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On Free Locally Convex Spaces

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Abstract. Let $L(X)$ be the free locally convex space over a Tychonoff space X. We prove that the following assertions are equivalent: (i) every functionally bounded subset of X is finite, (ii) $L(X)$ is semi-reflexive, (iii) $L(X)$ has the Grothendieck property, (iv) $L(X)$ is semi-Montel. We characterize those spaces X, for which $L(X)$ is c_0-quasibarrelled, distinguished or a (df)-space. If X is a convergent sequence, then $L(X)$ has the Glicksberg property, but the space $L(X)$ endowed with its Mackey topology does not have the Schur property.

1. Introduction

The study of locally convex properties such as the Dunford–Pettis property, the Grothendieck property, numerous weak barrelledness conditions, the property of being a (DF)-space or a (df)-space, the property of being a complete, quasi-complete or locally complete space, and others, is one of the main direction of researches in the theory of locally convex spaces. These properties are studied mainly in the most important special classes of locally convex spaces as, for example, the class of spaces $C(X)$ of continuous functions on a Tychonoff space X endowed with the pointwise topology or the compact-open topology. We refer the reader to the classical books [16, 20, 24, 25] and the excellent recent survey [22].

One of the most important classes of locally convex spaces is the class of free locally convex spaces introduced by Markov in [17]. The *free locally convex space* $L(X)$ over a Tychonoff space X is a pair consisting of a locally convex space $L(X)$ and a continuous map $i : X \to L(X)$ such that every continuous map f from X to a locally convex space E gives rise to a unique continuous linear operator $\Psi_E(f) : L(X) \to E$ with $f = \Psi_E(f) \circ i$. The free locally convex space $L(X)$ always exists and is essentially unique.

The first description of the topology of the free locally convex space $L(X)$ over X was obtained by Raĭkov [21, Theorem 1′] (all relevant notions will be defined below).

Theorem 1.1 ([21]). *For every Tychonoff space X, the topology of the free locally convex space $L(X)$ is the polar topology on $L(X)$ defined by the family of all equicontinuous pointwise bounded subsets of $C(X)$.*

Theorem 1.1 gives a *polar* description of the topology of the space $L(X)$. For *topological* descriptions of the topology of $L(X)$, see [3, 8].

For a locally convex space E, we denote by E' the space of all continuous functionals of E and let $\sigma(E', E)$ denote the weak* topology on E'. The dual space E' endowed with the strong topology $\beta(E', E)$ is denoted

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by E'_β. In Proposition 3.1 we describe the strong dual of $L(X)$. Using this result and a result of Schmets [24], we characterize Tychonoff spaces X for which $L(X)$ is distinguished, see Theorem 3.3.

Recall that a locally convex space E is said to have the *Grothendieck property* if every weak* convergent sequence in E'_β is weakly convergent. It is proved in [12] that for a μ-space X, the space $L(X)$ has the Grothendieck property if and only if every compact subset of X is finite. In Theorem 3.5 we obtain a complete characterization of $L(X)$ with the Grothendieck property. Moreover, in this theorem we show that $L(X)$ has the Grothendieck property if and only if it is a feral (semi-Montel or semi-reflexive) space.

Weak barrelledness concepts and the properties of being a Mackey space, a (DF)-space or a (df)-space are the cornerstone in the study of general locally convex spaces and have been intensively studied by many authors, we refer the reader to [16, 20]. Characterizations of Tychonoff spaces X for which $L(X)$ has one of weak barrelledness properties or is a (DF)-space are given in [6, 9, 23]. Nevertheless, it was still unknown a characterization of X such that $L(X)$ is c_0-quasibarrelled or a (df)-space. We provide such characterizations in Theorem 3.8 and Theorem 3.9.

2. Preliminaries

We start with some necessary definitions and notations used in the article. Set $\mathbb{N} := \{1, 2, \dots\}$ and $\omega := \{0, 1, 2, \dots\}$. All topological vector spaces are over the field $\mathbb{F}$ of real or complex numbers. The closed unit ball of the field $\mathbb{F}$ is denoted by $\mathbb{D}$.

