this says $\left(\bigvee_{n\in\mathbb{N}} \bigvee \mathscr{U}_n\right)^* \leq A$, and so we deduce from Theorem 3.2 that $\Omega(X)$ is qM.

Conversely, suppose that $\Omega(X)$ is qM, and let F be a closed subset of X. Let $(\mathscr{V}_n)$ be a sequence of covers of F by open subsets of X. Put $A = X \smallsetminus F$. Then $A \in \Omega(X)$ and each $\mathscr{V}_n$ is a subset of $\Omega(X)$ with $A \cup \bigcup \mathscr{V}_n = X$, that is, in frame language, $A \vee \bigvee \mathscr{V}_n = 1_{\Omega(X)}$. Since $\Omega(X)$ is qM, Theorem 3.2 furnishes, for each n, a finite $\mathscr{U}_n \subseteq \mathscr{V}_n$ such that $\left(\bigvee_{n\in\mathbb{N}} \bigvee U_n\right)^* \leq A$. Translated to topological language, this says $X \smallsetminus \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathscr{U}_n} \subseteq A$, which, upon taking complements, gives $F \subseteq \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathscr{U}_n}$. This proves that X is qM. $\square$

Since a closed subset F of a topological space X is regular-closed if and only if the element $X \setminus F$ of $\Omega(X)$ is regular, an argument exactly as in the proof of Theorem 3.6 shows the following.

Theorem 3.7. *A topological space is* rqM *if and only if the frame of its open sets is* rqM*.*

Remark 3.8. Since a topological space and its sobrification have isomorphic frames of open sets, it follows that a topological space is qM (resp. rqM) if and only if its sobrification has the same property.

From Theorem 3.2 (or Corollary 3.4) we obtain the following regarding localic images of qM frames.

Corollary 3.9. *A subframe of a* qM *frame is* qM*. Hence, a localic image of a* qM *frame is* qM*.*

Proof. This follows immediately from Theorem 3.2 because the pseudocomplement of an element of a subframe taken in the subframe is below the pseudocomplement of that element taken in the mother frame. □

Note that we cannot use the same argument as in the foregoing proof to say a subframe of a rqM frame is rqM because we are not guaranteed that an element that is regular in a subframe is regular in the ambient frame. There is however a subframe which inherits (and "co-inherits") the property under discussion. That is the content of the next result.

Recall from [19] (see also [6]) that the *semiregularization* of a frame L is the subframe, denoted L_s, generated by the regular elements of L. For $x \in L_s$, denote the pseudocomplement of x in L_s by $x^{\circledast}$. It is shown in [6, p. 369] that, for any $a \in L_s$, $a^{\circledast} = a^*$. A consequence of this is that a regular element in L is regular in L_s, and vice versa.

Corollary 3.10. *A frame is* rqM *if and only if its semiregularization is* rqM*.*

Proof. Suppose, first, that L_s is rqM. Let a be a regular element in L, and let (V_n) be a sequence of directed subsets of L such that $a \vee \bigvee V_n = 1$ for every n. For each n, put

$$\widetilde{V_n} = \{x^{**} \mid x \in V_n\}.$$

Then $(\widetilde{V_n})$ is a sequence of directed subsets of L_s such that, for each n, $a \vee \bigvee \widetilde{V_n} = 1$. Since $a \in L_s$ and L_s is rqM, we can select, for each n, an element $v_n \in V_n$ such that $(\bigvee_{n\in\mathbb{N}} v_n^{**})^{\circledast} \leq a$. Therefore

$$\Big(\bigvee_{n\in\mathbb{N}} v_n\Big)^* = \bigwedge_{n\in\mathbb{N}} v_n^* = \bigwedge_{n\in\mathbb{N}} v_n^{***} = \Big(\bigvee_{n\in\mathbb{N}} v_n^{**}\Big)^* = \Big(\bigvee_{n\in\mathbb{N}} v_n^{**}\Big)^{\circledast} \leq a,$$

which shows that L is rqM.

Conversely, suppose that L is rqM. Let $a \in L_s$ be regular in L_s and let (V_n) be a sequence of directed subsets of L_s with $a \vee \bigvee V_n = 1$ for each n. Then a is regular in L, and so there exists, for each n, an element $v_n \in V_n$ such that $(\bigvee_{n\in\mathbb{N}} v_n)^* \leq a$. Since $(\bigvee_{n\in\mathbb{N}} v_n)^{\circledast} = (\bigvee_{n\in\mathbb{N}} v_n)^*$, it follows that L_s is rqM. □

Remark 3.11. Now that we have presented a proof that brings up the identity

$$\Big(\bigvee_{i\in I} x_i\Big)^* = \Big(\bigvee_{i\in I} x_i^{**}\Big)^*,$$

we point out that, in light of this identity, we can replace "open sublocales" with "regular-open sublocales" in Definition 3.1 for both qM frames and rqM frames. Similarly, in Theorem 3.2 the phrase "subsets of L" can be replaced with "subsets of L each consisting of regular elements".

In [16, Proposition 3.1], it is stated that clopen subsets of qM spaces are qM. We show that this result extends to frames. We shall thereafter show that regular-closed sublocales of a rqM frame are rqM. Thus, it seems that the rqM property is more accommodating regarding which closed subsets inherit the property. We say "seems" because we do not have an example of a regular-closed subset of a qM space which is not qM.

In the proofs of the results just mentioned we shall use the fact that, for any a and b in any frame,

$$(a \wedge b)^{**} = a^{**} \wedge b^{**}.$$

Let us also recall from [8, Lemma 4.5] that if a is regular, and we denote pseudocomplements in $\mathfrak{c}_L(a)$ by $(-)^{\#}$, then $t^{\#} = (t \wedge a^*)^*$ for every $t \in \mathfrak{c}_L(a)$.

Proposition 3.12. *The following results hold.*

(a) Every clopen sublocale of a qM *frame is* qM*.*
(b) Every regular-closed sublocale of a rqM *frame is* rqM*.*

Proof. (a) Let a be a complemented element of a qM frame L. We use Corollary 3.4 to show that $\mathfrak{c}_L(a)$ is qM. So, let $b \in \mathfrak{c}_L(a)$, and suppose that (V_n) is a sequence of directed subsets of $\mathfrak{c}_L(a)$ with $b \vee \bigvee V_n = 1$ for every n. For each n, set

$$U_n = \{a^* \wedge x \mid x \in V_n\}.$$

Then (U_n) is a sequence of directed subsets of L. Since $b \geq a$, $b \vee a^* = 1$ because a is complemented, and so, for each n,

$$b \vee \bigvee U_n = b \vee \bigvee \{a^* \wedge x \mid x \in V_n\} = b \vee \Big(a^* \wedge \bigvee \{x \mid x \in V_n\}\Big) = \Big(b \vee a^*\Big) \wedge \Big(b \vee \bigvee V_n\Big) = 1.$$

Since L is qM, we can find, for each n, a $u_n \in U_n$ such that $(\bigvee_{n\in\mathbb{N}} u_n)^* \leq b$. Pick $v_n \in V_n$ such that $u_n = a^* \wedge v_n$. Now, taking into account how pseudocomplements in $\mathfrak{c}_L(a)$ are calculated, for the element $\bigvee_{n\in\mathbb{N}} v_n$ of $\mathfrak{c}_L(a)$ we have

$$\Big(\bigvee_{n\in\mathbb{N}} v_n\Big)^{\#} = \Big(a^* \wedge \bigvee_{n\in\mathbb{N}} v_n\Big)^* = \Big(\bigvee_{n\in\mathbb{N}} (a^* \wedge v_n)\Big)^* = \Big(\bigvee_{n\in\mathbb{N}} u_n\Big)^* \leq b,$$

which shows that $\mathfrak{c}_L(a)$ is qM.

