Filomat 36:18 (2022), 6403–6426
https://doi.org/10.2298/FIL2218403L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Weighted Generalized Tensor Functions Based on the Tensor-Product and their Applications

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Abstract. There are three weighted decompositions of tensors proposed in this paper, and the corresponding definitions of the weighted generalized tensor functions are given. The Cauchy integral formula of the weighted Moore-Penrose inverse is developed for solving the tensor equations. Besides above, we give the weighted projection tensors to discuss the representations of the weighted generalized power of tensors. Finally, some special tensors are studied which can preserve the structural invariance under the tensor functions defined in this paper.

1. Introduction

Tensors are used to represent multi-dimensional arrays. In 2005, the eigenvalues of real tensors which defined by Qi provide an important theoretical basis for some applications of positive definiteness in polynomial form [25]. After that, the studies of tensors have entered a new stages, including tensor singular value, tensor product, tensor norm and tensor calculation and so on [6, 8, 17, 24, 26, 27, 30, 39]. The product operation of tensors is regarded as a representation of tensor. There are some product operations of tensors, such as Einstein product [21, 31, 32, 35], ϕ-product [29], T-product. In 2011, Kilmer and Martin gave a tensor representation based on a tensor multiplication which called the T-product [14], at the same time, the T-SVD is given and applied to the image deblurring. Then Kilmer et al. proposed the concepts of orthogonal projection and tensor characteristic formula in [13], and discussed the relationship between tensor characteristic formula and tensor characteristic group. In 2020, Wang studied the tensor neural network model under the T-SVD in [37]. In recent years, there are some research and applications on tensors via the T-product, which could be found in [4, 9, 14, 15, 18, 19, 22, 34, 40]. In [22], Miao first defined the generalized inverse of tensors via the T-product. In this content, the weighted decompositions of tensors are proposed for giving the expressions of the weighted Moore-Penrose inverse of tensors.

The proposal of matrix function is beneficial to deal with the problems in matrix theory and matrix calculation [11]. Matrix function is also widely used in various applications. According to different applications, matrix function can be defined in many ways. As for its definition on the square matrix, it can be defined by the expansion of matrix power series or Jordan canonical form. For the generalized matrix

2020 *Mathematics Subject Classification*. Primary 15A69; Secondary 15A09.

Keywords. T-product, weighted generalized tensor function, weighted Moore-Penrose inverse, weighted resolvent equation, structural preservation

Received: 25 November 2021; Accepted: 14 May 2022

Communicated by Yimin Wei

Research supported by the National Natural Science Foundation of China under grant 11971136 and 12001142, and Aid Foreign Project for education of Heilongjiang Provincial Education Department.

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function on a rectangular matrix, we can refer to the results given by Hawkins in [10]. Yang and Li proposed the weighted spectral decomposition and gave the definition of the weighted generalized matrix functions [38]. Tensors as high-order generalization of matrices are applied to multidimensional differential equations [7, 12]. Tensor function is regarded as a tool to study multidimensional array. According to the definition of T-product, Lund gives the concept of function on F-square tensor and introduces its calculation method in [20]. By taking advantage of the T-SVD, Miao extends the results given by Lund to the generalized tensor functions of rectangular tensors [23]. In this paper, the weighted generalized tensor function is written as T-WGTF, we mainly study the definition, properties and applications of T-WGTF.

The arrangement of paper as below. In section 2, the relative concepts about the T-product and the MN-SVD of matrices are reviewed. In section 3, the T-MN-SVD, the T-MN-CSVD of tensors are proposed, and the definition of T-WGTF are given. The Cauchy integral formula formula of T-WGTF is proposed for solving the tensor equation. Furthermore, the weighted generalized power of tensors are given by the weighted projection tensors. In section 4, the structural properties invariance of some special tensors under T-WGTF are studied.

2. Preliminaries

2.1. The Tensor T-product

It is generally to called that a is complex-value if $a \in \mathbb{C}$, and b is real-value as $b \in \mathbb{R}$. It is written that $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$ is a complex tensor of order 3, where p_1, p_2 and p_3 are arbitrary nonzero natural numbers. If all entries of a tensor are zeros, we call the tensor as zero tensor and denote it by $\mathcal{O}$. The discrete fourier transform matrix is abbreviated as the DFT matrix. The T-product is a closed multiplication operation which preserves the order of tensors. There are some operations which derive the definition of the T-product [9, 13, 14]. If $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$, and its every frontal slice is written as $p_1 \times p_2$ matrix $A^{(k)}$, $k = 1, 2, \cdots, p_3$, then

$$\mathrm{bcirc}(\mathcal{A}) = \begin{pmatrix} A^{(1)} & A^{(p_3)} & \cdots & A^{(2)} \\ A^{(2)} & A^{(1)} & \cdots & A^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ A^{(p_3)} & A^{(p_3-1)} & \cdots & A^{(1)} \end{pmatrix},$$

and the inverse operation $\mathrm{bcirc}^{-1}(\mathrm{bcirc}(\mathcal{A})) = \mathcal{A}$. The first block column of $\mathrm{bcirc}(\mathcal{A})$ is written by

$$\mathrm{unfold}(\mathcal{A}) = \begin{pmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(p_3)} \end{pmatrix},$$

the operation "unfold" transfers a $p_1 \times p_2 \times p_3$ tensor to a $p_1 p_3 \times p_2$ matrix. The operation "fold" takes the matrix $\mathrm{unfold}(\mathcal{A})$ back to a tensor, that is

$$\mathrm{fold}(\mathrm{unfold}(\mathcal{A})) = \mathcal{A}.$$

The $p_3 \times p_3$ DFT matrix is defined as follows [5],

$$F_{p_3} = \frac{1}{\sqrt{p_3}} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{p_3-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{p_3-1} & \omega^{2(p_3-1)} & \omega^{3(p_3-1)} & \cdots & \omega^{(p_3-1)(p_3-1)} \end{pmatrix},$$

where $\omega = e^{-2\pi i/p_3}$, i is imaginary unit and I_{p_1} and I_{p_2} are identity matrices. For any block-circulant matrix can be transformed to a block diagonal matrix, if $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$, then

$$(F_{p_3}^H \otimes I_{p_1})\text{bcirc}(\mathcal{A})(F_{p_3} \otimes I_{p_2}) = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_{p_3} \end{pmatrix},$$

where "$\otimes$" is the Kronecker product and $A_i \in \mathbb{C}^{p_1\times p_2}$, $i = 1, 2, \cdots, p_3$.

Here are some related concepts about the T-product which can refer to [14, 22, 23].

Definition 2.1. [14] *Suppose $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$, $\mathcal{B} \in \mathbb{C}^{p_2\times p_4\times p_3}$, the T-product $\mathcal{A} * \mathcal{B}$ is a tensor which defined by*

$$\mathcal{A} * \mathcal{B} = \text{fold}(\text{bcirc}(\mathcal{A})\text{unfold}(\mathcal{B})) \in \mathbb{C}^{p_1\times p_4\times p_3}.$$

Example 2.2. *If $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times 3}$ and $\mathcal{B} \in \mathbb{C}^{p_2\times p_4\times 3}$. Then*

$$\mathcal{A} * \mathcal{B} = \text{fold}\left(\begin{pmatrix} A^{(1)} & A^{(3)} & A^{(2)} \\ A^{(2)} & A^{(1)} & A^{(3)} \\ A^{(3)} & A^{(2)} & A^{(1)} \end{pmatrix}\begin{pmatrix} B^{(1)} \\ B^{(2)} \\ B^{(3)} \end{pmatrix}\right) \in \mathbb{C}^{p_1\times p_4\times 3}.$$

Definition 2.3. [14] *Let $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$, the transpose of $\mathcal{A}$ is defined as*

$$\mathcal{A}^T = \text{fold}\begin{pmatrix} A^{(1)^T} \\ A^{(p_3)^T} \\ \vdots \\ A^{(2)^T} \end{pmatrix},$$

and the conjugate transpose of $\mathcal{A}$ is defined by

$$\mathcal{A}^H = \text{fold}\begin{pmatrix} A^{(1)^H} \\ A^{(p_3)^H} \\ \vdots \\ A^{(2)^H} \end{pmatrix}.$$

Definition 2.4. [14]*The identity tensor $\mathcal{I}_{p_2p_2p_3}$ is defined as*

$$\mathcal{I}_{p_2p_2p_3} = \text{fold}((I_{p_2}, O, \cdots, O)^T),$$

where I_{p_2} is a $p_2 \times p_2$ identity matrix, and zero matrix $O \in \mathbb{R}^{p_2\times p_2}$.

Definition 2.5. [14] *Assume $\mathcal{A} \in \mathbb{C}^{p_2\times p_2\times p_3}$, the unique inverse of $\mathcal{A}$ is $\mathcal{B} = \mathcal{A}^{-1}$ if*

$$\mathcal{A} * \mathcal{B} = \mathcal{I}_{p_2p_2p_3} \text{ and } \mathcal{B} * \mathcal{A} = \mathcal{I}_{p_2p_2p_3}.$$

Definition 2.6. [14] *$\mathcal{N} \in \mathbb{R}^{p_2\times p_2\times p_3}$ is orthogonal if $\mathcal{N}^T * \mathcal{N} = \mathcal{N} * \mathcal{N}^T = \mathcal{I}_{p_2p_2p_3}$. $\mathcal{M} \in \mathbb{C}^{p_2\times p_2\times p_3}$ is unitary if $\mathcal{M}^H * \mathcal{M} = \mathcal{M} * \mathcal{M}^H = \mathcal{I}_{p_2p_2p_3}$.*

Definition 2.7. [23] *If $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$, the T-range space of $\mathcal{A}$ is defined as*

$$\mathcal{R}(\mathcal{A}) = \text{Ran}((F_{p_3}^H \otimes I_{p_1})\text{bcirc}(\mathcal{A})(F_{p_3} \otimes I_{p_2})),$$

*where "*Ran*" is column space of matrix. Moreover, the T-null space of $\mathcal{A}$ is defined as*

$$\mathcal{N}(\mathcal{A}) = \text{Null}((F_{p_3}^H \otimes I_{p_1})\text{bcirc}(\mathcal{A})(F_{p_3} \otimes I_{p_2})),$$

*where "*Null*" is written as null space of matrix.*

Definition 2.8. [23] *Suppose that* $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$*, we design the T-norm of* $\mathcal{A}$ *as*

$$\|\mathcal{A}\| = \|\text{bcirc}(\mathcal{A})\|,$$

where "$\| \cdot \|$*" is a unitary invariant matrix norm.*

The T-SVD as a new representation of tensors is proposed by Kilmer in [14], which is developed for raising the T-WGTF.

Lemma 2.9. [14] *If* $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$*, then*

$$\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^H, \tag{1}$$

where $\mathcal{U} \in \mathbb{C}^{p_1 \times p_1 \times p_3}$*,* $\mathcal{V} \in \mathbb{C}^{p_2 \times p_2 \times p_3}$ *are unitary, the F-diagonal tensor (each frontal slice is diagonal matrix)* $\mathcal{S}$ *is* $p_1 \times p_2 \times p_3$*.*

The representation (1) is called as the T-SVD of $\mathcal{A}$.

Lemma 2.10. [9, 13, 14] *Suppose* $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$*,* $\mathcal{B} \in \mathbb{C}^{p_2 \times p_4 \times p_3}$ *and* $\mathcal{C} \in \mathbb{C}^{p_4 \times p_2 \times p_3}$*, then the following equations hold,*
(1) $(\mathcal{A} * \mathcal{B}) * \mathcal{C} = \mathcal{A} * (\mathcal{B} * \mathcal{C})$,
(2) $\text{bcirc}(\mathcal{A} * \mathcal{B}) = \text{bcirc}(\mathcal{A})\text{bcirc}(\mathcal{B})$,
(3) $(\mathcal{A} * \mathcal{B})^H = \mathcal{B}^H * \mathcal{A}^H$,
(4) $\text{bcirc}(\mathcal{A}^T) = \text{bcirc}(\mathcal{A})^T$, $\text{bcirc}(\mathcal{A}^H) = \text{bcirc}(\mathcal{A})^H$.