Let X be a set. It is well known that the dual space $(\mathbb{F}^X)'$ of the Tychonoff product $\mathbb{F}^X$ can be identified with the space of functions $\mu : X \to \mathbb{F}$ that have finite support $\mathrm{supp}(\mu) := \{x \in X : \mu(x) \neq 0\}$. Such functions μ will be called *finitely supported sign-measures* on X. The action of a sign-measure μ on a function $f : X \to \mathbb{F}$ is defined by the formula

$$\mu(f) = \sum_{x \in \mathrm{supp}(\mu)} \mu(x) \cdot f(x).$$

The *norm* of μ is defined as

$$\|\mu\| = \sum_{x \in \mathrm{supp}(\mu)} |\mu(x)|$$

(as usual, for $\mu = 0$ we set $\mathrm{supp}(\mu) = \emptyset$ and $\|\mu\| = 0$). For a subset $\mathcal{M} \subseteq (\mathbb{F}^X)'$, put

$$\mathrm{supp}(\mathcal{M}) := \bigcup_{\mu \in \mathcal{M}} \mathrm{supp}(\mu) \;\text{ and }\; \|\mathcal{M}\| := \sup\big(\{\|\mu\| : \mu \in \mathcal{M}\} \cup \{0\}\big) \in [0, +\infty].$$

For a point $x \in X$, we denote by δ_x the *Dirac measure* $\delta_x : \mathbb{F}^X \to \mathbb{F}$, $\delta_x : f \mapsto f(x)$.

Let X be a Tychonoff space. The closure of a subset A of X is denoted by $\overline{A}$ or $\mathrm{cl}(A)$. We denote by $C(X)$ the space of all continuous $\mathbb{F}$-valued functions on X. The Hewitt realcompactification of X is denoted by υX. It is well known that the restriction operator $R : C(\upsilon X) \to C(X)$, $R : f \mapsto f{\restriction}_X$, is bijective. A subset A of X is called *functionally bounded* if for every $f \in C(X)$, the image $f[A] = \{f(x) : x \in A\}$ of A has compact closure in $\mathbb{F}$. A Tychonoff space X is called *pseudocompact* if X is functionally bounded in X. We denote by $\mathcal{FB}(X)$ the family of all functionally bounded subsets of X.

A Tychonoff space X is called a *μ-space* if every functionally bounded subset of X has compact closure. By [4, Proposition 3], a Tychonoff space X is a μ-space if and only if $X = X''$ where

$$X'' = \bigcup_{B \in \mathcal{FB}(X)} \overline{B} \subseteq \beta X$$

is the *bidual* of X, which is defined as the union of closures in βX of functionally bounded subsets of X.

The Stone-Čech extension βX of X is a μ-space and so are the Hewitt realcompactification υX and the Dieudonné completion $\mathcal{D}X$ of X. For a Tychonoff space X its *μ-envelope* μX is defined as the smallest μ-subspace of βX that contains X. Schmets [24, Definition II.6.1] calles μ-envelope *le μ-espace associé à X*. The μ-envelope of X is equal to the intersection of all μ-subpaces of βX that contain X. It is also equal to the union $\bigcup_\alpha X_\alpha$ of the transfinite sequence $(X_\alpha)_\alpha$ of subspaces of βX defined by the recursive formula: $X_0 = X$ and

$$X_\alpha = \bigcup_{\beta<\alpha} (X_\beta)''$$

for any nonzero ordinal α, where for a subset $A \subseteq \beta X$ its bidual $A'' = \bigcup_{B\in\mathcal{FB}(A)} \overline{B}$ is the union of βX-closures of functionally bounded subsets of A.

Recall that a Tychonoff space X is *Dieudonné complete* if the universal uniformity $\mathcal{U}_X$ on X is complete. For numerous characterizations of Dieudonné complete spaces, see Section 8.5.13 of [5]. The *Dieudonné completion* $\mathcal{D}X$ of X is the completion of the uniform space $(X, \mathcal{U}_X)$. Since the Dieudonné completion of any Tychonoff space is a μ-space, we have the following inclusions:

$$X \subseteq \mu X \subseteq \mathcal{D}X \subseteq \upsilon X \subseteq \beta X.$$

Let X be a Tychonoff space. A topology $\mathcal{T}$ on $C(X)$ is called *locally convex* if $C_{\mathcal{T}}(X) := (C(X), \mathcal{T})$ is a locally convex topological vector space. For a function $f : X \to \mathbb{F}$ and subset $A \subseteq X$, let

$$\|f\|_A := \sup(\{|f(x)| : x \in A\} \cup \{0\}) \in [0, \infty].$$

For a subfamily $\mathcal{F} \subseteq \mathbb{F}^X$ and $\varepsilon > 0$, we put

$$[A; \varepsilon]_{\mathcal{F}} := \{f \in \mathcal{F} : \|f\|_A \le \varepsilon\}.$$

If the family $\mathcal{F}$ is clear from the context, then we shall omit the subscript $\mathcal{F}$ and write $[A; \varepsilon]$ instead of $[A; \varepsilon]_{\mathcal{F}}$.