(b) Let a be a regular element of a rqM frame L. Again, we use Corollary 3.4 to show that $\mathfrak{c}_L(a)$ is rqM. Let b be a regular element of $\mathfrak{c}_L(a)$, and let (V_n) be a sequence of directed subsets of $\mathfrak{c}_L(a)$ such that $b \vee \bigvee V_n = 1$ for each n. We show that b is also regular as an element of L. Indeed,

$$\begin{aligned} b = b^{\#\#} = \Big((b \wedge a^*)^*\Big)^{\#} &= \Big((b \wedge a^*)^* \wedge a^*\Big)^* \\ &= \Big(\Big((b \wedge a^*) \vee a\Big)^*\Big)^* \\ &= \Big((b \vee a) \wedge (a^* \vee a)\Big)^{**} = (b \vee a)^{**} \wedge (a \vee a^*)^{**} = b^{**}; \end{aligned}$$

the last part because $b \geq a$ and $(a \vee a^*)^{**} = 1$. Since L is rqM, we can select, for each n, an element $v_n \in V_n$ such that $(\bigvee_{n\in\mathbb{N}} v_n)^* \leq b$. For brevity, write $v = \bigvee_{n\in\mathbb{N}} v_n$, and note that $b^* \leq v^{**}$. Now,

$$v^{\#} = (v \wedge a^*)^* = (v \wedge a^*)^{***} = (v^{**} \wedge a^*)^* \leq (b^* \wedge a^*)^* = (b \vee a)^{**} = b^{**} = b,$$

which then shows that $\mathfrak{c}_L(a)$ is rqM. □

Corollary 3.13. *Every regular-closed subspace of a* rqM *space is* rqM*.*

We end this section by saying a few words about coproducts. We do not recall the construction of coproducts as it is adequately treated in [20]. We however recite some pertinent properties that we shall need. If L and M are frames, then:

– the elements $a \oplus b$ generate $L \oplus M$.

– $a \oplus b = 0_{L\oplus M}$ if and only if $a = 0$ or $b = 0$.

– $(a \oplus b)^* = (a^* \oplus 1) \vee (1 \oplus b^*)$ and $(a \oplus b)^{**} = a^{**} \oplus b^{**}$ (see, for instance, [2]).

– for any family $\{a_i \mid i \in I\}$ of elements of L and any $b \in M$, $\bigvee_{i\in I}(a_i \oplus b) = \left(\bigvee_{i\in I} a_i\right) \oplus b$.

Theorem 3.14. *If $L \oplus M$ is* qM *(resp.* rqM*), then both L and M are* qM *(resp.* rqM*).*

Proof. We show that L is qM. Let $a \in L$, and suppose that (V_n) is a sequence of directed subsets of L such that $a \vee \bigvee V_n = 1$ for each n. Without loss of generality, we may assume that none of the sets V_n contains 0. For each n, let $\widehat{V_n}$ be the subset of $L \oplus M$ given by

$$\widehat{V_n} = \{x \oplus 1 \mid x \in V_n\}.$$

Then $(\widehat{V_n})$ is a sequence of directed subsets of $L \oplus M$ such that, for each n,

$$(a \oplus 1) \vee \bigvee \widehat{V_n} = (a \oplus 1) \vee \bigvee_{x\in V_n} (x \oplus 1) = \left(a \vee \bigvee V_n\right) \oplus 1 = 1_{L\oplus M}.$$

Since $L \oplus M$ is qM, for each n we can select some $x_n \in V_n$ such that $\left(\bigvee_{n\in\mathbb{N}}(x_n \oplus 1)\right)^* \le (a \oplus 1)$. Now, note that

$$\Big(\bigvee_{n\in\mathbb{N}} (x_n \oplus 1)\Big)^* = \Big(\Big(\bigvee_{n\in\mathbb{N}} x_n\Big) \oplus 1\Big)^* = \Big(\Big(\bigvee_{n\in\mathbb{N}} x_n\Big)^* \oplus 1\Big) \vee \Big(\Big(\bigvee_{n\in\mathbb{N}} x_n\Big) \oplus 0\Big) = \Big(\bigvee_{n\in\mathbb{N}} x_n\Big)^* \oplus 1,$$

as a consequence of which the inequality in the previous sentence says

$$\Big(\bigvee_{n\in\mathbb{N}} x_n\Big)^* \oplus 1 \le a \oplus 1. \tag{‡}$$

If $(\bigvee_{n\in\mathbb{N}} x_n)^* = 0$, then, of course $(\bigvee_{n\in\mathbb{N}} x_n)^* \le a$. If $(\bigvee_{n\in\mathbb{N}} x_n)^* \ne 0$, we deduce from (‡) that $(\bigvee_{n\in\mathbb{N}} x_n)^* \le a$. It follows therefore that L is qM. That M is also qM is shown similarly.

The proof of the rqM case is similar because if a is regular in L, then $a \oplus 1$ is regular in $L \oplus M$ since $(a \oplus 1)^{**} = a^{**} \oplus 1 = a \oplus 1$. □

4. Concerning spectra

Recall that an element $p \in L$ is called a *point* (or a *prime*) if it satisfies the property that

$$p < 1 \quad \text{and} \quad (\forall x, y \in L)(x \wedge y \le p \implies x \le p \text{ or } y \le p).$$

We write $\mathrm{Pt}(L)$ for the set of points of L. A frame is *spatial* if it is isomorphic to $\Omega(X)$ for some space X. This is the case precisely when every element is a meet of primes.

We view the *spectrum* of L as the topological space

$$\Big(\mathrm{Pt}(L), \{\Sigma_a \mid a \in L\}\Big)$$

where, for each $a \in L$,

$$\Sigma_a = \{p \in \mathrm{Pt}(L) \mid a \not\le p\}.$$

The map $\eta_L\colon L \to \Omega(\Sigma L)$ given by $\eta_L(a) = \Sigma_a$ is an onto homomorphism, and is the reflection map from L to spatial frames.