Next, we give the concepts of some special tensors as follows, which including the Hermite tensor, positive definite tensor, the weighted conjugate transform of tensor and $\mathcal{N}$-unitary tensor.

Definition 2.11. $\mathcal{A} \in \mathbb{C}^{p_2 \times p_2 \times p_3}$ *is Hermite tensor if* $\mathcal{A}^H = \mathcal{A}$*, and* $\mathcal{B} \in \mathbb{R}^{p_2 \times p_2 \times p_3}$ *is real symmetric tensor if* $\mathcal{B}^T = \mathcal{B}$*, respectively.*

Definition 2.12. *A* $p_1 \times p_1 \times p_3$ *complex tensor* $\mathcal{M}$ *is called the positive definite tensor, if*

$$\mathcal{M} = \text{fold}\begin{pmatrix} M \\ O \\ \vdots \\ O \end{pmatrix},$$

where $M \in \mathbb{C}^{p_1 \times p_1}$ *is a positive definite matrix and "O" is a* $p_1 \times p_1$ *zero matrix.*

Definition 2.13. *If* $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$*,* $\mathcal{M}$ *and* $\mathcal{N}$ *are* $p_1 \times p_1 \times p_3$ *and* $p_2 \times p_2 \times p_3$ *Hermite positive definite tensors, denote the weighted conjugate transpose of* $\mathcal{A}$ *as* $\mathcal{A}^\#$*,*

$$\mathcal{A}^\# = \mathcal{N}^{-1} * \mathcal{A}^H * \mathcal{M}. \tag{2}$$

The block diagonal matrices of $(F_{p_3}^H \otimes I_{p_2})\text{bcirc}(\mathcal{N})(F_{p_3} \otimes I_{p_2})$ are Hermite positive definite matrices, which are invertible matrices, then the Hermite definite tensor $\mathcal{N}$ in (2) is also invertible.

Definition 2.14. *Let* $\mathcal{N} \in \mathbb{C}^{p_2 \times p_2 \times p_3}$ *be an Hermite positive definite tensor. A tensor* $\mathcal{Q} \in \mathbb{C}^{p_2 \times p_2 \times p_3}$ *is* $\mathcal{N}$*-unitary if* $\mathcal{Q}^H * \mathcal{N} * \mathcal{Q} = \mathcal{I}_{p_2 p_2 p_3}$*. Suppose* $\mathcal{N}$ *and* $\mathcal{Q}$ *are* $p_2 \times p_2 \times p_3$ *real tensors,* $\mathcal{N}$ *is the real symmetric positive definite tensor,* $\mathcal{Q}$ *is* $\mathcal{N}$*-orthogonal if* $\mathcal{Q}^T * \mathcal{N} * \mathcal{Q} = \mathcal{I}_{p_2 p_2 p_3}$*, respectively.*

Analogy with the operation of matrix multiplicative blocks, if there exist multiplicity in tensor blocks, a result of tensors which similar to that on the matrix may be gained [23].

Lemma 2.15. [23] *If* $\mathcal{A} \in \mathbb{C}^{t_1 \times s_1 \times p_3}$*,* $\mathcal{B} \in \mathbb{C}^{t_1 \times s_2 \times p_3}$*,* $\mathcal{C} \in \mathbb{C}^{t_2 \times s_1 \times p_3}$*,* $\mathcal{D} \in \mathbb{C}^{t_2 \times s_2 \times p_3}$*,* $\mathcal{E} \in \mathbb{C}^{s_1 \times q_1 \times p_3}$*,* $\mathcal{F} \in \mathbb{C}^{s_1 \times q_2 \times p_3}$*,* $\mathcal{G} \in \mathbb{C}^{s_2 \times q_1 \times p_3}$ *and* $\mathcal{H} \in \mathbb{C}^{s_2 \times q_2 \times p_3}$*, then,*

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} * \begin{pmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{G} & \mathcal{H} \end{pmatrix} = \begin{pmatrix} \mathcal{A} * \mathcal{E} + \mathcal{B} * \mathcal{G} & \mathcal{A} * \mathcal{F} + \mathcal{B} * \mathcal{H} \\ \mathcal{C} * \mathcal{E} + \mathcal{D} * \mathcal{G} & \mathcal{C} * \mathcal{F} + \mathcal{D} * \mathcal{H} \end{pmatrix}.$$

2.2. The Weighted Matrix Function

In this section, it is reviewed the MN-SVD, MN-CSVD of matrices and the concept of the weighted matrix functions.

Lemma 2.16. [36, 38] *If $A \in \mathbb{C}_r^{p_1\times p_2}$, $M \in \mathbb{C}^{p_1\times p_1}$ and $N \in \mathbb{C}^{p_2\times p_2}$ are coefficient matrices of A, where r =rank(A). Then $U^H MU = I_{p_1}$ and $V^H N^{-1} V = I_{p_2}$ hold with $U \in \mathbb{C}^{p_1\times p_1}$ and $V \in \mathbb{C}^{p_2\times p_2}$, furthermore,*

$$A = U\begin{pmatrix} \widehat{\Sigma} & O \\ O & O \end{pmatrix} V^H = U\Sigma V^H,$$

where $\widehat{\Sigma} = diag(\sigma_1, \cdots, \sigma_r)$, and $\sigma_1 \geq \cdots \geq \sigma_r > 0$ are written as the nonzero weighted singular values of A. Furthermore, suppose $U = (U_r, U')$ and $V = (V_r, V')$, where $U_r \in \mathbb{C}^{p_1\times r}$ and $V_r \in \mathbb{C}^{p_2\times r}$. The CSVD of A is written as

$$A = U_r\widehat{\Sigma}V_r^H. \tag{3}$$

Lemma 2.17. [28, 33]. *If A, Σ, U, V, U_r, V_r are tensors in Lemma* 2.16. *Then, the weighted Moore-Penrose inverse $A_{MN}^\dagger \in \mathbb{C}^{p_2\times p_1}$ is factorized by $A_{MN}^\dagger = N^{-1}V_r\widehat{\Sigma}^{-1}U_r^H M$.*

Definition 2.18. [38] *Suppose $A \in \mathbb{C}_r^{p_1\times p_2}$, and the scalar function $f : \mathbb{C} \to \mathbb{C}$, the MN-SVD of A has the form as* (3)*. The weighted matrix function $f_{MN} : \mathbb{C}^{p_1\times p_2} \to \mathbb{C}^{p_1\times p_2}$ is defined in terms of $f : \mathbb{C} \to \mathbb{C}$ as*

$$f_{MN}(A) = U_r f_{MN}(\widehat{\Sigma})V_r^H,\ f_{MN}(\widehat{\Sigma}) = \begin{pmatrix} f(\sigma_1) & & \\ & \ddots & \\ & & f(\sigma_r) \end{pmatrix}, \tag{4}$$

where each σ_i is nonzero weighted singular value of A, $i = 1, 2, \cdots, r$.

3. T-MN-SVD and T-WGTF

3.1. T-MN-SVD and T-MN-CSVD

According to the MN-SVD of matrices and the weighted generalized matrix functions in Lemma 2.16 and Definition 2.18, the T-MN-SVD and T-MN-CSVD of tensors are given for defining the T-WGTF.

Theorem 3.1. *(T-MN-SVD) If $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$, $\mathcal{M} \in \mathbb{C}^{p_1\times p_1\times p_3}$ and $\mathcal{N} \in \mathbb{C}^{p_2\times p_2\times p_3}$ are two Hermite positive definite tensors, then $\mathcal{A}$ is expressed as*

$$\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^H \tag{5}$$

with $\mathcal{U} \in \mathbb{C}^{p_1\times p_1\times p_3}$ is $\mathcal{M}$-unitary, $\mathcal{V} \in \mathbb{C}^{p_2\times p_2\times p_3}$ is $\mathcal{N}^{-1}$-unitary, $\mathcal{S} \in \mathbb{C}^{p_1\times p_2\times p_3}$ is F-diagonal.

Proof. Transform bcirc($\mathcal{M}$) and bcirc($\mathcal{N}$) into the Fourier domain,

$$(F_{p_3}^H \otimes I_{p_1})\text{bcirc}(\mathcal{M})(F_{p_3} \otimes I_{p_1}) = \begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_{p_3} \end{pmatrix}, (F_{p_3}^H \otimes I_{p_2})\text{bcirc}(\mathcal{N})(F_{p_3} \otimes I_{p_2}) = \begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_{p_3} \end{pmatrix},$$

where $M_i \in \mathbb{C}^{p_1\times p_1}$ and $N_i \in \mathbb{C}^{p_2\times p_2}$ are Hermite definite matrices. Let $M_i = L_i L_i^H$ and $N_i = K_i K_i^H$ be the Cholesky factorizations of M_i and N_i, $i = 1, 2, \cdots, p_3$. We define tensors $\mathcal{L}$ and $\mathcal{K}$ as

$$\text{bcirc}(\mathcal{L}) = (F_{p_3} \otimes I_{p_1})\begin{pmatrix} L_1 & & \\ & \ddots & \\ & & L_{p_3} \end{pmatrix}(F_{p_3}^H \otimes I_{p_1}), \text{bcirc}(\mathcal{K}) = (F_{p_3} \otimes I_{p_2})\begin{pmatrix} K_1 & & \\ & \ddots & \\ & & K_{p_3} \end{pmatrix}(F_{p_3}^H \otimes I_{p_2}),$$

and $\mathcal{C} = \mathcal{L}^H * \mathcal{A} * (\mathcal{K}^{-1})^H \in \mathbb{C}^{p_1\times p_2\times p_3}$, according to the T-SVD of $\mathcal{C}$, there exist two unitary tensors $\widetilde{\mathcal{U}} \in \mathbb{C}^{p_1\times p_1\times p_3}$ and $\widetilde{\mathcal{V}} \in \mathbb{C}^{p_2\times p_2\times p_3}$ satisfy

$$\widetilde{\mathcal{U}}^H * \mathcal{C} * \widetilde{\mathcal{V}} = \mathcal{S},$$

where $\mathcal{S}$ is an F-diagonal tensor. We define $\mathcal{U} = (\mathcal{L}^{-1})^H * \widetilde{\mathcal{U}}$ and $\mathcal{V} = \mathcal{K} * \widetilde{\mathcal{V}}$, therefore, $\mathcal{U}$ is $\mathcal{M}$-unitary and $\mathcal{V}$ is $\mathcal{N}^{-1}$-unitary, and $\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^H$. □

The T-MN-eigenvalues of $\mathcal{A}$ are elements of the set $\mathrm{spec}((F_{p_3}^H \otimes I_{p_1})\mathrm{bcirc}(\mathcal{S} * \mathcal{S}^{\#})(F_{p_3} \otimes I_{p_1}) = \{|\sigma_j^i|^2,\ 1 \le i \le p_3,\ 1 \le j \le p_1\})$, we write σ_j^i as the weighted singular values of $\mathcal{A}$. In the following content, we describe Hermite positive definite tensors $\mathcal{M} \in \mathbb{C}^{p_1\times p_1\times p_3}$ and $\mathcal{N} \in \mathbb{C}^{p_2\times p_2\times p_3}$ as the weighted coefficient tensors of $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$.

A specific example is given to illustrate the T-MN-SVD.