A family $\mathcal{S}$ of subsets of X is *directed* if for any sets $A, B \in \mathcal{S}$ the union $A \cup B$ is contained in some set $C \in \mathcal{S}$. Each directed family $\mathcal{S}$ of functionally bounded sets in a Tychonoff space X induces a locally convex topology $\mathcal{T}_{\mathcal{S}}$ on $C(X)$ whose neighborhood base at zero consists of the sets $[S; \varepsilon]$ where $S \in \mathcal{S}$ and $\varepsilon > 0$. The topology $\mathcal{T}_{\mathcal{S}}$ is called *the topology of uniform convergence on sets of the family $\mathcal{S}$*. The topology $\mathcal{T}_{\mathcal{S}}$ is Hausdorff if and only if the union $\bigcup \mathcal{S}$ is dense in X. If $\mathcal{S}$ is the family of all finite, compact or functionally bounded subsets of X, respectively, then the topology $\mathcal{T}_{\mathcal{S}}$ will be denoted by $\mathcal{T}_p$, $\mathcal{T}_k$ or $\mathcal{T}_b$, and the function space $C_{\mathcal{T}_{\mathcal{S}}}(X)$ will be denoted by $C_p(X)$, $C_k(X)$ or $C_b(X)$, respectively.

Since $C_p(X)$ is dense in $\mathbb{F}^X$, each linear continuous functional on $C_p(X)$ has a unique extension to a linear continuous functional on the space $\mathbb{F}^X$, which allows us to identify the dual space $C_p(X)'$ of $C_p(X)$ with the dual space $(\mathbb{F}^X)'$ of the locally convex space $\mathbb{F}^X$. The following assertion is proved in Proposition 4.10 of [11].

Proposition 2.1. *Let X be a Tychonoff space. A subset $\mathcal{M} \subseteq C_p(X)'_{w^*}$ is bounded if and only if its support* $\mathrm{supp}(\mathcal{M})$ *is functionally bounded in X and its norm $\|\mathcal{M}\|$ is finite.*

The definition of $L(X)$ implies that the dual space $L(X)'$ of $L(X)$ is linearly isomorphic to the space $C(X)$. Indeed, every function $f \in C(X)$ is a continuous function from X to the locally convex space $\mathbb{F}$. Therefore f can be uniquely extended from X to $L(X)$ as follows

$$\Psi_{\mathbb{F}}(f)(\chi) := a_1 f(x_1) + \cdots + a_n f(x_n) \quad \text{for} \quad \chi = a_1 x_1 + \cdots + a_n x_n \in L(X).$$

Observe that the inverse operator $R : L(X)' \to C(X)$ to the extension operator $\Psi_{\mathbb{F}}$ is the restriction operator $R : \chi \mapsto \chi{\restriction}_X$, where $\chi \in L(X)'$. Via the pairing $(L(X)', L(X)) = (C(X), L(X))$ we note that $C_p(X)'_{w^*} = L(X)_w$ and $C_p(X) = L(X)'_{w^*}$, where $E_w := (E, \sigma(E, E'))$ and $E'_{w^*} = (E', \sigma(E', E))$ for an lcs E. Usually the space $L(X)_w$ is denoted by $L_p(X)$.

Let E be a locally convex space. A *barrel* in E is an absolutely convex closed subset B of E such that $E = \bigcup_{n\in\omega} nB$. A sequence $\{B_n\}_{n\in\omega}$ in E is called a *fundamental bounded sequence* if for every bounded subset B of E there is $n \in \omega$ such that $B \subseteq B_n$. Denote by $\beta(E, E')$ the topology on E whose neighborhood base at zero consists of barrels (for more details about this topology, see § 8.4 of [16]) and set $E_\beta := \big(E, \beta(E, E')\big)$. Following [16, 8.4.3.C], we denote by $\beta^*(E', E)$ the topology E' of uniform convergence on $\beta(E, E')$-bounded subsets of E and put $E'_{\beta^*} := \big(E', \beta^*(E', E)\big)$. A subset D of E is *bornivorous* if it absorbs the bounded sets, i.e., for any bounded set $B \subseteq E$, there is $a > 0$ such that $B \subseteq \lambda D$ for $|\lambda| \geq a$. Recall that E is

- *(quasi)barrelled* if every (bornivorous) barrel in E is a neighborhood of zero;
- c_0*-quasibarrelled* if every $\beta(E', E)$-null sequence is equicontinuous;
- a (df)*-space* if it has a fundamental bounded sequence and is c_0-quasibarrelled;
- a *semi-Montel space* if every bounded set of E has compact closure;
- a *semi-reflexive* if $E = (E'_\beta)'(= E'')$;
- *distinguished* if its strong dual E'_β is barrelled;
- *feral* if every bounded subset of E is finite-dimensional.