The contravariant functors $\Omega\colon$ **Top** $\to$ **Frm** and $\Sigma\colon$ **Frm** $\to$ **Top** do not, in general, behave similarly with regard to preserving or reflecting properties. We saw in the previous section that Ω preserves and reflects both the qM and the rqM properties. We shall show that, subject to some condition on the meet of primes, Σ reflects the rqM property. As mentioned in the Introduction, we propose to do this directly, using topological arguments because in the course of the proof some properties of the spectral topology come to the fore. We will then show how the result can also be achieved via frame-theoretic arguments.

We start by describing the closure of an open set in the spectral topology. For any $a \in L$, let us write Σ'_a for the closed set $\Sigma L \smallsetminus \Sigma_a$, so that

$$\Sigma'_a = \{p \in \mathrm{Pt}(L) \mid a \leq p\}.$$

Note that, in any frame L, if $u \leq v$, then $\Sigma'_v \subseteq \Sigma'_u$. Let us write m_L for the meet of all points of L. The understanding is, of course, that $m_L = 1$ if L has no points. We remark, in passing, that if L is spatial then $m_L = 0$, but not conversely, as shown in [7, Example 3.2]. The following lemma characterizes when the reflection map $L \to \Omega(\Sigma L)$ is dense in terms of m_L.

Lemma 4.1. *For any L, $m_L = 0$ if and only if the homomorphism $\eta_L\colon L \to \Omega(\Sigma L)$ is dense.*

Proof. To see the left-to-right implication, note first that if $m_L = 0$, then the spectrum of L is nonempty. Now consider any $a \in L$ with $\eta_L(a) = 0_{\Omega(\Sigma L)}$. Then $\Sigma_a = \emptyset$, so that a is below every point of L, hence $a = 0$.

Conversely, suppose that $\eta_L\colon L \to \Omega(\Sigma L)$ is dense. Then L has at least one point, else every element of L would be mapped to the bottom of $\Omega(\Sigma L)$. Since m_L is below every point of L, $\Sigma_{m_L} = \emptyset$, which says $\eta_L(m_L) = 0_{\Omega(\Sigma L)}$, which then implies $m_L = 0$. □

Recall the Heyting implication $\to$ in any frame, given by

$$a \to b = \bigvee\{x \in L \mid x \wedge a \leq b\}.$$

Lemma 4.2. *Let L be a frame with nonempty spectrum. For any $a \in L$, $\overline{\Sigma_a} = \Sigma'_{a\to m_L}$.*

Proof. Let $p \in \Sigma_a$. Then $a \nleq p$. Since

$$a \wedge (a \to m_L) = a \wedge m_L \leq m_L \leq p,$$

and since p is a point with $a \nleq p$, it follows that $a \to m_L \leq p$, so that $p \in \Sigma'_{a\to m_L}$. Therefore $\Sigma_a \subseteq \Sigma'_{a\to m_L}$, and since $\Sigma'_{a\to m_L}$ is a closed set, it follows that $\overline{\Sigma_a} \subseteq \Sigma'_{a\to m_L}$. Now consider any closed set, say Σ'_b, with $\Sigma_a \subseteq \Sigma'_b$. Then $\Sigma_a \cap \Sigma_b = \emptyset$, that is $\Sigma_{a\wedge b} = \emptyset$, which implies that $a \wedge b$ is below every point of L, and hence $a \wedge b \leq m_L$, whence $b \leq a \to m_L$, thence $\Sigma'_{a\to m_L} \subseteq \Sigma'_b$. Therefore $\Sigma'_{a\to m_L}$ is the smallest closed set containing Σ_a, which is exactly what we are to prove. □

Corollary 4.3. *If the meet of all points of L is 0, then $\overline{\Sigma_a} = \Sigma'_{a^*}$ for every $a \in L$.*

Let us note that, for a dense homomorphism h, if $h(a) \leq h(b)$ with b regular, then $a \leq b$. This is so because

$$h(a \wedge b^*) = h(a) \wedge h(b^*) \leq h(b) \wedge h(b^*) = 0,$$

implying $a \wedge b^* = 0$ as h is dense, and hence $a \leq b^{**} = b$. Let us also note that, in any frame, if $a \vee b = 1$, then $\Sigma'_a \subseteq \Sigma_b$.

We shall use the notation that, for any $A \subseteq L$,

$$\Sigma_A = \{\Sigma_a \mid a \in A\}.$$

Theorem 4.4. *If the points of L meet at the bottom and ΣL is* rqM*, then L is* rqM.

Proof. Let a be a regular element in L, and suppose that (V_n) is a sequence of subsets of L such that $a \vee \bigvee V_n = 1$ for every n. Then, for each n, $\Sigma'_a \subseteq \Sigma_{\bigvee V_n} = \bigcup \Sigma_{V_n}$. Therefore (Σ_{V_n}) is a sequence of covers of the closed set Σ'_a by sets open in ΣL. Since $m_L = 0$, we know from Corollary 4.3 that $\Sigma'_a = \Sigma'_{(a^*)^*} = \overline{\Sigma_{a^*}}$, and so Σ'_a is a regular-closed subset of ΣL. Since ΣL is rqM, there therefore exists, for each n, a finite $U_n \subseteq V_n$ such that $\Sigma'_a \subseteq \overline{\bigcup_{n\in\mathbb{N}} \bigcup \Sigma_{U_n}}$. Putting $u_n = \bigvee U_n$, this containment says

$$\Sigma'_a \subseteq \overline{\bigcup_{n\in\mathbb{N}} \bigcup \Sigma_{U_n}} = \overline{\bigcup_{n\in\mathbb{N}} \Sigma_{u_n}} = \overline{\Sigma_{\bigvee_{n\in\mathbb{N}} u_n}} = \Sigma'_{(\bigvee_{n\in\mathbb{N}} u_n)^*};$$

the last equality in view of Corollary 4.3. Taking complements in ΣL, we have $\Sigma_{(\bigvee_{n\in\mathbb{N}} u_n)^*} \subseteq \Sigma_a$, which, in terms of the homomorphism η_L, says $\eta_L((\bigvee_{n\in\mathbb{N}} u_n)^*) \leq \eta_L(a)$. Since η_L is dense, by Lemma 4.1, and a is a regular element, we therefore have $(\bigvee_{n\in\mathbb{N}} u_n)^* \leq a$, and it follows that L is rqM. □

Remark 4.5. The crucial part played by hypothesizing a to be regular should be noted. In consequence, this same proof cannot be used to come to a similar conclusion for qM frames.

As mentioned above, the result in Theorem 4.4 can also be obtained differently by first establishing the following proposition. Recall from [2] that a homomorphism h is called *nearly open* if $h(a^*) = h(a)^*$ for every a in the domain of h. It is known that h is nearly open if and only if $h(a^{**}) = h(a)^{**}$ for every a. Therefore nearly open homomorphisms send regular elements to regular elements. Honor to whom honor is due: nearly open homomorphisms were first considered by Johnstone [12] under the appellation "weakly open" homomorphisms.