Example 3.2. *Let $\mathcal{A} \in \mathbb{C}^{3\times2\times3}$, the frontal slices of $\mathcal{A}$ have the following forms,*

$$A^{(1)} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \\ 7 & 5 \end{pmatrix}, A^{(2)} = \begin{pmatrix} 3 & -1 \\ 5 & 2 \\ 0.1i & 8 \end{pmatrix}, A^{(3)} = \begin{pmatrix} 5 & 9 \\ 3 & -0.1i \\ 5 & -6 \end{pmatrix}.$$

The weighted coefficient tensors $\mathcal{M} \in \mathbb{C}^{3\times3\times3}$ and $\mathcal{N} \in \mathbb{C}^{2\times2\times3}$ are given by

$$M^{(1)} = \begin{pmatrix} 5 & -1 & 2 \\ -1 & 3 & 0 \\ 2 & 0 & 4 \end{pmatrix}, M^{(2)} = M^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, N^{(1)} = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}, N^{(2)} = N^{(3)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

Transform bcirc($\mathcal{M}$) and bcirc($\mathcal{N}$) into the Fourier domain, since the Cholesky decomposition of matrix, it is known that

$$L^{(1)} = \begin{pmatrix} 2.2361 & 0.0000 & 0.0000 \\ -0.4472 & 1.6733 & 0.0000 \\ 0.8944 & 0.2390 & 1.7728 \end{pmatrix}, L^{(2)} = L^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and

$$K^{(1)} = \begin{pmatrix} 3.4142 & 0.0000 \\ -0.7071 & 4.1213 \end{pmatrix}, K^{(2)} = K^{(3)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

By equation $\mathcal{C} = \mathcal{L}^H * \mathcal{A} * (\mathcal{K}^{-1})^H$ and the T-SVD of $\mathcal{C}$, we have

$$S^{(1)} = \begin{pmatrix} 9.0432 & 0 \\ 0 & 2.9151 \\ 0 & 0 \end{pmatrix}, S^{(2)} = \begin{pmatrix} 2.2337 & 0 \\ 0 & -1.2492 \\ 0 & 0 \end{pmatrix}, S^{(3)} = \begin{pmatrix} 2.2337 & 0 \\ 0 & -1.2492 \\ 0 & 0 \end{pmatrix}.$$

By equations $\mathcal{U} = (\mathcal{L}^{-1})^H * \widetilde{\mathcal{U}}$ and $\mathcal{V} = \mathcal{K} * \widetilde{\mathcal{V}}$, we get that $\mathcal{U}$ and $\mathcal{V}$ have the following forms,

$$U^{(1)} = \begin{pmatrix} -0.3379 - 0.0001i & -0.0652 - 0.0058i & -0.0430 + 0.0012i \\ -0.0538 + 0.0002i & 0.3427 + 0.0119i & -0.1555 - 0.0004i \\ 0.2083 - 0.0008i & -0.0196 + 0.0027i & 0.2101 - 0.0005i \end{pmatrix},$$

$$U^{(2)} = \begin{pmatrix} 0.1249 - 0.0001i & 0.0190 - 0.0058i & -0.0432 + 0.0012i \\ -0.0571 + 0.0002i & -0.0267 + 0.0119i & 0.1788 - 0.0004i \\ -0.3381 - 0.0008i & 0.2264 + 0.0027i & 0.1326 - 0.0005i \end{pmatrix},$$

$$U^{(3)}=\begin{pmatrix}-0.0278-0.0001i & -0.2044-0.0058i & -0.3032+0.0012i\\ -0.0896+0.0002i & 0.1042+0.0119i & -0.4048-0.0004i\\ -0.1812-0.0008i & -0.0586+0.0027i & 0.1039-0.0005i\end{pmatrix},$$

and

$$V^{(1)}=\begin{pmatrix}-0.2777-0.0000i & -1.5394+0.0000i\\ -0.1450-0.0036i & -0.6460-0.0053i\end{pmatrix},$$

$$V^{(2)}=\begin{pmatrix}-1.2754-0.0000i & 1.7258+0.0000i\\ -3.0107-0.0036i & -2.2611-0.0053i\end{pmatrix},$$

$$V^{(3)}=\begin{pmatrix}-1.2754-0.0000i & 1.7258+0.0000i\\ 1.4333-0.0036i & -0.9031-0.0053i\end{pmatrix}.$$

The MN-CSVD is applied in the weighted Moore-Penrose inverse theory, the T-MN-CSVD of tensors is introduced as below. Let $\mathcal{A}\in\mathbb{C}^{p_1\times p_2\times p_3}$ with $(\mathcal{M},\mathcal{N})$ weighted coefficient tensors. The T-MN-SVD of $\mathcal{A}$ as (5) with $\mathcal{U}\in\mathbb{C}^{p_1\times p_1\times p_3}$, $\mathcal{V}\in\mathbb{C}^{p_2\times p_2\times p_3}$ and F-diagonal tensor $\mathcal{S}\in\mathbb{C}^{p_1\times p_2\times p_3}$, where $\mathcal{U}$ and $\mathcal{V}$ satisfying $\mathcal{U}^H*\mathcal{M}*\mathcal{U}=\mathcal{I}_{p_1p_1p_3}$ and $\mathcal{V}^H*\mathcal{N}^{-1}*\mathcal{V}=\mathcal{I}_{p_2p_2p_3}$. Suppose $\text{rank}(\Sigma_i)=r_i$, where Σ_i is the block diagonal matrices obtained by the discrete Fourier diagonalization of bcirc($\mathcal{S}$), we denote $U_i=(x_1^i,x_2^i,\cdots,x_{p_1}^i)$ and $V_i=(y_1^i,y_2^i,\cdots,y_{p_2}^i)$, $i=1,2,\cdots,p_3$. Besides, r is the maximum value of r_i which written as the T-tubal-rank of $\mathcal{A}$ and denoted as $rank_t(\mathcal{A})$ [14]. The T-MN-CSVD is given by deleting the zero weighted singular values. In other words,

$$(\Sigma_i)_r=diag(c_1^i,c_2^i,\cdots,c_r^i)\in\mathbb{R}^{r\times r},(V_i)_r=(y_1^i,y_2^i,\cdots,y_r^i)\in\mathbb{C}^{p_2\times r},(U_i)_r=(x_1^i,x_2^i,\cdots,x_r^i)\in\mathbb{C}^{p_1\times r},$$

thus,

$$\begin{aligned}\text{bcirc}(\mathcal{A})&=(F_{p_3}\otimes I_{p_1})\begin{pmatrix}(U_1)_r(\Sigma_1)_r(V_1^H)_r & & \\ & \ddots & \\ & & (U_{p_3})_r(\Sigma_{p_3})_r(V_{p_3}^H)_r\end{pmatrix}(F_{p_3}^H\otimes I_{p_2})\\ &=\text{bcirc}(\mathcal{U}_\text{r})\text{bcirc}(\mathcal{S}_\text{r})\text{bcirc}(\mathcal{V}_\text{r}^\text{H}),\end{aligned}$$

where $\mathcal{U}_r\in\mathbb{C}^{p_1\times r\times p_3}$, $\mathcal{S}_r\in\mathbb{R}^{r\times r\times p_3}$, $\mathcal{V}_r\in\mathbb{C}^{p_2\times r\times p_3}$. Thus, $\mathcal{A}$ has the following expression,

$$\mathcal{A}=\mathcal{U}_r*\mathcal{S}_r*\mathcal{V}_r^H. \tag{6}$$

The factorization (6) is called the T-MN-CSVD of $\mathcal{A}$.

Remark 3.3. *In matrix theory, the MN-CSVD of A has represented as $A=U_rS_rV_r^H$, where $S_r=diag(\sigma_1,\sigma_2,\cdots,\sigma_r)$, and $\sigma_i\neq 0$ for i is a positive integer from 1 to r, as for the T-MN-CSVD of $\mathcal{A}$, there are some σ_j^i of $\mathcal{A}$ are zeros as the choose of the T-tubal-rank of $\mathcal{A}$. In the following description, denote the nonzero weighted singular values of tensors as c_j^i, $j=1,2,\cdots,r$ and $i=1,2,\cdots,p_3$.*

3.2. T-W-MP Inverse of Tensor

The weighted Moore-Penrose inverse of tensors is written as T-W-MP inverse and defined by four equations.

Definition 3.4. [1] *Let $\mathcal{A}\in\mathbb{C}^{p_1\times p_2\times p_3}$ with $(\mathcal{M},\mathcal{N})$ weighted coefficient tensors. The T-W-MP inverse $\mathcal{X}\in\mathbb{C}^{p_2\times p_1\times p_3}$ of $\mathcal{A}$ such that*

$$\mathcal{A}*\mathcal{X}*\mathcal{A}=\mathcal{A},\mathcal{X}*\mathcal{A}*\mathcal{X}=\mathcal{X},(\mathcal{M}*\mathcal{A}*\mathcal{X})^H=\mathcal{M}*\mathcal{A}*\mathcal{X},(\mathcal{N}*\mathcal{X}*\mathcal{A})^H=\mathcal{N}*\mathcal{X}*\mathcal{A} \tag{7}$$

hold.

Corollary 3.5. *Suppose $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$ with $(\mathcal{M},\mathcal{N})$ weighted coefficient tensors, the T-MN-CSVD of $\mathcal{A}$ is $\mathcal{A} = \mathcal{U}_r * \mathcal{S}_r * \mathcal{V}_r^H$. Then T-W-MP inverse of $\mathcal{A}$ is given by*

$$\mathcal{A}^{\dagger}_{\mathcal{MN}} = \mathcal{N}^{-1} * \mathcal{V}_r * \mathcal{S}_r^{\dagger} * \mathcal{U}_r^H * \mathcal{M},$$

where $\mathcal{S}_r^{\dagger} = \text{bcirc}^{-1}\left((F_{p_3}\otimes I_r)\begin{pmatrix}(\Sigma_1)_r^{\dagger} & & & \\ & (\Sigma_2)_r^{\dagger} & & \\ & & \ddots & \\ & & & (\Sigma_{p_3})_r^{\dagger}\end{pmatrix}(F_{p_3}^H\otimes I_r)\right).$

In view of the concept of T-W-MP inverse of tensors, it is obtained the following inferences.

Corollary 3.6. *Suppose $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$ with $(\mathcal{M},\mathcal{N})$ weighted coefficient tensors and the T-MN-CSVD of $\mathcal{A}$ is $\mathcal{A} = \mathcal{U}_r * \mathcal{S}_r * \mathcal{V}_r^H$, then*
(1) $\text{bcirc}(\mathcal{U}_r * \mathcal{U}_r^H * \mathcal{M}) = P_{\mathcal{R}(\mathcal{A})} = \text{bcirc}(\mathcal{A} * \mathcal{A}^{\dagger}_{\mathcal{MN}})$,
(2) $\text{bcirc}(\mathcal{N}^{-1} * \mathcal{V}_r * \mathcal{V}_r^H) = P_{\mathcal{R}(\mathcal{A}^{\#})} = \text{bcirc}(\mathcal{A}^{\dagger}_{\mathcal{MN}} * \mathcal{A})$,
(3) *The tensor $\mathcal{E} := \mathcal{U}_r * \mathcal{V}_r^H$ is the weighted projection tensor which makes* $\text{bcirc}(\mathcal{E} * \mathcal{E}^{\#}) = P_{\mathcal{R}(\mathcal{E})} = P_{\mathcal{R}(\mathcal{A})}$ *and* $\text{bcirc}(\mathcal{E}^{\#} * \mathcal{E}) = P_{\mathcal{R}(\mathcal{E}^{\#})} = P_{\mathcal{R}(\mathcal{A}^{\#})}$ *hold.*

The $p_1 \times p_2 \times p_3$ weighted partial isometry tensors of $\mathcal{A}$ are defined by satisfying

$$\mathcal{E}_j^i * \mathcal{E}_l^{k^{\#}} = \mathcal{O}, \mathcal{E}_l^k * \mathcal{E}_j^{i^{\#}} = \mathcal{O} \tag{8}$$

for $i \neq k$ or $j \neq l$, and

$$\mathcal{E}_j^i * \mathcal{E}^{\#} * \mathcal{A} = \mathcal{A} * \mathcal{E}^{\#} * \mathcal{E}_j^i.$$

By the concept of $\mathcal{E}$,

$$\text{bcirc}(\mathcal{E}) = (F_{p_3}\otimes I_{p_1})\begin{pmatrix}(U_1)_r & & \\ & \ddots & \\ & & (U_{p_3})_r\end{pmatrix}\begin{pmatrix}(V_1)_r & & \\ & \ddots & \\ & & (V_{p_3})_r\end{pmatrix}^H (F_{p_3}^H\otimes I_{p_2}),$$

where $(U_i)_r \in \mathbb{C}^{p_1\times r}$ and $(V_i)_r \in \mathbb{C}^{p_2\times r}$, then

$$\text{bcirc}(\mathcal{E}_j^{(i)}) = (F_{p_3}\otimes I_{p_1})diag(O, O, \cdots, u_j^i v_j^{i^H}, \cdots, O)(F_{p_3}^H\otimes I_{p_2}),$$

where u_j^i and v_j^i means the j-th column of $(U_i)_r$ and $(V_i)_r$, $i = 1, 2, \cdots, p_3$, $j = 1, 2, \cdots, r$, then we have

$$\mathcal{E} = \sum_{i,j} \mathcal{E}_j^i. \tag{9}$$

Furthermore, $(\mathcal{E}_j^i)^{\dagger}_{\mathcal{MN}} = (\mathcal{E}_j^i)^{\#}$.