We say that a locally convex space (lcs for short) E is *b-feral* if every barrel-bounded subset of E is finite-dimensional, where a subset A of E is called *barrel-bounded* if A is a bounded subset of the space E endowed with the topology $\beta(E, E')$ on E defined by barrels. We shall use the next theorem which is proved on page 4 in [6].

Theorem 2.2. *For every Tychonoff space X, the free locally convex space $L(X)$ is b-feral.*

3. Main results

We start with the following description of the strong dual $L(X)'_\beta$ of $L(X)$.

Proposition 3.1. *For every Tychonoff space X, the restriction map*

$$R : L(X)'_\beta \to C_b(X), \quad R : F \mapsto F{\restriction}_X,$$

is a topological isomorphism.

Proof. As we explained above in the previous section, the map R is a linear isomorphism. Recall that the family of sets

$$[A; \varepsilon], \quad \text{where } \varepsilon > 0 \text{ and } A \in \mathcal{FB}(X),$$

form a base at zero of $\mathcal{T}_b$, and, by Proposition 2.1, the family of the polars $B^\circ_{A,\varepsilon}$, where

$$B_{A,\varepsilon} := \left\{\chi \in L(X) : A \in \mathcal{FB}(X),\ \mathrm{supp}(\chi) \subseteq A, \text{ and } \|\chi\| \leq \tfrac{1}{\varepsilon}\right\}$$

form a base at zero of $\beta\big(L(X)', L(X)\big)$. Therefore to prove the proposition it remains to show that $R\big(B^\circ_{A,\varepsilon}\big) = [A; \varepsilon]$. If $F \in B^\circ_{A,\varepsilon}$, then for every $x \in A$ we have $\left|\frac{1}{\varepsilon}R(F)(x)\right| = |F(\frac{1}{\varepsilon}\delta_x)| \leq 1$ and hence $|R(F)(x)| \leq \varepsilon$. Therefore $R(F) \in [A; \varepsilon]$ and hence $R\big(B^\circ_{A,\varepsilon}\big) \subseteq [A; \varepsilon]$. Conversely, let $f \in [A; \varepsilon]$. Then for every $\chi \in B_{A,\varepsilon}$, we have

$$|R^{-1}(f)(\chi)| = \Big|\sum_{x\in\mathrm{supp}(\chi)} \chi(x) \cdot f(x)\Big| \leq \sum_{x\in\mathrm{supp}(\chi)} \big|\chi(x)\big| \cdot |f(x)| \leq \|\chi\| \cdot \varepsilon \leq 1,$$

and hence $R^{-1}(f) \in B^\circ_{A,\varepsilon}$. Thus $[A; \varepsilon] \subseteq R\big(B^\circ_{A,\varepsilon}\big)$ and hence $R\big(B^\circ_{A,\varepsilon}\big) = [A; \varepsilon]$. $\square$

By Proposition 3.1, the strong dual $L(X)'_\beta$ is topologically isomorphic to the function space $C_b(X)$. This reduces the problem of recognizing distinguished free locally convex spaces to the problem of recognizing barrelled spaces among the function spaces $C_b(X)$. This problem has been studied and resolved by Schmets [25, Theoreme III.3.13] who proved the following characterization.

Theorem 3.2 (Schmets). *For a Tychonoff space X, the space $C_b(X) = L(X)'_\beta$ is barrelled if and only if every functionally bounded subset $A \subseteq X''$ is contained in the μX-closure of some functionally bounded subset of X.*

It is known (see [12, Proposition 3.4] or [7, Theorem 27]) that if X is a μ-space, then $L(X)$ is distinguished. Essentially using the Schmets Theorem 3.2 we obtain below a complete characterization of Tychonoff spaces X for which $L(X)$ is a distinguished space.

Theorem 3.3. *For a Tychonoff space X the following conditions are equivalent:*

(i) *$L(X)$ is distinguished;*
(ii) *$C_b(X)$ is barelled;*
(iii) *for every compact subset $A \subseteq \mu X$ there is a compact set $B \subseteq \mu X$ such that $A \subseteq B$ and $B \cap X$ is dense in B;*
(iv) *the restriction operator $R : C_k(\mu X) \to C_b(X)$, $R : f \mapsto f{\restriction}_X$, is a topological isomorphism.*

Proof. The equivalence (i)⇔(ii) follows from Proposition 3.1 and the definition of a distinguished space.