Proposition 4.6. *If $h\colon L \to M$ is a dense nearly open homomorphism and M is rqM, then L is rqM.*

Proof. Let a be a regular element in L, and suppose that (V_n) is a sequence of directed subsets of L such that $a \vee \bigvee V_n = 1$. Then $h(a)$ is a regular element in M and $(h[V_n])$ is a sequence of directed subsets of M such that $h(a) \vee \bigvee h[V_n] = 1$ for each n. Since M is rqM, for each n, there exists an element $v_n \in V_n$ such that $\left(\bigvee_{n\in\mathbb{N}} h(v_n)\right)^* \leq h(a)$. Using the fact that h is nearly open, we therefore have

$$h\Big(\Big(\bigvee_{n\in\mathbb{N}} v_n\Big)^*\Big) = h\Big(\bigvee_{n\in\mathbb{N}} v_n\Big)^* = \Big(\bigvee_{n\in\mathbb{N}} h(v_n)\Big)^* \leq h(a),$$

which implies $(\bigvee_{n\in\mathbb{N}} v_n)^* \leq a$ because a is regular and h is dense. It follows therefore that L is rqM. □

Now, by Corollary 4.1, $\eta_L\colon L \to \Omega(\Sigma L)$ is dense if and only if the points of L meet at the bottom. Also, by Theorem 3.6, ΣL is rqM if and only if $\Omega(\Sigma L)$ is rqM. Since dense onto homomorphisms are nearly open, it therefore follows from Proposition 4.6 that if the points of L meet at the bottom and ΣL is rqM, then L is rqM.

Having observed that a nearly open homomorphism reflects the rqM property when it is dense, we show that it preserves the rqM property when its right adjoint takes covers to covers. In fact, as the reader will notice, the proof we give can easily be mimicked to show that it also preserves the qM property. Of course the proof of Proposition 4.6 cannot be mimicked to establish the qM analogue of the result in that proposition because the regularity of the element a (as in the case of Theorem 4.4) is used to conclude the argument.

Proposition 4.7. *Let $h\colon L \to M$ be a nearly open homomorphism whose right adjoint takes covers to covers. If L is rqM (resp. qM), then M is also rqM (resp. qM).*

Proof. We prove the rqM case. Let b be a regular element in M, and suppose that (W_n) is a sequence of directed subsets of M such that $b \vee \bigvee W_n = 1$ for each n. Then $(h_*[W_n])$ is a sequence of directed subsets of L, and, for each n, $h_*(b) \vee \bigvee h_*[W_n] = 1$ because h_* takes covers to covers. Hence $h_*(b)^{**} \vee \bigvee h_*[W_n] = 1$. Since $h_*(b)^{**}$ is regular and L is rqM, for each n we can select an element w_n in W_n such that $\left(\bigvee_{n\in\mathbb{N}} h_*(w_n)\right)^* \leq h_*(b)^{**}$.

Now, in light of the fact that each $h(h_*(w_n)) \leq w_n$, we have

$$\begin{aligned}
\Big(\bigvee_{n\in\mathbb{N}} w_n\Big)^* \leq \Big(\bigvee_{n\in\mathbb{N}} h(h_*(w_n))\Big)^* &= \Big(h\Big(\bigvee_{n\in\mathbb{N}} h_*(w_n)\Big)\Big)^* \\
&= h\Big(\Big(\bigvee_{n\in\mathbb{N}} h_*(w_n)\Big)^*\Big) && \text{since } h \text{ is nearly open} \\
&\leq h\Big(h_*(b)^{**}\Big) \\
&= \Big(h(h_*(b)\Big)^{**} && \text{since } h \text{ is nearly open} \\
&\leq b^{**} \\
&= b.
\end{aligned}$$

It follows therefore that M is qM. □

5. Weakly Menger frames

We start by recalling the concept in spaces that we wish to import to frames. A topological space X is called weakly Menger if for every sequence $(\mathscr{V}_n)$ of open covers of X, there exists, for each n, a finite $\mathscr{U}_n \subseteq \mathscr{V}_n$ such that $\bigcup_{n\in\mathbb{N}}\bigcup\mathscr{U}_n$ is dense in X. These spaces are called "weakly Hurewicz" in [4]. Importing this definition almost verbatim to frames, we formulate the following. Recall that an element a of a frame L is *dense* if $a^* = 0$. This is equivalent to saying the open sublocale $\mathfrak{o}_L(a)$ is dense in L.

Definition 5.1. A frame L is *weakly Menger* (abbreviated wM) if for every sequence $(\mathscr{C}_n)$ of open coverings of L, there exists, for each n, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigvee\{T \mid T \in \bigcup_{n\in\mathbb{N}}\mathscr{D}_n\}$ is a dense sublocale of L. We shall say the sequence $(\mathscr{D}_n)$ is a *weakly Menger witness* (abbreviated wM-witness) for the sequence $(\mathscr{C}_n)$.

It is immediate from the definition that every Menger frame is wM. Also, by taking F to be the whole frame in the definition of qM frames, we see that every qM frame is wM. Using the bijections $\mathscr{C} \mapsto C^{\mathscr{C}}$ and $C \mapsto \mathscr{C}^{C}$, and passing to sequences of directed coverings or covers, arguments similar to those of the proof of [3, Proposition 2.7] yield the following characterizations.

Proposition 5.2. *The following are equivalent for a frame L.*

1. *L is* wM.
2. *For every sequence $(\mathscr{C}_n)$ of directed open coverings of L, there exists, for each n, some $C_n \in \mathscr{C}_n$ such that $\bigvee_{n\in\mathbb{N}}C_n$ is a dense sublocale of L.*
3. *For every sequence (C_n) of covers of L, there exists, for each n, a finite $D_n \subseteq C_n$ such that $\bigvee D$ is a dense element in L, where $D = \bigcup_{n\in\mathbb{N}}D_n$.*
4. *For every sequence (C_n) of directed covers of L, there exists, for each n, some $c_n \in C_n$ such that $\bigvee_{n\in\mathbb{N}}c_n$ is a dense element in L.*

Since a cover of a subframe is a cover of the ambient frame, and since an element of the ambient frame that also belongs to a subframe is dense in the subframe if it is dense in the frame, we have the following corollary.

Corollary 5.3. *A subframe of a* wM *frame is* wM*. Hence, a localic image of a* wM *frame is* wM*.*

Recall that a frame is called *almost compact* if each of its covers has a finite subset the join of which is a dense element. As another corollary of Proposition 5.2, one checks routinely that every almost compact frame is wM.

The wM property for frames, as we have defined it, is conservative. To see this, observe that, as in frames, a space X is wM if and only if for every sequence $(\mathscr{V}_n)$ of directed open covers of X, there exists, for each n, a set $V_n \in \mathscr{V}_n$ such that $\bigcup_{n\in\mathbb{N}}V_n$ is dense in X.