The weighted spectral decomposition of tensors is proposed by using the weighted singular values of tensors and the weighted partial isometry tensors as followed.

Theorem 3.7. *Suppose $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$ with $(\mathcal{M},\mathcal{N})$ weighted coefficient tensors, then the T-MN-spectral decomposition of $\mathcal{A}$ is*

$$\mathcal{A} = \sum_{i,j} c_j^i \mathcal{E}_j^i, \tag{10}$$

where c_j^i and $\mathcal{E}_j^i$ are the weighted singular values and the weighted partial isometry tensors of $\mathcal{A}$, $j = 1, 2, \cdots, r$ and $i = 1, 2, \cdots, p_3$.

Proof. By the weighted spectral decomposition of matrices, it is obtained that the following equation holds,

$$\mathrm{bcirc}(\mathcal{A}) = (F_{p_3} \otimes I_{p_1}) \begin{pmatrix} \sum_j c_j^1 E_j^1 & & \\ & \ddots & \\ & & \sum_j c_j^{p_3} E_j^{p_3} \end{pmatrix} (F_{p_3}^H \otimes I_{p_2}) = \sum_{i,j} c_j^i \mathrm{bcirc}(\mathcal{E}_j^i),$$

then the equation (10) can be obtained by taking "bcirc^{-1}" on the above equation. □

By the fact $\mathrm{bcirc}(\mathcal{A}_{\mathcal{MN}}^{\dagger}) = (\mathrm{bcirc}(\mathcal{A}))_{\mathrm{MN}}^{\dagger}$ that $\mathcal{A}_{\mathcal{MN}}^{\dagger}$ could be expressed as

$$\mathcal{A}_{\mathcal{MN}}^{\dagger} = \sum_{i,j} \frac{1}{c_j^i} \mathcal{E}_j^{i^{\#}}.$$

3.3. T-WGTF

With the T-MN-SVD and the T-MN-CSVD, the T-WGTF could be defined as follows.

Theorem 3.8. *If $f : \mathbb{C} \to \mathbb{C}$, and $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$ with $(\mathcal{M}, \mathcal{N})$ weighted coefficient tensors, then the corresponding T-WGTF $f_{\mathcal{MN}}^{\diamond} : \mathbb{C}^{p_1 \times p_2 \times p_3} \to \mathbb{C}^{p_1 \times p_2 \times p_3}$ is defined by*

$$f_{\mathcal{MN}}^{\diamond}(\mathcal{A}) = \mathcal{U} * \widehat{f_{\mathcal{MN}}}(\mathcal{S}) * \mathcal{V}^H,$$

where the function $\widehat{f_{\mathcal{MN}}}(\mathcal{S})$ is given by

$$\widehat{f_{\mathcal{MN}}}(\mathcal{S}) = \mathrm{bcirc}^{-1}\left((F_{p_3} \otimes I_{p_1}) \begin{pmatrix} f_{MN}(\Sigma_1) & & \\ & \ddots & \\ & & f_{MN}(\Sigma_{p_3}) \end{pmatrix} (F_{p_3}^H \otimes I_{p_2}) \right),$$

$f_{MN}(\Sigma_i)$ are defined in (4) and each Σ_i is the diagonal block obtained by the discrete Fourier diagonalization of $\mathrm{bcirc}(\mathcal{S})$*, $i = 1, 2, \cdots, p_3$.*

Proof. According to the T-MN-SVD of $\mathcal{A}$, by taking "bcirc" on (1) we have

$$\mathrm{bcirc}(\mathcal{A}) = \mathrm{bcirc}(\mathcal{U} * \mathcal{S} * \mathcal{V}^{\mathrm{H}}) = \mathrm{bcirc}(\mathcal{U}) \cdot \mathrm{bcirc}(\mathcal{S}) \cdot \mathrm{bcirc}(\mathcal{V}^{\mathrm{H}}),$$

where $\mathrm{bcirc}(\mathcal{U}) \in \mathbb{C}^{p_1p_3 \times p_1p_3}$ and $\mathrm{bcirc}(\mathcal{V}^H) \in \mathbb{C}^{p_2p_3 \times p_2p_3}$ are matrices which satisfying $\mathrm{bcirc}(\mathcal{U}^H) \cdot \mathrm{bcirc}(\mathcal{M}) \cdot \mathrm{bcirc}(\mathcal{U}) = \mathrm{bcirc}(\mathcal{I}_{p_1p_1p_3})$, $\mathrm{bcirc}(\mathcal{V}^H) \cdot \mathrm{bcirc}(\mathcal{N}^{-1}) \cdot \mathrm{bcirc}(\mathcal{V}) = \mathrm{bcirc}(\mathcal{I}_{p_2p_2p_3})$. Besides, $\mathrm{bcirc}(\mathcal{S}) \in \mathbb{C}^{p_1p_3 \times p_2p_3}$ is factorized as

$$\mathrm{bcirc}(\mathcal{S}) = (F_{p_3} \otimes I_{p_1}) \begin{pmatrix} \Sigma_1 & & \\ & \ddots & \\ & & \Sigma_{p_3} \end{pmatrix} (F_{p_3}^H \otimes I_{p_2}).$$

By the expression of the weighted GMF in equation (4), the induced function on $\mathcal{S}$ is defined by

$$\widehat{f_{\mathcal{MN}}}(\mathcal{S}) = \mathrm{bcirc}^{-1}\left((F_{p_3} \otimes I_{p_1}) \begin{pmatrix} f_{MN}(\Sigma_1) & & \\ & \ddots & \\ & & f_{MN}(\Sigma_{p_3}) \end{pmatrix} (F_{p_3}^H \otimes I_{p_2}) \right),$$

we define

$$\mathrm{bcirc}(f_{\mathcal{MN}}^{\diamond}(\mathcal{A})) = \mathrm{bcirc}(\mathcal{U})\mathrm{bcirc}(\widehat{f_{\mathcal{MN}}}(\mathcal{S}))\mathrm{bcirc}(\mathcal{V}^{\mathrm{H}}). \tag{11}$$

then the above equation turns out

$$f_{\mathcal{MN}}^{\diamond}(\mathcal{A}) = \mathcal{U} * \widehat{f_{\mathcal{MN}}}(\mathcal{S}) * \mathcal{V}^H.$$

□

The following corollary could be easily obtained from Theorem 3.8.

Corollary 3.9. *Suppose $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$ with $(\mathcal{M},\mathcal{N})$ weighted coefficient tensors. Let $f : \mathbb{C} \to \mathbb{C}$ be a scalar function, $f^{\diamond}_{\mathcal{MN}} : \mathbb{C}^{p_1\times p_2\times p_3} \to \mathbb{C}^{p_1\times p_2\times p_3}$ is the corresponding T-WGTF, then*
(1) $[f^{\diamond}_{\mathcal{MN}}(\mathcal{A})]^H = f^{\diamond}_{\mathcal{MN}}(\mathcal{A}^H)$ *and* $[f^{\diamond}_{\mathcal{MN}}(\mathcal{A})]^{\#} = f^{\diamond}_{\mathcal{MN}}(\mathcal{A}^{\#})$*,*
(2) $f^{\diamond}_{\mathcal{MN}}(\mathcal{P} * \mathcal{A} * \mathcal{Q}) = \mathcal{P} * f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) * \mathcal{Q}$*, where* $\mathcal{P} \in \mathbb{C}^{p_1\times p_1\times p_3}$ *and* $\mathcal{Q} \in \mathbb{C}^{p_2\times p_2\times p_3}$ *are unitary tensors.*

Since the T-MN-CSVD, the T-WGTF of tensor could be derived as followed without proof.

Theorem 3.10. *Suppose $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$ with $(\mathcal{M},\mathcal{N})$ weighted coefficient tensors and the T-MN-CSVD of $\mathcal{A}$ is $\mathcal{A} = \mathcal{U}_r * \mathcal{S}_r * \mathcal{V}_r^H$. If $f : \mathbb{C} \to \mathbb{C}$, then $f^{\diamond}_{\mathcal{MN}} : \mathbb{C}^{p_1\times p_2\times p_3} \to \mathbb{C}^{p_1\times p_2\times p_3}$ is defined by*

$$f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) = \mathcal{U}_r * \widehat{f_{\mathcal{MN}}}(\mathcal{S}_r) * \mathcal{V}_r^H,$$

where "$\widehat{f_{\mathcal{MN}}}$" is given in Theorem 3.8.

According to the weighted projection tensor $\mathcal{E}$, the following results could be obtained.

Corollary 3.11. *Let $f, g, h : \mathbb{C} \to \mathbb{C}$, and $f^{\diamond}_{\mathcal{MN}}, g^{\diamond}_{\mathcal{MN}}, h^{\diamond}_{\mathcal{MN}} : \mathbb{C}^{p_1\times p_2\times p_3} \to \mathbb{C}^{p_1\times p_2\times p_3}$ are induced T-WGTF. Suppose $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$ with $(\mathcal{M},\mathcal{N})$ weighted coefficient tensors and the T-MN-CSVD of $\mathcal{A}$ is $\mathcal{A} = \mathcal{U}_r * \mathcal{S}_r * \mathcal{V}_r^H$,*
(1) *If $f(z) = k$, then $f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) = k\mathcal{E}$,*
(2) *If $f(z) = z$, then $f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) = \mathcal{A}$,*
(3) *If $f(z) = g(z) + h(z)$, then $f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) = g^{\diamond}_{\mathcal{MN}}(\mathcal{A}) + h^{\diamond}_{\mathcal{MN}}(\mathcal{A})$,*
(4) *If $f(z) = g(z)h(z)$, then $f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) = g^{\diamond}_{\mathcal{MN}}(\mathcal{A}) * \mathcal{E}^{\#} * h^{\diamond}_{\mathcal{MN}}(\mathcal{A})$.*

For the non-zero weighted singular values of tensor, there may be some same c^{i}_{j}s. The same c^{i}_{j}s will not be distinguished in the following theorem, we make different $c^i_j = \gamma^l_k$, that is, γ^l_k is differ from one another, where $1 \le k \le j$ and $1 \le l \le i$.

Next, we need the function $f : \mathbb{C} \to \mathbb{C}$ to content the conditions as below. Suppose each Γ^l_k is a Jordan curve and f is analytic on and inside Γ^l_k, then
(1) $f(c^i_j) = 0$ if $c^i_j = 0$,
(2) Each Γ^l_k only contain one γ^l_k, and there is no other $\gamma^{l'}_{k'}$ on or inside Γ^l_k.