(ii)⇒(iii) By Schmets Theorem 3.2, if $C_b(X)$ is barrelled, then for every functionally bounded subset $A \subseteq X''$ there exists a functionally bounded subset $B \subseteq X$ such that $A \subseteq \overline{B}$ and hence $\overline{A} \subseteq \overline{B}$, which means that $(X'')'' = X''$ and hence $\mu X = X''$. Consequently, for every compact subset $A \subseteq \mu X = X''$, there exists a functionally bounded set $B \subseteq X$ such that $A \subseteq \overline{B}$. The set $\overline{B} \subseteq X'' = \mu X$ is compact, contains A and the intersection $\overline{B} \cap X \supseteq B$ is dense in $\overline{B}$.

(iii)⇒(iv) To prove that the restriction operator $R : C_k(\mu X) \to C_b(X)$ is a topological isomorphism, it suffices to prove that for any neighborhood $U \subseteq C_k(\mu X)$ of zero, its image $R[U]$ is a neighborhood of zero in $C_b(X)$. By the definition of the topology of $C_k(\mu X)$, there exist a compact set $A \subseteq \mu X$ and $\varepsilon > 0$ such that $[A; \varepsilon] \subseteq U$. By (iii), there exists a compact set $B \subseteq \mu X$ such that $A \subseteq B$ and $B \cap X$ is dense in B. The compactness of B and the bijectivity of the restriction operator R imply that the set $B \cap X$ is functionally bounded in X. It remains to prove that $[B \cap X; \varepsilon] \subseteq R([A; \varepsilon])$. Choose any function $f \in [B \cap X; \varepsilon] \subseteq C_b(X)$ and let $\bar{f} = R^{-1}(f) \in C_k(\mu X)$ be a unique continuous extension of f to μX. It follows from $f \in [B \cap X; \varepsilon]$ and the density of $B \cap X$ in B that $\bar{f} \in [B; \varepsilon] \subseteq [A; \varepsilon] \subseteq U$.

(iv)⇒(ii) Assume that the spaces $C_k(\mu X)$ and $C_b(X)$ are topologically isomorphic. By Nachbin–Shirota Theorem [16, 11.7.5], the function space $C_k(\mu X)$ is barrelled and so is its isomorphic copy $C_b(X)$. □

In the proof of the implication (ii)⇒(iii) in Theorem 3.2 we showed that the barrelledness of the function space $C_b(X)$ implies the equality $\mu X = X''$. This remark suggests the following problem.

Problem 3.4. *Is there a Tychonoff space X such that $\mu X = X''$ but $C_b(X)$ is not barrelled?*

Let X be a Tychonoff space. We denote by $L_{\mathcal{T}}(X)$ the vector space $L(X)$ endowed with some locally convex topology $\mathcal{T}$ on $L(X)$. If $\mathcal{T}$ coincides with the topology of the free locally convex space $L(X)$, then we shall omit the subscript $\mathcal{T}$ and write simply $L(X)$. A locally convex topology $\mathcal{T}$ on $L(X)$ is called *compatible* if $L_{\mathcal{T}}(X)' = L(X)'$. Although the equivalence (i)⇔(ii) in the next theorem is proved in [6], we provide its simple proof for the reader convenience and because it helps to simplify the proof of other equivalences.

Theorem 3.5. *Let X be a Tychonoff space, and let $\mathcal{T}$ be a compatible locally convex topology on $L(X)$. Then the following assertions are equivalent:*

(i) *every functionally bounded subset of X is finite;*
(ii) *$L_{\mathcal{T}}(X)$ is feral;*
(iii) $\beta^*\big(L_{\mathcal{T}}(X)', L_{\mathcal{T}}(X)\big) = \beta\big(L_{\mathcal{T}}(X)', L_{\mathcal{T}}(X)\big)$;

(iv) $L_{\mathcal{T}}(X)$ *is semi-reflexive;*
(v) $L_{\mathcal{T}}(X)$ *has the Grothendieck property;*
(vi) $L_{\mathcal{T}}(X)$ *is semi-Montel.*

Proof. (i)⇒(ii) Let $\mathcal{M}$ be a bounded subset of E. Then, by Proposition 2.1, the support $\mathrm{supp}(\mathcal{M})$ of $\mathcal{M}$ is functionally bounded in X. By (i), $\mathrm{supp}(\mathcal{M})$ is finite and hence $\mathcal{M}$ is finite-dimensional, witnessing that $L_{\mathcal{T}}(X)$ is a feral space.

(ii)⇒(iii) Let $E := L_{\mathcal{T}}(X)$ and let B be a bounded subset of E. By (ii), B is finite-dimensional and hence B is also barrel-bounded. Since every barrel-bounded subset is bounded it follows that $\beta^*(E',E) = \beta(E',E)$.