Proposition 5.4. *A space X is* wM *if and only if* $\Omega(X)$ *is* wM*.*

Proof. Suppose that X is wM and let $(\mathscr{C}_n)$ be a sequence of directed covers of $\Omega(X)$. Viewed topologically, each $\mathscr{C}_n$ is an open cover of X. So we can find, for each n, some $C_n \in \mathscr{C}_n$ such that $\bigcup_{n\in\mathbb{N}} C_n$ is dense in X. Frame-theoretically, this says $\bigvee_{n\in\mathbb{N}} C_n$ is a dense element in $\Omega(X)$. Therefore $\Omega(X)$ is wM.

Conversely, suppose that $\Omega(X)$ is wM, and let $(\mathscr{V}_n)$ be a sequence of directed open covers of X. Viewed frame-theoretically, each $\mathscr{V}_n$ is a directed cover of the frame $\Omega(X)$, so we can select, for each n, some $V_n \in \mathscr{V}_n$ such that $\bigvee_{n\in\mathbb{N}} V_n$ is a dense element in $\Omega(X)$. Topologically, this says $\bigcup_{n\in\mathbb{N}} V_n$ is dense in X. Therefore X is wM. □

For comparison, let us recall from [3] that a frame L is called *almost Menger* (abbreviated aM) if for every sequence $(\mathscr{C}_n)$ of open coverings of L, there exists, for each n, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigvee\{\overline{D} \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n\} = L$. In this case, the sequence $(\mathscr{D}_n)$ is called an *aM-witness* for the sequence $(\mathscr{C}_n)$. It is perhaps not so immediate that every aM frame is wM. To prove this, we draw the attention of the reader to the fact that a sublocale of any frame is dense if and only if it has non-void intersection with every non-void open sublocale (see [9, Lemma 9.2]).

Proposition 5.5. *Every* aM *frame is* wM*.*

Proof. Let $(\mathscr{C}_n)$ be a sequence of open coverings of an aM frame L, and let $(\mathscr{D}_n)$ be an aM-witness for $(\mathscr{C}_n)$, so that $\bigvee\{\overline{D} \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n\} = L$. We argue that $(\mathscr{D}_n)$ is a wM-witness for $(\mathscr{C}_n)$. Suppose, by way of contradiction, that $\bigvee\{D \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n\}$ is not a dense sublocale of L. Then there exists a non-void open sublocale U of L such that

$$\mathsf{O} = U \cap \bigvee\Big\{D \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n\Big\} = \bigvee\Big\{U \cap D \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n\Big\};$$

which implies $U \cap D = \mathsf{O}$ for every $D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n$. We know from [20, Lemma VIII. 4.2.1] that an open sublocale misses a sublocale if and only if it misses the closure of that sublocale, so

$$U = U \cap L = U \cap \bigvee\Big\{\overline{D} \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n\Big\} = \bigvee\Big\{U \cap \overline{D} \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n\Big\} = \mathsf{O};$$

yielding a contradiction. Therefore L is wM. □

Remark 5.6. From the equivalence of statements (1) and (4) in Proposition 5.2, we have the following "sublocale-free" reaffirmation of the fact that aM frames are wM. Let L be an aM frame, and let (C_n) be a sequence of directed covers of L. By [3, Corollary 3.7], there exists, for each n, an element $c_n \in C_n$ such that $0 = \bigwedge_{n\in\mathbb{N}} t_n$, for some elements t_n with $c_n^* \le t_n$ for each n. Therefore

$$0 \le \Big(\bigvee_{n\in\mathbb{N}} c_n\Big)^* = \bigwedge_{n\in\mathbb{N}} c_n^* \le \bigwedge_{n\in\mathbb{N}} t_n = 0,$$

which implies that $\bigvee_{n\in\mathbb{N}} c_n$ is dense, and therefore L is wM.

Remark 5.7. Abbreviate by M the Menger property. We then have the non-reversible (already in spaces) implications:

$$\mathsf{M} \implies \mathsf{aM} \implies \mathsf{wM} \qquad \text{and} \qquad \mathsf{M} \implies \mathsf{qM} \implies \mathsf{wM}.$$

In the case of Boolean frames all these implications are equivalences. To see this, observe that if L is Boolean and wM, then it is Menger because the only dense element in a Boolean frame is the top.

We now present some characterizations of wM frames, which include localic versions of [18, Theorem 7]. To that end, let us introduce some terminology which is lifted verbatim from spaces.

Definition 5.8. We say an open covering $\mathscr{U}$ of a frame L is a *κ-covering* if $L \notin \mathscr{U}$ and for every compact sublocale K of L, there exists some $U \in \mathscr{U}$ such that $K \subseteq U$.

Extending earlier usage, let us say a sequence (D_n) of covers of L is a wM-witness for a sequence (C_n) of covers of L if each D_n is a finite subset of C_n and $\bigvee\{d \mid d \in \bigcup_{n\in\mathbb{N}} D_{n\in\mathbb{N}}\}$ is a dense element of L.

Note that dense homomorphisms reflect density of elements. That is, if $h\colon L \to M$ is a dense homomorphism and $a \in L$ is such that $h(a)$ is dense, then a is dense. For, if $x \wedge a = 0$, for any $x \in L$, then $h(x) \wedge h(a) = 0$, which implies $h(x) = 0$, whence $x = 0$, showing that a is dense.

Theorem 5.9. *The following are equivalent for a frame L.*

1. *L is* wM*.*
2. *L_s is* wM*.*
3. *Every sequence of κ-coverings of L has wM-witness.*
4. *Whenever $h\colon M \to L$ is a dense homomorphism, then M is* wM*.*

Proof. (1) ⇔ (2): If L is wM, then, being a subframe of L, L_s is wM. Conversely, suppose that L_s is wM, and let (C_n) be a sequence of directed covers of L. For each n, let $D_n = \{x^{**} \mid x \in C_n\}$. Then (D_n) is a sequence of directed covers of L_s, so for each n there exists some $c_n \in C_n$ such that $\bigvee_{n\in\mathbb{N}} c_n^{**}$ is dense in L_s. Therefore

$$\Big(\bigvee_{n\in\mathbb{N}} c_n\Big)^* = \Big(\bigvee_{n\in\mathbb{N}} c_n^{**}\Big)^* = \Big(\bigvee_{n\in\mathbb{N}} c_n^{**}\Big)^{\circledast} = 0,$$

so that $\bigvee_n c_n$ is a dense element in L. Therefore L is wM.