Besides, we suppose $\overline{\mathcal{E}}$ as

$$\overline{\mathcal{E}} = \sum_{c^l_k=\gamma^l_k} \mathcal{E}^l_k.$$

These assumptions lead to the following result.

Theorem 3.12. *Suppose $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$ is represented by*

$$\mathcal{A} = \sum_{i,j} c^i_j \mathcal{E}^i_j.$$

Suppose Γ^l_k is needed to make above conditions hold.
(1) *The connection of $\overline{\mathcal{E}^l_k}$ and Γ^l_k is*

$$\overline{\mathcal{E}^l_k}^{\#} = \frac{1}{2\pi i}\int_{\Gamma^l_k} (z\mathcal{E} - \mathcal{A})^{\dagger}_{\mathcal{MN}} \mathrm{d}z,$$

where the complex-value z stands for the integral variable of function f on the contour Γ_k^l.
(2) *Suppose the scalar function* $f : \mathbb{C} \to \mathbb{C}$ *is analytic in a domain which containing the set* $\Gamma = \bigcup_{k,l} \Gamma_k^l$, *then*

$$\sum_{i,j} f(c_j^i)\mathcal{E}_k^{l^\#} = \frac{1}{2\pi i}\int_\Gamma f(z)(z\mathcal{E} - \mathcal{A})^\dagger_{\mathcal{M}\mathcal{N}}\mathrm{d}z, \tag{12}$$

In particular, if $c_j^i \neq 0$, *then*

$$\mathcal{A}^\dagger_{\mathcal{M}\mathcal{N}} = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z}(z\mathcal{E} - \mathcal{A})^\dagger_{\mathcal{M}\mathcal{N}}\mathrm{d}z.$$

Proof. (1) From (9) and (10),

$$z\mathcal{E} - \mathcal{A} = \sum_{i,j}(z - c_j^i)\mathcal{E}_j^i, \tag{13}$$

Then,

$$(z\mathcal{E} - \mathcal{A})^\dagger_{\mathcal{M}\mathcal{N}} = \sum_{i,j}\frac{1}{z - c_j^i}\mathcal{E}_j^{i^\#}.$$

By (8) and the assumptions on Γ_k^l, we get

$$\frac{1}{2\pi i}\int_{\Gamma_k^l}(z\mathcal{E} - \mathcal{A})^\dagger_{\mathcal{M}\mathcal{N}}\mathrm{d}z = \frac{1}{2\pi i}\int_{\Gamma_k^l}\sum_{i,j}\frac{1}{z - c_j^i}\mathcal{E}_j^{i^\#}\mathrm{d}z = \sum_{c_k^l = \gamma_k^l}\mathcal{E}_j^{i^\#} = \overline{\mathcal{E}_k^l}^\#.$$

(2) Similarly,

$$\frac{1}{2\pi i}\int_\Gamma f(z)(z\mathcal{E} - \mathcal{A})^\dagger_{\mathcal{M}\mathcal{N}}\mathrm{d}z = \sum_{i,j}\sum_{k,l}\frac{1}{2\pi i}\left(\int_{\Gamma_k^l}\frac{f(z)}{z - c_j^i}\mathrm{d}z\right)\mathcal{E}_j^{i^\#} = \sum_{i,j} f(c_j^i)\mathcal{E}_j^{i^\#}.$$

Finally, $\mathcal{A}^\dagger_{\mathcal{M}\mathcal{N}}$ could be obtained immediately since the above result and $f(z) = \frac{1}{z}$,

$$\mathcal{A}^\dagger_{\mathcal{M}\mathcal{N}} = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z}(z\mathcal{E} - \mathcal{A})^\dagger_{\mathcal{M}\mathcal{N}}\mathrm{d}z = \sum_{i,j}\frac{1}{c_j^i}\mathcal{E}_j^{i^\#}. \tag{14}$$

□

If $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$, *the following example is given to describe the process of solving* $\mathcal{A}^\dagger_{\mathcal{M}\mathcal{N}} \in \mathbb{C}^{p_2\times p_1\times p_3}$.

Example 3.13. *Let* $\mathcal{A} \in \mathbb{C}^{3\times 2\times 3}$, $\mathcal{M} \in \mathbb{C}^{3\times 3\times 3}$ *and* $\mathcal{N} \in \mathbb{C}^{2\times 2\times 3}$ *have the following forms,*

$$A^{(1)} = \begin{pmatrix} 2 & -3 \\ 0.1i & 3 \\ 1 & -1 \end{pmatrix}, A^{(2)} = \begin{pmatrix} 3 & 0.03i \\ 5 & -2 \\ 0.1i & 4 \end{pmatrix}, A^{(3)} = \begin{pmatrix} 3 & 2 \\ -1 & 5 \\ 3 & 4 \end{pmatrix},$$

$$M^{(1)} = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 5 & 3 \\ -2 & 3 & 6 \end{pmatrix}, M^{(2)} = M^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, N^{(1)} = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}, N^{(2)} = N^{(3)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

by the T-MN-SVD of $\mathcal{A}$, the weighted T-singular values are $c_1^1 = 20.0826$, $c_2^1 = 5.0818$, $c_1^2 = 9.9738$, $c_2^2 = 3.5559$, $c_1^3 = 9.9738$, $c_2^3 = 3.5559$, and the weighted partial isometry tensors $\mathcal{E}_j^i$ of $\mathcal{A}$ have the following forms, $j = 1, 2$, $i = 1, 2, 3$,

$$E_1^{1^{(1)}} = E_1^{1^{(2)}} = E_1^{1^{(3)}} = \begin{pmatrix} 0.0277 + 0.0004i & 0.0469 - 0.0006i \\ 0.0608 + 0.0016i & 0.1029 - 0.0000i \\ 0.0681 + 0.0018i & 0.1152 - 0.0001i \end{pmatrix},$$

$$E_2^{1^{(1)}} = E_2^{1^{(2)}} = E_2^{1^{(3)}} = \begin{pmatrix} 0.4153 - 0.0015i & -0.2508 + 0.0045i \\ 0.0220 + 0.0000i & -0.0133 + 0.0002i \\ -0.0066 - 0.0006i & 0.0040 + 0.0003i \end{pmatrix},$$

$$E_1^{2^{(1)}} = \begin{pmatrix} 0.0114 - 0.0717i & -0.1087 + 0.0726i \\ -0.0858 - 0.0676i & 0.0155 + 0.1960i \\ 0.0379 - 0.0555i & -0.1197 + 0.0177i \end{pmatrix}, E_1^{2^{(2)}} = \begin{pmatrix} 0.0564 + 0.0457i & -0.0086 - 0.1304i \\ 0.1014 - 0.0405i & -0.1775 - 0.0846i \\ 0.0291 + 0.0606i & 0.0445 - 0.1125i \end{pmatrix},$$

$$E_1^{2^{(3)}} = \begin{pmatrix} -0.0678 + 0.0260i & 0.1172 + 0.0578i \\ -0.0156 + 0.1081i & 0.1620 - 0.1114i \\ -0.0670 - 0.0051i & 0.0752 + 0.0948i \end{pmatrix}, E_2^{2^{(1)}} = \begin{pmatrix} -0.1256 + 0.2012i & -0.0677 - 0.0428i \\ 0.0532 - 0.2881i & 0.0972 + 0.0185i \\ -0.1451 + 0.3946i & -0.1329 - 0.0497i \end{pmatrix},$$

$$E_2^{2^{(2)}} = \begin{pmatrix} -0.1114 - 0.2093i & 0.0709 - 0.0372i \\ 0.2229 + 0.1901i & -0.0646 + 0.0749i \\ -0.2692 - 0.3229i & 0.1095 - 0.0903i \end{pmatrix}, E_2^{2^{(3)}} = \begin{pmatrix} 0.2370 + 0.0082i & -0.0032 + 0.0800i \\ -0.2761 + 0.0980i & -0.0326 - 0.0934i \\ 0.4142 - 0.0716i & 0.0234 + 0.1400i \end{pmatrix},$$

$$E_1^{3^{(1)}} = \begin{pmatrix} 0.0114 + 0.0717i & -0.1087 - 0.0726i \\ -0.0858 + 0.0676i & 0.0155 - 0.1960i \\ 0.0379 + 0.0555i & -0.1197 - 0.0177i \end{pmatrix}, E_1^{3^{(2)}} = \begin{pmatrix} 0.0564 - 0.0457i & -0.0086 + 0.1304i \\ 0.1014 + 0.0405i & -0.1775 + 0.0846i \\ 0.0291 - 0.0606i & 0.0445 + 0.1125i \end{pmatrix},$$

$$E_1^{3^{(3)}} = \begin{pmatrix} -0.0678 - 0.0260i & 0.1172 - 0.0578i \\ -0.0156 - 0.1081i & 0.1620 + 0.1114i \\ -0.0670 + 0.0051i & 0.0752 - 0.0948i \end{pmatrix}, E_2^{3^{(1)}} = \begin{pmatrix} -0.1256 - 0.2012i & -0.0677 + 0.0428i \\ 0.0532 + 0.2881i & 0.0972 - 0.0185i \\ -0.1451 - 0.3946i & -0.1329 + 0.0497i \end{pmatrix},$$

$$E_2^{3^{(2)}} = \begin{pmatrix} -0.1114 + 0.2093i & 0.0709 + 0.0372i \\ 0.2229 - 0.1901i & -0.0646 - 0.0749i \\ -0.2692 + 0.3229i & 0.1095 + 0.0903i \end{pmatrix}, E_2^{3^{(3)}} = \begin{pmatrix} 0.2370 - 0.0082i & -0.0032 - 0.0800i \\ -0.2761 - 0.0980i & -0.0326 + 0.0934i \\ 0.4142 + 0.0716i & 0.0234 - 0.1400i \end{pmatrix}.$$

By (14), we get $\mathcal{A}^{\dagger}_{MN}$ has the following frontal slices,

$$A^{\dagger^{(1)}}_{MN} = \begin{pmatrix} 0.0556 - 0.0000i & -0.0349 - 0.0001i & -0.1237 - 0.0001i \\ 0.0289 - 0.0005i & -0.0391 - 0.0004i & -0.0942 + 0.0004i \end{pmatrix},$$

$$A^{\dagger^{(2)}}_{MN} = \begin{pmatrix} -0.0609 - 0.0000i & 0.0454 - 0.0001i & 0.1566 - 0.0001i \\ -0.0548 - 0.0005i & 0.0693 - 0.0004i & 0.1553 + 0.0004i \end{pmatrix},$$

$$A^{\dagger^{(3)}}_{MN} = \begin{pmatrix} 0.1154 - 0.0000i & 0.0887 - 0.0001i & -0.1021 - 0.0001i \\ -0.0419 - 0.0005i & -0.0120 - 0.0004i & 0.0565 + 0.0004i \end{pmatrix}.$$

The Cauchy integral formula for the T-WGTF in following corollary is developed for solving tensor equation.