(iii)⇒(i) Let $E := L_{\mathcal{T}}(X)$ and assume that $\beta^*(E',E) = \beta(E',E)$. Let D be a functionally bounded subset of X. Then, by Proposition 2.1, the set

$$\mathcal{M} := \{\mu \in E : \mathrm{supp}(\mu) \subseteq D \ \text{ and } \ \|\mu\| \leq 1\}$$

is a bounded subset of E. Since $\beta^*(E',E) = \beta(E',E)$, there are a closed barrel-bounded set B in E and $\varepsilon > 0$ such that $[B;\varepsilon] \subseteq [\mathcal{M};1]$. Observe that, by Theorem 2.2, the set B is finite-dimensional and hence $\mathrm{supp}(B)$ is a finite (in particular, closed) subset of X. Therefore to show that D is finite it suffices to prove that $D \subseteq \mathrm{supp}(B)$. Suppose for a contradiction and $D \nsubseteq \mathrm{supp}(B)$. Then we can find a point $z \in D\setminus\mathrm{supp}(B)$ and an open neighborhood O_z of z such that $O_z \cap \mathrm{supp}(B) = \emptyset$. Take $f \in C(X) = E'$ such that $f(X\setminus O_z) \subseteq \{0\}$ and $f(z) > 1$. It is clear that $f \in [B;\varepsilon]$. Since $\delta_z \in \mathcal{M}$ and $\delta_x(f) = f(z) > 1$, we obtain that $f \notin [\mathcal{M};1]$, a contradiction. Thus $D \subseteq \mathrm{supp}(B)$ is finite, as desired.

(i)⇒(iv) By Proposition 3.1, $L_{\mathcal{T}}(X)'_\beta = C_b(X)$. Since every functionally bounded subset of X is finite, we have $L_{\mathcal{T}}(X)'_\beta = C_p(X)$ and hence $L_{\mathcal{T}}(X)'' = L(X)$. Therefore $L_{\mathcal{T}}(X)$ is semi-reflexive.

(iv)⇒(v) Since E is semi-reflexive (so $E = (E'_\beta)'$), every weak* convergent sequence in E'_β is trivially weakly convergent. Thus E has the Grothendieck property.

(v)⇒(i) Assume that $L(X)$ has the Grothendieck property and suppose for a contradiction that X has an infinite functionally bounded subset A. By Lemma 11.7.1 of [16], there exist a one-to-one sequence $\{x_n\}_{n\in\omega} \subseteq A$ and a sequence $\{U_n\}_{n\in\omega}$ of pairwise disjoint open sets such that $x_n \in U_n$ for every $n \in \omega$. For every $n \in \omega$, choose $f_n \in C(X)$ such that $f_n(X\setminus U_n) = \{0\}$ and $f_n(x_n) = 2^n$. Since the sets U_n are disjoint, we have $f_n \to 0$ in the pointwise topology and hence in the weak* topology of the dual pair $(C(X), L(X))$. On the other hand, consider the linear functional $\mu = \sum_{n\in\omega} 2^{-n}\delta_{x_n}$ on $C(X)$. Since $\{x_n\}_{n\in\omega} \subseteq A$, we have

$$|\mu(h)| \leq 1 \qquad \text{for every } h \in [A;1] \subseteq C(X).$$

and hence $\mu \in [A;1]^\circ$. Since the set $[A;1]$ is a neighborhood of zero in $L_{\mathcal{T}}(X)'_\beta = C_b(X)$ (Proposition 3.1), we obtain $\mu \in L_{\mathcal{T}}(X)''$. But since $\mu(f_n) = 2^{-n}\cdot f_n(x_n) = 1 \not\to 0$ it follows that $f_n \not\to 0$ in the weak topology of $L_{\mathcal{T}}(X)'_\beta$. Thus $L_{\mathcal{T}}(X)$ does not have the Grothendieck property, a contradiction.

(ii)⇒(vi) is trivial.

(vi)⇒(iv) is clear, see Proposition 11.5.1 of [16]. □

A Tychonoff space X is called *(sequentially) Ascoli* if every compact subset (resp. convergent sequence) in $C_k(X)$ is equicontinuous. In other words, X is Ascoli if and only if the compact-open topology of $C_k(X)$ is Ascoli in the sense of [18, p.45]. Ascoli and sequentially Ascoli spaces were thoroughly studied in [1, 2, 12, 13].

To characterize Tychonoff spaces X for which $L(X)$ is c_0-quasibarrelled or a (df)-space we introduce below new classes of Tychonoff spaces.