(1) ⇔ (3): It is immediate that (1) implies (3). Conversely, suppose that (3) holds, and let $(\mathscr{C}_n)$ be a sequence of directed open coverings of L. If $L \in \mathscr{C}_{n_0}$ for some index n_0, then, choosing $C_{n_0} = L$ and C_n to be any member of $\mathscr{C}_n$ for $n \neq n_0$, we see that the sequence $(\mathscr{C}_n)$ has a wM-witness. So we may assume that L does not belong to any of the coverings $\mathscr{C}_n$. Let K be a compact sublocale of L. Since in the lattice $\mathcal{S}(L)$ binary meets distribute over joins consisting of open sublocales, for any n we have

$$K = K \cap \bigvee\{C \mid C \in \mathscr{C}_n\} = \bigvee\{K \cap C \mid C \in \mathscr{C}_n\},$$

and so, since K is compact, there is a positive integer k_n and sublocales $C_1, \dots, C_{k_n}$ in $\mathscr{C}_n$ such that

$$K = (K \cap C_1) \vee \cdots \vee (K \cap C_{k_n}) = K \cap (C_1 \vee \cdots \vee C_{k_n}),$$

so that $K \subseteq C_1 \vee \cdots \vee C_{k_n}$ and hence K is contained in some member of $\mathscr{C}_n$ because the collection $\mathscr{C}_n$ is directed. Therefore $(\mathscr{C}_n)$ is a sequence of κ-coverings of L, and hence has a wM-witness by hypothesis. In all then, L is wM.

(1) ⇔ (4): Assume that L is wM, and let $h\colon M \to L$ be a dense homomorphism. Let (C_n) be a sequence of directed covers of M. Then $(h[C_n])$ is a sequence of directed covers of L, and so, for each n, we can select an element $c_n \in C_n$ such that $\bigvee_{n\in\mathbb{N}} h(c_n)$ is a dense element in L. This says $h(\bigvee_{n\in\mathbb{N}} c_n)$ is dense, which therefore makes $\bigvee_{n\in\mathbb{N}} c_n$ dense because h is dense. It follows there fore that M is wM.

Conversely, assume that the condition in (4) holds. Since the identical map $\mathrm{id}_L\colon L \to L$ is dense, L is wM. □

Remark 5.10. Without the density condition, the implication (1) ⇒ (4) fails. Indeed, let X be any nonempty topological space which is not wM, and consider any $p \in X$. For the homomorphism $\xi_p\colon \Omega(X) \to \mathbf{2}$, induced by the point $X \smallsetminus \overline{\{p\}}$ of $\Omega(X)$, the codomain of ξ_p is wM whereas its domain is not.

Since any sublocale is a dense sublocale of its closure, the implication (1) ⇒ (4) in Theorem 5.9 gives us the following result.

Corollary 5.11. *Let L be a frame.*

(a) The closure of any wM *sublocale of L is* wM*.*
(b) If the smallest dense sublocale of L is wM*, then L is* wM*.*

There are frames which are wM precisely when their smallest dense sublocales are wM. In [1], Arietta calls a frame *infinitely extremally disconnected* if $(\bigvee_{i\in I} a_i)^{**} = \bigvee_{i\in I} a_i^{**}$ for all families $\{a_i \mid i \in I\}$ of elements of the frame.

Proposition 5.12. *An infinitely extremally disconnected frame is* wM *if and only if its smallest dense sublocale is* wM*.*

Proof. Only one implication needs to be proved. Suppose L is an infinitely extremally disconnected frame which is wM. Denote the join in $\mathfrak{B}L$ by $\bigsqcup$. Let (C_n) be a sequence of directed covers of $\mathfrak{B}L$. Then, for each n,

$$\bigsqcup C_n = \Big(\bigvee C_n\Big)^{**} = \bigvee\{x^{**} \mid x \in C_n\} = \bigvee\{x \mid x \in C_n\} = \bigvee C_n,$$

so that (C_n) is a sequence of directed covers of L. Since L is wM, we can select, for each n, an element $c_n \in C_n$ such that $\bigvee_{n\in\mathbb{N}} c_n$ is a dense in L. This implies

$$1 = \Big(\bigvee_{n\in\mathbb{N}} c_n\Big)^{**} = \bigsqcup_{n\in\mathbb{N}} c_n,$$

showing that the latter is a dense element in $\mathfrak{B}L$. Therefore $\mathfrak{B}L$ is wM. □

In general, we are able to characterize when the smallest dense sublocale is wM. Recall that a subset of a frame is called a *quasi-cover* if its join is a dense element in the frame.

Theorem 5.13. *The following are equivalent for a frame L.*

1. *$\mathfrak{B}L$ is* wM*.*
2. *For every sequence (C_n) of directed quasi-covers of L, we can select, for each n, an element $c_n \in C_n$ such that $\{c_n \mid n \in \mathbb{N}\}$ is a quasi-cover of L.*

Proof. Assume, first, that $\mathfrak{B}L$ is wM, and let (C_n) be a sequence of directed quasi-covers of L. For each n, put $D_n = \{x^{**} \mid x \in C_n\}$. As before, denote the join in $\mathfrak{B}L$ by $\bigsqcup$. Then, since $\bigvee C_n$ is a dense element in L,

$$\bigsqcup D_n = \Big(\bigvee\{x^{**} \mid x \in C_n\}\Big)^{**} \geq \Big(\bigvee\{x \mid x \in C_n\}\Big)^{**} = 1.$$

Thus, (D_n) is a sequence of directed covers of $\mathfrak{B}L$, so, by hypothesis, we can select, for each n, some $c_n \in C_n$ such that $\bigsqcup_{n\in\mathbb{N}} c_n^{**} = 1$ because the only dense element in a Boolean frame is the top. Thus,

$$1 = \Big(\bigvee_{n\in\mathbb{N}} c_n^{**}\Big)^{**} = \Big(\bigwedge_{n\in\mathbb{N}} c_n^{***}\Big)^{*} = \Big(\bigwedge_{n\in\mathbb{N}} c_n^{*}\Big)^{*} = \Big(\bigvee_{n\in\mathbb{N}} c_n\Big)^{**},$$

which says $\bigvee_{n\in\mathbb{N}} c_n$ is a dense element in L. Therefore L is wM.

Conversely, suppose that L is wM, and let (C_n) be a sequence of directed covers of $\mathfrak{B}L$. Then (C_n) is a sequence of directed quasi-covers of L, so, by the present hypothesis, we can select, for each n, some $c_n \in C_n$ such that $\bigvee_{n\in\mathbb{N}} c_n$ is a dense element in L. This certainly makes $\bigsqcup_{n\in\mathbb{N}} c_n$ the top element (hence a dense element) in $\mathfrak{B}L$. Therefore $\mathfrak{B}L$ is wM. □

Another corollary of the implication (1) ⇒ (4) in Theorem 5.9 is with regard to coproducts. It also gives us a result in topological spaces.

Corollary 5.14. *Let $(L_i \mid i \in I)$ be a family of frames, and $(X_j \mid j \in J)$ a family of topological spaces.*

(a) If the coproduct $\bigoplus_{i\in I} L_i$ is wM*, then each L_i is* wM*.*
(b) If the product $\prod_{j\in J} X_j$ is wM*, then each X_j is* wM*.*

Proof. (a) This follows from the fact that each coproduct injection is dense because it is one-one.