Corollary 3.14. *Let $\mathcal{A}$, $\mathcal{E}$, Γ and f have the forms in Theorem 3.12, if $f^{\lozenge}_{MN} : \mathbb{C}^{p_1 \times p_2 \times p_3} \to \mathbb{C}^{p_1 \times p_2 \times p_3}$, then*

$$f^{\lozenge}_{\mathcal{MN}}(\mathcal{A}) = \mathcal{E} * (\frac{1}{2\pi i}(\int_{\Gamma} f(z)(z\mathcal{E} - \mathcal{A})^{\dagger}_{\mathcal{MN}} \mathrm{d}z) * \mathcal{E}. \tag{15}$$

Proof. It follows from (9), (10) and (12) that (15) holds. □

Theorem 3.15. *The weighted generalized tensor resolvent of $\mathcal{A}$ is denoted by $\widehat{R}(z, \mathcal{A})$ and defined as*

$$\widehat{R}(z, \mathcal{A}) = (z\mathcal{E} - \mathcal{A})^{\dagger}_{\mathcal{MN}},$$

then for any $\lambda, \mu \neq c^i_j$, we have

$$\widehat{R}(\lambda, \mathcal{A}) - \widehat{R}(\mu, \mathcal{A}) = (\mu - \lambda)\widehat{R}(\lambda, \mathcal{A}) * \mathcal{E} * \widehat{R}(\mu, \mathcal{A}). \tag{16}$$

Proof. Since

$$(z\mathcal{E}-\mathcal{A})^{\dagger}_{\mathcal{MN}} = \sum_{i,j}\frac{1}{z-c^i_j}\mathcal{E}^{i^{\#}}_j$$

that the left-hand side of (16) comes to

$$\begin{aligned}\widehat{R}(\lambda,\mathcal{A})-\widehat{R}(\mu,\mathcal{A}) &= \sum_{i,j}(\frac{1}{\lambda-c^i_j}-\frac{1}{\mu-c^i_j})\mathcal{E}^{i^{\#}}_j\\ &= (\mu-\lambda)(\sum_{i,j}\frac{1}{\lambda-c^i_j}\mathcal{E}^{i^{\#}}_j)*\mathcal{E}*(\sum_{k,l}\frac{1}{\mu-c^k_l}\mathcal{E}^{k^{\#}}_l)\\ &= (\mu-\lambda)\widehat{R}(\lambda,\mathcal{A})*\mathcal{E}*\widehat{R}(\mu,\mathcal{A}).\end{aligned}$$

□

The results in Corollary 3.14 and Theorem 3.15 are used to solve the following tensor equation,

$$\mathcal{A}*\mathcal{X}*\mathcal{B}=\mathcal{D}. \tag{17}$$

Here $\mathcal{A}\in\mathbb{C}^{p_1\times p_2\times p_3}$ with $(\mathcal{M},\mathcal{N})$ weighted coefficient tensors, $\mathcal{B}\in\mathbb{C}^{k_1\times k_2\times p_3}$ with $(\mathcal{P},\mathcal{Q})$ weighted coefficient tensors and $\mathcal{D}\in\mathbb{C}^{p_1\times k_2\times p_3}$. The T-MN-Spectral decomposition and the relative weighted partial isometry tensors of $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\mathcal{A}=\sum_{i,j=1}^{p_3,r^{\mathcal{A}}}c^{i^{\mathcal{A}}}_j\mathcal{E}^{i^{\mathcal{A}}}_j,\mathcal{E}^{\mathcal{A}}=\sum_{i,j=1}^{p_3,r^{\mathcal{A}}}\mathcal{E}^{i^{\mathcal{A}}}_j,\mathcal{B}=\sum_{i,j=1}^{p_3,r^{\mathcal{B}}}c^{i^{\mathcal{B}}}_j\mathcal{E}^{i^{\mathcal{A}}}_j,\mathcal{E}^{\mathcal{B}}=\sum_{i,j=1}^{p_3,r^{\mathcal{B}}}\mathcal{E}^{i^{\mathcal{B}}}_j,$$

where $r^{\mathcal{A}}$ =rank$_t(\mathcal{A})$ and $r^{\mathcal{B}}$ =rank$_t(\mathcal{B})$.

Theorem 3.16. *If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ have the above forms, the curve Γ_1 surrounds $c(\mathcal{A})=\{c^{i^{\mathcal{A}}}_j,\ i=1,2,\cdots,p_3,\ j=1,2,\cdots,r^{\mathcal{A}}\}$and the curve Γ_2 surrounds $c(\mathcal{B})=\{c^{i^{\mathcal{B}}}_j,\ i=1,2,\cdots,p_3,\ j=1,2,\cdots,r^{\mathcal{B}}\}$. Then the solution of* (17) *is*

$$X=-\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2}\frac{\widehat{R}_{\mathcal{MN}}(\lambda,\mathcal{A})*\mathcal{D}*\widehat{R}_{\mathcal{PQ}}(\mu,\mathcal{B})}{\lambda\mu}\mathrm{d}\mu\mathrm{d}\lambda.$$

Proof. It follows from (15) that

$$\mathcal{A}=\mathcal{E}^{\mathcal{A}}*(\frac{1}{2\pi i}\int_{\Gamma_1}\lambda\widehat{R}_{\mathcal{MN}}(\lambda,\mathcal{A})\mathrm{d}\lambda)*\mathcal{E}^{\mathcal{A}},$$

$$\mathcal{B}=\mathcal{E}^{\mathcal{B}}*(\frac{1}{2\pi i}\int_{\Gamma_2}\mu\widehat{R}_{\mathcal{PQ}}(\mu,\mathcal{B})\mathrm{d}\mu)*\mathcal{E}^{\mathcal{B}}.$$

Therefore,

$$\begin{aligned}\mathcal{A}*\mathcal{X}*\mathcal{B} &= \mathcal{E}^{\mathcal{A}}*(\frac{1}{2\pi i}\int_{\Gamma_1}\widehat{R}_{\mathcal{MN}}(\lambda,\mathcal{A})\mathrm{d}\lambda)*\mathcal{D}*(\frac{1}{2\pi i}\int_{\Gamma_2}\widehat{R}_{\mathcal{PQ}}(\mu,\mathcal{B})\mathrm{d}\mu)*\mathcal{E}^{\mathcal{B}}\\ &= \mathcal{E}^{\mathcal{A}}*(\mathcal{E}^{\mathcal{A}})^{\#}*\mathcal{D}*(\mathcal{E}^{\mathcal{B}})^{\#}*(\mathcal{E}^{\mathcal{B}})=\mathcal{A}*\mathcal{A}^{\dagger}_{\mathcal{MN}}*\mathcal{D}*\mathcal{B}^{\dagger}_{\mathcal{PQ}}*\mathcal{B},\end{aligned}$$

while $\mathcal{A}*\mathcal{A}^{\dagger}_{\mathcal{MN}}*\mathcal{D}*\mathcal{B}^{\dagger}_{\mathcal{PQ}}*\mathcal{B}=\mathcal{D}$ is equal to the solution of (17) is existent, which means that a solution of (17) is $\mathcal{X}=\mathcal{A}^{\dagger}_{\mathcal{MN}}*\mathcal{D}*\mathcal{B}^{\dagger}_{\mathcal{PQ}}$ if (17) holds. □

Next, we give the following example for solving the equation (17).

Example 3.17. *Let $\mathcal{A}$ with the weighted coefficient tensors $\mathcal{M}$ and $\mathcal{N}$ have the following forms,*

$$A^{(1)} = \begin{pmatrix} 3 & -0.2i \\ 1 & 2 \\ -1 & 4 \end{pmatrix}, A^{(2)} = \begin{pmatrix} 0.1i & 2 \\ -1 & 3 \\ 5 & -2 \end{pmatrix}, A^{(3)} = \begin{pmatrix} 1 & 2 \\ -3 & 2 \\ -1 & 0 \end{pmatrix},$$

$$M^{(1)} = \begin{pmatrix} 4 & 2 & -1 \\ 2 & 5 & -1 \\ -1 & -1 & 6 \end{pmatrix}, M^{(2)} = M^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, N^{(1)} = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}, N^{(2)} = N^{(3)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The frontal slices of $\mathcal{B}$ with the weighted coefficient tensors $\mathcal{P}$ and $\mathcal{Q}$ are as follows,

$$B^{(1)} = \begin{pmatrix} 2 & 0.1i \\ -3 & 5 \\ 2 & 1 \end{pmatrix}, B^{(2)} = \begin{pmatrix} 3 & 0 \\ -2 & 1 \\ 0.2i & 5 \end{pmatrix}, B^{(3)} = \begin{pmatrix} 5 & -2 \\ -1 & 2 \\ 3 & 0 \end{pmatrix},$$

$$P^{(1)} = \begin{pmatrix} 3 & 2 & -0.01i \\ 2 & 4 & 2 \\ -0.01i & 2 & 7 \end{pmatrix}, P^{(2)} = P^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q^{(1)} = \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}, Q^{(2)} = Q^{(3)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Considering the existence of the solution of (17)*, we suppose the frontal slices of $\mathcal{D}$ as follows,*

$$D^{(1)} = \begin{pmatrix} 0.0776 + 0.0146i & -0.0526 + 0.0040i \\ -0.6927 - 0.0068i & 0.5402 + 0.0209i \\ 0.1308 + 0.0073i & 0.2673 + 0.0069i \end{pmatrix},$$

$$D^{(2)} = \begin{pmatrix} 0.3265 + 0.0146i & 0.1078 + 0.0040i \\ -0.0431 - 0.0068i & 0.2579 + 0.0209i \\ 0.0991 + 0.0073i & -0.4260 + 0.0069i \end{pmatrix},$$

$$D^{(3)} = \begin{pmatrix} 0.3449 + 0.0146i & -0.0265 + 0.0040i \\ 0.3204 - 0.0068i & 0.1381 + 0.0209i \\ 0.3176 + 0.0073i & 0.0833 + 0.0069i \end{pmatrix}.$$

By Theorem 3.16, the solution of (17) is that $\mathcal{X} = \mathcal{A}^{\dagger}_{\mathcal{MN}} * \mathcal{D} * \mathcal{B}^{\dagger}_{\mathcal{PQ}}$. Thus, the solution $\mathcal{X} \in \mathbb{C}^{2\times 3\times 3}$ has the following frontal slices,

$$X^{(1)} = \begin{pmatrix} 0.0168 + 0.0002i & -0.0072 + 0.0000i & 0.0243 - 0.0002i \\ -0.0107 - 0.0000i & -0.0174 + 0.0002i & -0.0082 + 0.0002i \end{pmatrix},$$

$$X^{(2)} = \begin{pmatrix} -0.0342 + 0.0003i & -0.0374 + 0.0000i & -0.0229 + 0.0001i \\ -0.0134 - 0.0000i & -0.0046 + 0.0002i & -0.0156 + 0.0003i \end{pmatrix},$$

$$X^{(3)} = \begin{pmatrix} 0.0296 + 0.0003i & 0.0363 + 0.0000i & -0.0010 - 0.0001i \\ 0.0251 - 0.0001i & 0.0281 + 0.0002i & 0.0320 + 0.0002i \end{pmatrix}.$$

3.4. Weighted Generalized Power of Tensor

The definition of the weighted generalized power of the tensor is given by the projection tensor $\mathcal{E}$.

Definition 3.18. *Suppose $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$ with $(\mathcal{M}, \mathcal{N})$ weighted coefficient tensors and the T-MN-CSVD of $\mathcal{A}$ is $\mathcal{A} = \mathcal{U}_r * \mathcal{S}_r * \mathcal{V}_r^H$. The weighted generalized power $\mathcal{A}^{(k)}$ of $\mathcal{A}$ can be factorized as,*

$$\mathcal{A}^{(k)} = \mathcal{A}^{(k-1)} * \mathcal{E}^{\#} * \mathcal{A}, k \geq 1,$$

$$\mathcal{A}^{(k)} = \mathcal{A}^{(k+1)} * \mathcal{E}^{\#} * \mathcal{A}^{(-1)}, k < 1,$$

where

$$\mathcal{A}^{(0)} = \mathcal{E} = \mathcal{U}_r * \mathcal{V}_r^H, \mathcal{A}^{(-1)} = \mathcal{U}_r * \mathcal{S}_r^{\dagger} * \mathcal{V}_r^H.$$

Here are expressions of the weighted generalized odd power and even power of tensor which obtained directly from Definition 3.18.

Corollary 3.19. *Suppose $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$ with $(\mathcal{M}, \mathcal{N})$ coefficient tensors, then*

$$\mathcal{A}^{(2k+1)} = (\mathcal{A} * \mathcal{A}^{\#})^k * \mathcal{A},$$

$$\mathcal{A}^{(2k)} = (\mathcal{A} * \mathcal{A}^{\#})^k * \mathcal{E}.$$

The Taylor expansion of the T-WGTF which induced by $f : \mathbb{C} \to \mathbb{C}$ could be obtained as follows.