Definition 3.6. A Tychonoff space X is called (*sequentially*) *b-Ascoli* if every compact subset (resp. convergent sequence) in $C_b(X)$ is equicontinuous. □

Proposition 3.7. *Let X be a Tychonoff space.*

(i) *If X is a (sequentially) Ascoli space, then it is a (sequentially) b-Ascoli space. The converse is true if X is a μ-space.*
(ii) *Every pseudocompact space is a sequentially b-Ascoli space.*

Proof. (i) Since the identity map $C_b(X) \to C_k(X)$ is continuous, every compact subset (resp. convergent sequence) in $C_b(X)$ is equicontinuous because it is a compact subset (resp. convergent sequence) in $C_k(X)$ and X is assumed to be a (sequentially) Ascoli space. If in addition X is a μ-space, we have $C_k(X) = C_b(X)$ (see Proposition 3 in [4]) and it is clear that if X is a (sequentially) b-Ascoli space then it is a (sequentially) Ascoli space.

(ii) If X is pseudocompact, then $C_b(X)$ is a Banach space. Therefore any null sequence in $C_b(X)$ is trivially equicontinuous. □

Note that there are pseudocompact spaces which are not sequentially Ascoli, see [14].

Theorem 3.8. *For a Tychonoff space X, the space $L(X)$ is c_0-quasibarrelled if and only if X is a sequentially b-Ascoli space.*

Proof. Assume that $L(X)$ is c_0-quasibarrelled. Take an arbitrary null-sequence $\{f_n\}_{n\in\omega}$ in $C_b(X)$. Since, by Proposition 3.1, $L(X)'_\beta = C_b(X)$, we obtain that $\{f_n\}_{n\in\omega}$ is equicontinuous as a sequence in $L(X)'_\beta$. Now Proposition 2.3 of [12] guarantees that $\{f_n\}_{n\in\omega}$ is an equicontinuous sequence of functions. Thus X is a sequentially b-Ascoli space.

Conversely, assume that X is a sequentially b-Ascoli space. Fix a null sequence $S = \{f_n\}_{n\in\omega}$ in $L(X)'_\beta$. By Proposition 3.1 we have $L(X)'_\beta = C_b(X)$ and hence S is a null sequence in $C_b(X)$. Since X is sequentially b-Ascoli, S is equicontinuous. Clearly S is also pointwise bounded. Therefore, by Theorem 1.1, the polar S° of S is a neighborhood of zero in $L(X)$. The inclusion $S \subseteq S^{\circ\circ}$ implies that S is equicontinuous as a subset of the locally convex space $L(X)$. Thus $L(X)$ is c_0-quasibarrelled. □

A Tychonoff space X is defined to have a *fundamental functionally bounded sequence* if there exists a sequence $\{B_n\}_{n\in\omega}$ of functionally bounded subsets in X such that every functionally bounded subset $B \subseteq X$ is contained in some set B_n.

Theorem 3.9. *Let X be a Tychonoff space. Then $L(X)$ is a (df)-space if and only if X has a fundamental functionally bounded sequence and is a sequentially b-Ascoli space. In particular, $L(X)$ is a (df)-space for every pseudocompact space X.*

Proof. By Proposition 2.12 of [12], the space $L(X)$ has a fundamental bounded sequence if and only if X has a fundamental functionally bounded sequence. Now the theorem follows from this result and Theorem 3.8. The last assertion follows from (ii) of Proposition 3.7. □

Recall that a locally convex space E is said to have

- the *Schur property* if E and $\big(E, \sigma(E,E')\big)$ have the same convergent sequences;
- the *Glicksberg property* if E and $\big(E, \sigma(E,E')\big)$ have the same compact sets.

It is clear that the Glicksberg property implies the Schur property, but the converse is not true in general (see, for example, Proposition 3.5 of [10]). It is evident that every lcs having its weak topology has the Glicksberg property trivially. If an lcs E has the Glicksberg property, then for every locally convex topology τ on E which is stronger than $\sigma(E,E')$ but weaker than the topology $\mathcal{T}$ of E, the space (E,τ) also has the Glicksberg property. These remarks suggest the following problem. Does there exist an lcs $(E,\mathcal{T})$ without the Glicksberg property, but such that there is a locally convex topology τ on E such that $\sigma(E,E') \subsetneq \tau \subsetneq \mathcal{T}$ and the space (E,τ) has the Glicksberg property? We answer this question in the affirmative in Example 3.12 below. We need the following two assertions.