(b) If $\prod_{j\in J} X_j$ is wM, then $\Omega\bigl(\prod_{j\in J} X_j\bigr)$ is wM by Proposition 5.4. As shown in [20, Chapter IV, Section 5.4], there is a dense (actually, dense onto) homomorphism

$$\bigoplus_{j\in J} \Omega(X_j) \to \Omega\Bigl(\prod_{j\in J} X_j\Bigr).$$

Therefore $\bigoplus_{j\in J} \Omega(X_j)$ is wM. So, by the first part, each $\Omega(X_j)$ is wM, which implies that each X_j is wM, by Proposition 5.4 again. □

We have seen that a dense homomorphism reflects the wM property. We now identify some homomorphisms that preserve it. A homomorphism is called *weakly perfect* if its right adjoint preserves directed covers. This is strictly weaker than requiring the homomorphism to be *perfect*, which is defined by requiring that the right adjoint preserve all directed joins. Recall that a homomorphism is called *skeletal* if it maps dense elements to dense elements. This term is borrowed from topology. Indeed, a continuous map is skeletal precisely when $f^{-1}[-]$ sends dense open sets to dense (open) sets.

Proposition 5.15. *If $h\colon L \to M$ is a weakly perfect skeletal homomorphism and L is* wM*, then M is* wM*.*

Proof. Let (C_n) be a sequence of directed covers of M. Since h is weakly perfect, $(h_*[C_n])$ is a sequence of directed covers of L. Since L is weakly Menger, we can select, for each n, some $c_n \in C_n$ such that $\bigvee_{n\in\mathbb{N}} h_*(c_n)$ is a dense element in L. Since h is skeletal, $h(\bigvee_{n\in\mathbb{N}} h_*(c_n))$ is dense, that is, $\bigvee_{n\in\mathbb{N}} h(h_*(c_n))$ is dense, and so $\bigvee_{n\in\mathbb{N}} c_n$ is dense because each c_n is above $h(h_*(c_n))$. Therefore M is wM. □

As a corollary, we have the following result about binary coproducts. Recall that if L and M are frames, then the coproduct injections

$$L \xrightarrow{\ i_L\ } L \oplus M \xleftarrow{\ i_M\ } M$$

are *open maps*, which is to say they preserve meets and the Heyting implication. Furthermore, as shown in [10, Lemma 2], if L is compact then $(i_M)_*$, the right adjoint of i_M, preserves directed joins. Note that a nearly open map is skeletal.

Corollary 5.16. *If L is compact and M is* wM*, then $L \oplus M$ is* wM*.*

Proof. This follows from Proposition 5.15 because the injection $i_M\colon M \to L \oplus M$ is nearly open (being open), hence skeletal, and its right adjoint preserves directed joins, hence it is weakly perfect. □

Let us apply Proposition 5.15 to topological spaces. As usual, by a *filtered* subset of a poset we mean a down-directed one. Call a collection $\mathscr{C}$ of closed subsets of a topological space a *co-cover* if $\bigcap \mathscr{C} = \emptyset$.

Corollary 5.17. *Let $f\colon X \to Y$ be a skeletal closed continuous map whose induced direct-image map $f[-]$ preserves filtered co-covers. If Y is* wM*, then so is X. In particular, a skeletal perfect continuous map reflects the* wM *property.*

Proof. In light of the fact that a topological space is wM if and only if its frame of open sets is wM, we need only show that the homomorphism $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ is weakly perfect and skeletal. The latter holds because f is skeletal. Let $\{U_\alpha \mid \alpha \in A\}$ be a directed cover of $\Omega(X)$. Then $\{X \smallsetminus U_\alpha \mid \alpha \in A\}$ is a filtered co-cover of X. Since $f[-]$ preserves filtered co-covers, $\bigcap_{\alpha\in A} f[X \smallsetminus U_\alpha] = \emptyset$. Recall that the right adjoint of $\Omega(f)$ is given by

$$\Omega(f)_*(V) = Y \smallsetminus \overline{f[X \smallsetminus V]}.$$

Now, in light of the fact that f is a closed map,

$$\begin{aligned}1_{\Omega(Y)} = Y = Y \smallsetminus \emptyset = Y \smallsetminus \bigcap_{\alpha\in A} f[X \smallsetminus U_\alpha] \\ = Y \smallsetminus \bigcap_{\alpha\in A} \overline{f[X \smallsetminus U_\alpha]} = \bigcup_{\alpha\in A}\Big(Y \smallsetminus \overline{f[X \smallsetminus U_\alpha]}\Big) = \bigvee_{\alpha\in A} \Omega(f)_*(U_\alpha),\end{aligned}$$

showing that $\Omega(f)$ preserves directed covers. The rest follows from Proposition 5.15. $\square$

Remark 5.18. Note that in the proof we have actually shown that, for a closed map $f\colon X \to Y$, if $f[-]$ preserves filtered co-covers, then $\Omega(f)_*$ preserves directed covers. The converse also holds, as can be seen from the last displayed string of equalities in the proof.

We now give a criterion for a sublocale of a frame L to be wM in terms of collections of open sublocales of L. Recall that if S is a sublocale of L, then joins in $\mathcal{S}(S)$ are exactly joins in $\mathcal{S}(L)$. We use the notation that if S is a sublocale of L and $\mathscr{C}$ is a collection of sublocales of L, then

$$S \cap \mathscr{C} = \{S \cap C \mid C \in \mathscr{C}\}.$$

Recall that collections of open sublocales are *distributive*, meaning that if $\mathscr{U}$ is a collection of open sublocales of L, then for any sublocale T of L,

$$T \cap \bigvee\{U \mid U \in \mathscr{U}\} = \bigvee\{T \cap U \mid U \in \mathscr{U}\}.$$

This is due to Isbell [11].

Theorem 5.19. *The following are equivalent for a sublocale S of a frame L.*

1. *S is* wM.
2. *For every sequence $(\mathscr{U}_n)$ of directed collections of open sublocales of L with $S \subseteq \bigvee \mathscr{U}_n$ for every n, there exists, for each n, some $U_n \in \mathscr{U}_n$ such that $S \subseteq \overline{S \cap \bigvee_{n\in\mathbb{N}} U_n}$.*

Proof. (1) ⇒ (2): We shall reserve the overline for the closure in L. Suppose that S is wM and $(\mathscr{U}_n)$ is a sequence with each $\mathscr{U}_n$ directed and consisting of open sublocales of L such that $S \subseteq \bigvee \mathscr{U}_n$ for each n. Since $\mathscr{U}_n$ consists entirely of open sublocales of L,

$$S = S \cap \bigvee \mathscr{U}_n = \bigvee\{S \cap U \mid U \in \mathscr{U}_n\},$$

which then implies that $(S \cap \mathscr{U}_n)$ is a sequence of directed open coverings of S. Since S is wM, we deduce from Proposition 5.2 that there exists, for each n, some $U_n \in \mathscr{U}_n$ such that $\bigvee_{n\in\mathbb{N}}(S \cap U_n)$ is a dense sublocale of S. Therefore

$$S = \mathrm{cl}_S\Big(\bigvee_{n\in\mathbb{N}}(S \cap U_n)\Big) = S \cap \overline{\bigvee_{n\in\mathbb{N}}(S \cap U_n)} = S \cap \overline{S \cap \bigvee_{n\in\mathbb{N}} U_n},$$

showing that $S \subseteq \overline{S \cap \bigvee_{n\in\mathbb{N}} U_n}$.