Theorem 3.20. *Suppose $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$ with $(\mathcal{M}, \mathcal{N})$ coefficient tensors, the complex-valued function $f : \mathbb{C} \to \mathbb{C}$ is introduced by*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

for $|z - z_0| < R$. Then the T-WGTF $f^{\diamond}_{\mathcal{MN}} : \mathbb{C}^{p_1 \times p_2 \times p_3} \to \mathbb{C}^{p_1 \times p_2 \times p_3}$ is

$$f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(\mathcal{A} - z_0\mathcal{E})^k$$

for $|c^i_j - z_0| < R$, $i = 1, 2, \cdots, p_3$, $j = 1, 2, \cdots, r_i$.

Proof. By Definition 3.18 and the T-MN-CSVD, we get

$$(\mathcal{A} - z_0\mathcal{E})^k = \mathcal{U}_r * (\mathcal{S}_r - z_0\mathcal{I})^k * \mathcal{V}^H_r, k = 0, 1, \cdots$$

For $n = 0, 1, \cdots$, we define

$$\begin{aligned} f^{\diamond}_{\mathcal{MN}_n}(\mathcal{A}) &= \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(\mathcal{A} - z_0\mathcal{E})^k \\ &= \mathcal{U}_r * \left(\sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(\mathcal{S}_r - z_0\mathcal{I})^k \right) * \mathcal{V}^H_r, \end{aligned}$$

such that

$$\|f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) - f^{\diamond}_{\mathcal{MN}_n}(\mathcal{A})\| \leq \|\mathcal{U}_r\| \| \sum_{k=n+1}^{\infty} \frac{f^{(k)}(z_0)}{k!}(\mathcal{S}_r - z_0\mathcal{I})^k \| \|\mathcal{V}^H_r\|,$$

According to Definition 2.8, then

$$\|f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) - f^{\diamond}_{\mathcal{MN}_n}(\mathcal{A})\| \to 0, (n \to \infty).$$

□

Example 3.21. *Let $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$ with $(\mathcal{M}, \mathcal{N})$ coefficient tensors, $f : \mathbb{C} \to \mathbb{C}$ is given by*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(z)^k$$

for $|z| < R$. The functions $f_1(z)$ and $f_2(z)$ are defined as

$$f_1(z) = \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!}(z)^k, f_2(z) = \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!}(z)^k.$$

According to Corollary 3.19 and Theorem 3.20, the related T-WGTF $f^{\diamond}_{\mathcal{MN}} : \mathbb{C}^{p_1\times p_2\times p_3} \to \mathbb{C}^{p_1\times p_2\times p_3}$ comes to

$$f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) = f^{\diamond}_{1_{\mathcal{MN}}}(\mathcal{A} * \mathcal{A}^{\#}) * \mathcal{E} + f^{\diamond}_{2_{\mathcal{MN}}}(\mathcal{A} * \mathcal{A}^{\#}) * \mathcal{A}$$

for $|c^i_j| < R$. From Maclaurin formulas of complex functions, the Maclaurin formulas of the exponential, logarithmic and trigonometric T-WGTFs are induced to

$$\begin{aligned}
exp^{\diamond}_{\mathcal{MN}}(\mathcal{A}) &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}(\mathcal{A} * \mathcal{A}^{\#})^k * \mathcal{E} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(\mathcal{A} * \mathcal{A}^{\#})^k * \mathcal{A},\\
ln^{\diamond}_{\mathcal{MN}}(\mathcal{I} + \mathcal{A}) &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(\mathcal{A} * \mathcal{A}^{\#})^k * \mathcal{A} - \sum_{k=0}^{\infty} \frac{1}{(2k)!}(\mathcal{A} * \mathcal{A}^{\#})^k * \mathcal{E},\\
sin^{\diamond}_{\mathcal{MN}}(\mathcal{A}) &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!}(\mathcal{A} * \mathcal{A}^{\#})^k * \mathcal{A},\\
cos^{\diamond}_{\mathcal{MN}}(\mathcal{A}) &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!}(\mathcal{A} * \mathcal{A}^{\#})^k * \mathcal{E},\\
sinh^{\diamond}_{\mathcal{MN}}(\mathcal{A}) &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(\mathcal{A} * \mathcal{A}^{\#})^k * \mathcal{A},\\
cosh^{\diamond}_{\mathcal{MN}}(\mathcal{A}) &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}(\mathcal{A} * \mathcal{A}^{\#})^k * \mathcal{E}.
\end{aligned}$$

If $\mathcal{A} \in \mathbb{C}^{p_1\times p_1\times p_3}$ or $\mathcal{A} \in \mathbb{C}^{p_2\times p_2\times p_3}$, we call it the F-square tensor. The conclusion of applying the T-WGTF to an F-square block tensor as follows.

Remark 3.22. *Let $\mathcal{A} \in \mathbb{C}^{p_1\times p_2\times p_3}$ with $(\mathcal{M}, \mathcal{N})$ coefficient tensors and*

$$\mathcal{B} = \begin{pmatrix} \mathcal{O} & \mathcal{A} \\ \mathcal{A}^{\#} & \mathcal{O} \end{pmatrix}.$$

Assume that $f : \mathbb{C} \to \mathbb{C}$ is an odd function, the T-WGTF could be expressed as

$$f_{(\mathcal{M}+\mathcal{N})(\mathcal{M}+\mathcal{N})}(\mathcal{B}) = \begin{pmatrix} \mathcal{O} & f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) \\ f^{\diamond}_{\mathcal{MN}}(\mathcal{A}^{\#}) & \mathcal{O} \end{pmatrix}.$$

Actually, since Lemma 2.15, $\mathcal{B}$ is factorized as

$$\mathcal{B} = \frac{1}{2}\begin{pmatrix} \mathcal{I}_{p_1p_1p_3} & \mathcal{O} \\ \mathcal{O} & \mathcal{N}^{-1} \end{pmatrix} * \begin{pmatrix} \mathcal{U}_r & -\mathcal{U}_r \\ \mathcal{V}_r & \mathcal{V}_r \end{pmatrix} * \begin{pmatrix} \mathcal{S}_r & \mathcal{O} \\ \mathcal{O} & -\mathcal{S}_r \end{pmatrix} * \begin{pmatrix} \mathcal{U}_r^H & -\mathcal{V}_r^H \\ -\mathcal{U}_r^H & \mathcal{V}_r^H \end{pmatrix} * \begin{pmatrix} \mathcal{M} & \mathcal{O} \\ \mathcal{O} & \mathcal{I}_{p_2p_2p_3} \end{pmatrix},$$

then

$$\begin{aligned}
&2f_{(\mathcal{M}+\mathcal{N})(\mathcal{M}+\mathcal{N})}(\mathcal{B})\\
&= \begin{pmatrix} \mathcal{I}_{p_1p_1p_3} & \mathcal{O} \\ \mathcal{O} & \mathcal{N}^{-1} \end{pmatrix} * \begin{pmatrix} \mathcal{U}_r & -\mathcal{U}_r \\ \mathcal{V}_r & \mathcal{V}_r \end{pmatrix} * \begin{pmatrix} f_{\mathcal{MN}}(\mathcal{S}_r) & \mathcal{O} \\ \mathcal{O} & -f_{\mathcal{MN}}(\mathcal{S}_r) \end{pmatrix} * \begin{pmatrix} \mathcal{U}_r^H & -\mathcal{V}_r^H \\ -\mathcal{U}_r^H & \mathcal{V}_r^H \end{pmatrix} * \begin{pmatrix} \mathcal{M} & \mathcal{O} \\ \mathcal{O} & \mathcal{I}_{p_2p_2p_3} \end{pmatrix}\\
&= \begin{pmatrix} \mathcal{O} & 2\mathcal{U}_r * f_{\mathcal{MN}}(\mathcal{S}_r) * \mathcal{V}_r^H \\ 2\mathcal{N}^{-1} * \mathcal{V}_r * f_{\mathcal{MN}}(\mathcal{S}_r) * \mathcal{U}_r^H * \mathcal{M} & \mathcal{O} \end{pmatrix},
\end{aligned}$$

which equals to

$$f_{(\mathcal{M}+\mathcal{N})(\mathcal{M}+\mathcal{N})}(\mathcal{B}) = \begin{pmatrix} \mathcal{O} & f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) \\ f^{\diamond}_{\mathcal{MN}}(\mathcal{A}^{\#}) & \mathcal{O} \end{pmatrix}.$$

If a complex-value function can be expanded as $f(z) = \sum_{k=0}^{\infty} a_k z^k$, we denote the odd part f_{odd} and even part f_{even} as following forms,

$$f_{odd}(z) = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1}, \; f_{even}(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k}.$$

Therefore, for $\mathcal{B}$ in Remark 3.22,

$$f_{(\mathcal{M}+\mathcal{N})(\mathcal{M}+\mathcal{N})}(\mathcal{B}) = \begin{pmatrix} f_{\mathcal{M}\mathcal{N}_{even}}(\sqrt{\mathcal{A} * \mathcal{A}^{\#}}) & f^{\Diamond}_{\mathcal{M}\mathcal{N}odd}(\mathcal{A}) \\ f^{\Diamond}_{\mathcal{M}\mathcal{N}odd}(\mathcal{A}) & f_{\mathcal{M}\mathcal{N}_{even}}(\sqrt{\mathcal{A}^{\#} * \mathcal{A}}) \end{pmatrix}.$$

Here is an example showed by the T-WGTF and Lemma 2.15.

Example 3.23. *Let $\mathcal{B} \in \mathbb{C}^{(p_1+p_2)\times(p_1+p_2)\times p_3}$ be as Remark 3.22, then the weighted exponential function $exp_{(\mathcal{M}+\mathcal{N})(\mathcal{M}+\mathcal{N})}(\mathcal{B})$ is denoted by*

$$\begin{aligned} exp_{(\mathcal{M}+\mathcal{N})(\mathcal{M}+\mathcal{N})}(\mathcal{B}) &= \begin{pmatrix} cosh(\sqrt{\mathcal{A} * \mathcal{A}^{\#}}) & sinh^{\Diamond}_{\mathcal{N}\mathcal{N}}(\sqrt{\mathcal{A}^{\#} * \mathcal{A}}) \\ sinh^{\Diamond}_{\mathcal{N}\mathcal{N}}(\sqrt{\mathcal{A}^{\#} * \mathcal{A}}) & cosh(\sqrt{\mathcal{A}^{\#} * \mathcal{A}}) \end{pmatrix} \\ &= \begin{pmatrix} cosh(\sqrt{\mathcal{A} * \mathcal{A}^{\#}}) & \mathcal{A} * (\sqrt{\mathcal{A}^{\#} * \mathcal{A}})^{\dagger}_{\mathcal{N}\mathcal{N}} * sinh(\sqrt{\mathcal{A}^{\#} * \mathcal{A}}) \\ sinh(\sqrt{\mathcal{A}^{\#} * \mathcal{A}}) * (\sqrt{\mathcal{A}^{\#} * \mathcal{A}})^{\dagger}_{\mathcal{N}\mathcal{N}} * \mathcal{A}^{\#} & cosh(\sqrt{\mathcal{A}^{\#} * \mathcal{A}}) \end{pmatrix}. \end{aligned}$$

4. Function Invariance

4.1. Weighted GMF Invariance

It is known that the studying of matrix properties which are invariant is more efficient under matrix functions at accurate algorithms, and the matrix properties preservation under GMF is provided in [3]. The weighted GMF are introduced in [38], but the structural preservation of matrix under the weighted GMF has not been mentioned. Before studying the invariance of structural properties of tensors under the T-WGTF, the related concepts and properties of matrices are given first.