Lemma 3.10. *For every cardinal κ and each continuous linear map $T : \mathbb{F}^\kappa \to \ell_\infty$, the image of T is finite-dimensional.*

Proof. We assume that κ is infinite. Since T is continuous, there is a finite subset λ of κ such that $T\Big(\{0\}^\lambda \times \mathbb{F}^{\kappa\setminus\lambda}\Big)$ is contained in the unit ball B of ℓ_∞. Since B contains no linear subspaces we obtain that $\{0\}^\lambda \times \mathbb{F}^{\kappa\setminus\lambda}$ is contained in the kernel of T. Thus $T[\mathbb{F}^\kappa] = T[\mathbb{F}^\lambda]$ is finite-dimensional. □

Proposition 3.11. *For every infinite Tychonoff space X, the space $L(X)$ does not carry its weak topology.*

Proof. Suppose for a contradiction that $L(X)$ has its weak topology, so $L(X)$ is a dense linear subspace of $\mathbb{F}^\kappa$ for some infinite cardinal κ. Since X is infinite, there is a sequences $\{U_n\}_{n\in\omega}$ of pairwise disjoint open sets in X. For every $n \in \omega$, fix a point $x_n \in U_n$ and choose a continuous function $f_n : X \to [0, \frac{1}{n}]$ such that $f_n(x_n) = \frac{1}{n}$ and $f_n(X\setminus U_n) = \{0\}$. Consider the map $F : X \to \ell_\infty$, $F : x \mapsto (f_n(x))_{n\in\omega}$. By the choice of $\{U_n\}_{n\in\omega}$ and $\{f_n\}_{n\in\omega}$, the map F is continuous and the image $F[X]$ is infinite-dimensional because $F(x_n) = \frac{1}{n}e_n$ for all $n \in \omega$, where $\{e_n\}_{n\in\omega}$ is the standard unit basis of $c_0 \subseteq \ell_\infty$. By the definition of $L(X)$, there is a continuous linear map $G : L(X) \to \ell_\infty$ such that $G{\restriction}_X = F$. Therefore G can be extended to a continuous linear map from $\mathbb{F}^\kappa$ to ℓ_∞ which has an infinite-dimensional image. But this is impossible by Lemma 3.10. □

The space $L(X)$ endowed with the Mackey topology $\mu(L(X), C(X))$ is denoted by $L_\mu(X)$. Below we answer affirmatively the question posed above by showing that there exists an lcs $(E, \mathcal{T})$ without the Glicksberg property, but for which there is a locally convex topology τ on E such that $\sigma(E, E') \subsetneq \tau \subsetneq \mathcal{T}$ and the space (E, τ) is Glicksberg; where $(E, \mathcal{T}) = L_\mu(X)$ and $(E, \tau) = L(X)$.

Example 3.12. Let $X = [0, \omega]$ be a convergent sequence. Then the space $L(X)$ has the Glicksberg property, but $L_\mu(X)$ fails to have the Schur property.

Proof. By Theorem 1.2 of [11], the space $L(X)$ has the Glicksberg property. By Proposition 3.11, the topology of $L(X)$ is strictly stronger than the weak topology of $L(X)$. It remains to show that the space $L_\mu(X)$ is not Schur.

For every $x \in [0, \omega]$, let $\delta_x \in L(X)$ be the Dirac measure at x. Since $L(X)' = C(X)$, the sequence $(\delta_n - \delta_\omega)_{n\in\omega}$ is weakly null in $L(X)$.

For every $n \in \omega$, let $f_n : X \to \{0, 1\}$ be the unique function such that $f_n^{-1}(1) = \{n\}$. The sequence $S = \{f_n\}_{n\in\omega}$ tends to zero in the weak* topology on $C_p(X) = L(X)'_{w^*}$. It is easy to see that the closed absolutely convex hull $\overline{\mathrm{acx}}(S)$ of the set S in $\mathbb{F}^X$ consists of all functions $f : X \to \mathbb{F}$ such that $f(\omega) = 0$ and $\sum_{n\in\omega} |f(n)| \le 1$. Since any such function f is continuous, the set $\overline{\mathrm{acx}}(S) \subseteq C_p(X)$ is a compact disc in $L(X)'_{w^*} = C_p(X)$. Then the polar $\overline{\mathrm{acx}}(S)^\circ = S^\circ$ is a neighborhood of zero in the Mackey topology on $L(X)$. Observe that for every $n \in \omega$ we have $\delta_n(f_n) - \delta_\omega(f_n) = 1$ and hence $\delta_n - \delta_\omega \notin \frac{1}{2}S^\circ$ for all $n \in \omega$. Therefore, the weak null sequence $\{\delta_n - \delta_\omega\}_{n\in\omega}$ is not null in $L_\mu(X)$, and hence the locally convex space $L_\mu(X)$ does not have the Schur property. Since the space $L(X)$ has the Glicksberg property, the topology of $L(X)$ is strictly weaker than the Mackey topology of the space $L_\mu(X)$. □

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