(2) ⇒ (1): Let $(\mathscr{C}_n)$ be a sequence of directed open coverings of S. Then, for each n, $\mathscr{C}_n = S \cap \mathscr{U}_n$ for some collection $\mathscr{U}_n$ consisting of open sublocales of L with $S \subseteq \bigvee \mathscr{U}_n$. (Caution: $\mathscr{U}_n$ is not necessarily directed). For each n, put

$$\mathscr{V}_n = \Big\{\bigvee \mathscr{F} \mid \mathscr{F} \text{ is a finite subset of } \mathscr{U}_n\Big\},$$

and observe that $(\mathscr{V}_n)$ is a sequence of directed collections of open sublocales of L with $S \subseteq \bigvee \mathscr{V}_n$ for each n. Therefore, by the present hypothesis, there exists, for each n, some $V_n \in \mathscr{V}_n$ such that $S \subseteq \overline{S \cap \bigvee_{n\in\mathbb{N}} V_n}$. Now, for each n, there exists some $k_n \in \mathbb{N}$ and elements $F_n^{(1)}, \ldots, F_n^{(k_n)}$ of $\mathscr{U}_n$ such that

$$V_n = F_n^{(1)} \vee \cdots \vee F_n^{(k_n)}.$$

Since $\mathscr{C}_n = S \cap \mathscr{U}_n$,

$$\left\{S \cap F_n^{(1)}, \dots, S \cap F_n^{(k_n)}\right\} \subseteq \mathscr{C}_n,$$

and since $\mathscr{C}_n$ is directed, there exists an element of $\mathscr{C}_n$ which contains each of the sublocales $S \cap F_n^{(i)}$ for $i = 1, \dots, k_n$. Thus, there exists some $U_n \in \mathscr{U}_n$ such that

$$S \cap V_n = S \cap \left(F_n^{(1)} \vee \dots \vee F_n^{(k_n)}\right) = \left(S \cap F_n^{(1)}\right) \vee \dots \vee \left(S \cap F_n^{(k_n)}\right) \subseteq S \cap U_n.$$

Thus, putting $C_n = S \cap U_n$, we have that C_n is a sublocale of S belonging to $\mathscr{C}_n$ such that

$$S \subseteq S \cap \overline{S \cap \bigvee_{n\in\mathbb{N}} V_n} = S \cap \overline{\bigvee_{n\in\mathbb{N}} (S \cap V_n)} \subseteq S \cap \overline{\bigvee_{n\in\mathbb{N}} (S \cap U_n)} = S \cap \overline{\bigvee_{n\in\mathbb{N}} C_n} = \mathrm{cl}_S\left(\bigvee_{n\in\mathbb{N}} C_n\right) \subseteq S,$$

showing that $\bigvee_{n\in\mathbb{N}} C_n$ is a dense sublocale of S. Therefore S is wM. $\square$

This theorem gives us a sublocale-based verification of the assertion that the closure of a wM sublocale is wM. Indeed, let S be a wM sublocale of a frame L, and suppose that $(\mathscr{U}_n)$ is a sequence of directed collections of open sublocales of L such that $\overline{S} \subseteq \bigvee \mathscr{U}_n$ for each n. Then $S \subseteq \bigvee \mathscr{U}_n$, and so, by the theorem, there exists, for each n some $U_n \in \mathscr{U}_n$ such that $S \subseteq \overline{S \cap \bigvee_{n\in\mathbb{N}} U_n}$. Then, upon taking closures,

$$\overline{S} \subseteq \overline{S \cap \bigvee_{n\in\mathbb{N}} U_n} \subseteq \overline{\overline{S} \cap \bigvee_{n\in\mathbb{N}} U_n},$$

which then shows that $\overline{S}$ is wM.

Corollary 5.20. *The join of finitely many* wM *sublocales is* wM.

Proof. Let $S_1, \dots, S_k$ be finitely many wM sublocales of a frame L, and put $S = S_1 \vee \dots \vee S_k$. Let $(\mathscr{U}_n)$ be a sequence of directed collections of open sublocales of L with $S \subseteq \bigvee \mathscr{U}_n$ for every n. For each $i \in \{1, \dots, k\}$, $S_i \subseteq \bigvee \mathscr{U}_n$, and so, for each n, we can select $U_n^{(i)} \in \mathscr{U}_n$ such that $S_i \subseteq \overline{S_i \cap \bigvee_{n\in\mathbb{N}} U_n^{(i)}}$. Since $\mathscr{U}_n$ is directed, there exists some $U_n \in \mathscr{U}_n$ such that $U_n^{(1)} \vee \dots \vee U_n^{(k)} \subseteq U_n$, which then implies

$$S_i \subseteq \overline{S_i \cap \bigvee_{n\in\mathbb{N}} U_n} \subseteq \overline{S \cap \bigvee_{n\in\mathbb{N}} U_n}.$$

Taking joins over all i yields the containment $S \subseteq \overline{S \cap \bigvee_{n\in\mathbb{N}} U_n}$, whence we deduce that S is wM. $\square$

The sublocales of a wM frame that inherit the property include the regular-closed ones, as we now show. In the proof of Proposition 3.12 we observed that if a is a regular element of L, and we denote pseudocomplements in $\mathfrak{c}_L(a)$ by $(-)^\#$, then, for any $x \in \mathfrak{c}_L(a)$, $x^{\#\#} = x^{**}$. A consequence of this is that if x is dense as an element of L, then x is also dense as an element of $\mathfrak{c}_L(a)$.

Proposition 5.21. *A regular-closed sublocale of a* wM *frame is* wM.

Proof. Let a be a regular element of a wM frame L. Let (C_n) be a sequence of directed covers of $\mathfrak{c}_L(a)$. Then (C_n) is also a sequence of directed covers of L, and so, for each n, we can select $c_n \in C_n$ such that $\bigvee_{n\in\mathbb{N}} c_n$ is dense in L. By what we observed above, $\bigvee_{n\in\mathbb{N}} c_n$ is dense as an element of $\mathfrak{c}_L(a)$, and so it follows that $\mathfrak{c}_L(a)$ is wM. $\square$

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