Definition 4.1. [16] *$A \in \mathbb{C}^{p_1\times p_2}$ is centrohermitian (skew-centrohermitian) if $R_{p_1} A R_{p_2} = \overline{A}$ (respectively, $R_{p_1} A R_{p_2} = -\overline{A}$), where $R_{p_1} \in \mathbb{C}^{p_1\times p_1}$ and $R_{p_2} \in \mathbb{C}^{p_2\times p_2}$ are reverse matrices.*

Lemma 4.2. [2] *Let $A \in \mathbb{C}_r^{p_1\times p_2}$ with coefficient matrices M and N. The scalar function is $f : \mathbb{C} \to \mathbb{C}$ and the induced weighted GMF is $f_{MN} : \mathbb{C}^{p_1\times p_2} \to \mathbb{C}^{p_1\times p_2}$, then*
(1) $[f_{MN}(A)]^{\#} = f_{NM}(A^{\#})$,
(2) If $X \in \mathbb{C}^{p_1\times p_1}$ and $Y \in \mathbb{C}^{p_2\times p_2}$ are unitary, then $f_{MN}(XAY) = X f_{MN}(A) Y$,
(3) If $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, then $f_{MN}(A) = f_{MN}(A_1) \oplus f_{MN}(A_2) \oplus \cdots \oplus f_{MN}(A_k)$, where "$\oplus$" means the direct sum of matrices.

With the above lemma, the structure invariance under the weighted GMF has obtained as follows.

Lemma 4.3. *Let $A \in \mathbb{C}_r^{p_1\times p_2}$, M and N are coefficient matrices of A. The scalar function is $f : \mathbb{C} \to \mathbb{C}$ and $f_{MN} : \mathbb{C}^{p_1\times p_2} \to \mathbb{C}^{p_1\times p_2}$ is the induced weighted GMF.*
(1) If reverse matrices $R_{p_1} \in \mathbb{C}^{p_1\times p_1}$, $R_{p_2} \in \mathbb{C}^{p_2\times p_2}$ make $R_{p_1} A R_{p_2} = \overline{A}$ (or $R_{p_1} A R_{p_2} = -\overline{A}$) hold, then $R_{p_1} f_{MN}(A) R_{p_2} = \overline{f_{MN}(A)}$ (or $R_{p_1} f_{MN}(A) R_{p_2} = -\overline{f_{MN}(A)}$),
(2) If $AA^{\#} = A^{\#}A$, then $f_{MN}(A) f_{MN}(A)^{\#} = f_{MN}(A)^{\#} f_{MN}(A)$,
(3) If $A \in \mathbb{C}^{p_2\times p_2}$ is a circular matrix, then $f_{MN}(A)$ is also a circular matrix.

Proof. (1) Since A is centrohermitian, there exist unitary matrices R_{p_1} and R_{p_2} such that $R_{p_1}AR_{p_2} = \overline{A}$ hold. By Lemma 4.2, $R_{p_1}f_{MN}(A)R_{p_2} = f_{MN}(R_{p_1}AR_{p_2}) = f_{MN}(\overline{A}) = \overline{f_{MN}(A)}$. Similarly, if A is skew-centrohermitian, then $R_{p_1}AR_{p_2} = -\overline{A}$. Thus, $R_{p_1}f_{MN}(A)R_{p_2} = f_{MN}(R_{p_1}AR_{p_2}) = f_{MN}(-\overline{A}) = -f_{MN}(\overline{A}) = -\overline{f_{MN}(A)}$.

(2) According to the MN-SVD of A and $A^{\#}$, we get

$$AA^{\#} = U\Sigma V^H N^{-1} V \Sigma^H U^H M,$$

then

$$\begin{aligned} f_{MN}(AA^{\#}) &= f_{MN}(U\Sigma V^H N^{-1} V\Sigma^H U^H M) \\ &= Uf_{MN}(\Sigma)V^H N^{-1} V f_{MN}(\Sigma^H)U^H M = f_{MN}(A)f_{MN}(A^{\#}). \end{aligned}$$

Since $AA^{\#} = A^{\#}A$, $f_{MN}(A)f_{MN}(A)^{\#} = f_{MN}(A)^{\#}f_{MN}(A)$ holds directly.

(3) Note that circulant matrices are diagonalized by the DFT matrix, then A can be expressed as

$$A = F_{p_2}\Lambda F_{p_2}^H = F_{p_2}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{p_2} \end{pmatrix} F_{p_2}^H,$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{p_2} \geq 0$ are the eigenvalues of A. Note that F_{p_2} is unitary, then

$$f_{MN}(A) = F_{p_2}\begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_{p_2}) \end{pmatrix} F_{p_2}^H.$$

Hence, $f_{MN}(A)$ is also circulant. □

Definition 4.4. *Let $A \in \mathbb{R}^{p_1\times p_1}$, A is called the permutation matrix if the elements in each row and column contain the unique* 1*, and the others are* 0*.*

Lemma 4.5. *Let $A \in \mathbb{C}_r^{p_1\times p_2}$ with coefficient matrices M and N, $f : \mathbb{C} \to \mathbb{C}$ is a complex-value function, $f_{MN} : \mathbb{C}^{p_1\times p_2} \to \mathbb{C}^{p_1\times p_2}$ is weighted GMF.*
(1) If each element of a column(row) of A is 0*, then the corresponding column(row) of $f_{MN}(A)$ is also composed of* 0*,*
(2) If PAQ is block-diagonal, then $f_{MN}(A)$ is also block-diagonal, where $P \in \mathbb{R}^{p_1\times p_1}$ and $Q \in \mathbb{R}^{p_2\times p_2}$ are permutation matrices.

Proof. (1) We suppose the last column of A are zeros, for any permutation matrix $Q \in \mathbb{C}^{p_2\times p_2}$, $f_{MN}(AQ) = f_{MN}(A)Q$, denote $A = \begin{pmatrix} \widehat{A} & O \end{pmatrix}$, as $A = U_r\Sigma_r V_r^H$ and $\widehat{A} = \widehat{U_r}\widehat{\Sigma_r}\widehat{V_r^H}$, we have

$$A = \begin{pmatrix} \widehat{A} & O \end{pmatrix} = \widehat{U_r}\begin{pmatrix} \widehat{\Sigma_r} & O \end{pmatrix}\begin{pmatrix} \widehat{V_r^H} & O \\ O & 1 \end{pmatrix} = U_r\Sigma_r V_r^H,$$

where r =rank(A), thus, the elements of the last row of V_r^H are zeros, the result can be obtained.

(2) It is a straightforward result by Lemma 4.2(3). □

Lemma 4.6. *Let $A \in \mathbb{R}^{p_1\times p_2}$ be nonnegative, M and N are coefficient matrices of A, $f : \mathbb{R} \to \mathbb{R}$ is the odd part of a differentiable function and the Maclaurin expansion of f is $f(z) = \sum_{k=0}^{\infty} c_k z^k$ with $c_{2k+1} \geq 0$. Suppose $f(z)$ is convergent for $|z| < R$. Then $f_{MN}(A)$ is also nonnegative for $|\sigma_i - z| < R$, where σ_i is the weighted singular values of A, $i = 1, 2, ..., r$, r =* rank*(A).*

Proof. We assume that f is

$$f(x) = \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1} \quad and \quad c_{2k+1} \geq 0.$$

According to the MN-CSVD of A,

$$AA^{\#} = AN^{-1}A^H M = U_r \Sigma_r^2 U_r^{-1} \geq 0,$$

which means that $(AA^{\#})^k A = U_r \Sigma_r^{2k+1} V_r^H \geq 0$. Then

$$f_{MN}(A) = U_r f_{MN}(\Sigma_r) V_r^T = U_r \sum_{k=0}^{\infty} c_{2k+1} \Sigma_r^{2k+1} V_r^T = \sum_{k=0}^{\infty} c_{2k+1} U_r \Sigma_r^{2k+1} V_r^T \geq 0.$$

□

4.2. Tensor Function Invariance

With the study of structure invariance of the weighted GMF, the structure invariance of the T-WGTF of tensor could be obtained similarly in this section.

Definition 4.7. [23] *$\mathcal{R} \in \mathbb{C}^{p_1 \times p_1 \times p_3}$ is called the reverse tensor if*

$$\mathcal{R} = \text{fold}\begin{pmatrix} R_{p_1} \\ O \\ \vdots \\ O \end{pmatrix},$$

while $R_{p_1} \in \mathbb{C}^{p_1 \times p_1}$ is a reverse matrix.

Definition 4.8. [23] *Let $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$ be centrohermitian (Skew-centrohermitian) if $\mathcal{R}_{p_1} * \mathcal{A} * \mathcal{R}_{p_2} = \overline{\mathcal{A}}$ ($\mathcal{R}_{p_1} * \mathcal{A} * \mathcal{R}_{p_2} = -\overline{\mathcal{A}}$).*

Theorem 4.9. *Suppose $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$ with $(\mathcal{M}, \mathcal{N})$ coefficient tensors, if $f : \mathbb{C} \to \mathbb{C}$, and $f^{\diamond}_{\mathcal{MN}} : \mathbb{C}^{p_1 \times p_2 \times p_3} \to \mathbb{C}^{p_1 \times p_2 \times p_3}$ is corresponding T-WGTF.*
(1) If $\mathcal{R}_{p_1} * \mathcal{A} * \mathcal{R}_{p_2} = \overline{\mathcal{A}}$ (or $\mathcal{R}_{p_1} * \mathcal{A} * \mathcal{R}_{p_2} = -\overline{\mathcal{A}}$), then $\mathcal{R}_{p_1} * f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) * \mathcal{R}_{p_2} = \overline{f^{\diamond}_{\mathcal{MN}}(\mathcal{A})}$ (or $\mathcal{R}_{p_1} * \mathcal{A} * \mathcal{R}_{p_2} = -\overline{f^{\diamond}_{\mathcal{MN}}(\mathcal{A})}$), where $\mathcal{R}_{p_1}$ and $\mathcal{R}_{p_2}$ are reverse tensors,
(2) If $\mathcal{A}^{\#} * \mathcal{A} = \mathcal{A} * \mathcal{A}^{\#}$, then $f^{\diamond}_{\mathcal{MN}}(\mathcal{A}^{\#}) * f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) = f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) * f^{\diamond}_{\mathcal{MN}}(\mathcal{A}^{\#})$,
(3) If each frontal slice of $\mathcal{A} \in \mathbb{C}^{p_2 \times p_2 \times p_3}$ is a circular matrix, then the frontal slices of $f^{\diamond}_{\mathcal{NN}}(\mathcal{A})$ are also circular matrices,

Proof. (1) It follows from there exist reverse tensors $\mathcal{R}_{p_1} \in \mathbb{C}^{p_1 \times p_1 \times p_3}$ and $\mathcal{R}_{p_2} \in \mathbb{C}^{p_2 \times p_2 \times p_3}$ satisfying $\mathcal{R}_{p_1} * \mathcal{A} * \mathcal{R}_{p_2} = \overline{\mathcal{A}}$ that

$$\begin{aligned} \mathcal{R}_{p_1}^H * \mathcal{R}_{p_1} &= \text{fold}(\text{bcirc}(\mathcal{R}_{p_1}^H)\text{unfold}(\mathcal{R}_{p_1})) \\ &= \text{fold}\left(\begin{pmatrix} R_{p_1}^H & 0 & \cdots & 0 \\ 0 & R_{p_1}^H & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{p_1}^H \end{pmatrix}\begin{pmatrix} R_{p_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) = \mathcal{I}_{p_1 p_1 p_3}, \end{aligned}$$

which imply that $\mathcal{R}_{p_1}$ is a unitary tensor, by the same method that $\mathcal{R}_{p_2}$ is also unitary. According to Corollary 3.5,

$$\mathcal{R}_{p_1} * f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) * \mathcal{R}_{p_2} = f^{\diamond}_{\mathcal{MN}}(\mathcal{R}_{p_1} * \mathcal{A} * \mathcal{R}_{p_2}) = f^{\diamond}_{\mathcal{MN}}(\overline{\mathcal{A}}) = \overline{f^{\diamond}_{\mathcal{MN}}(\mathcal{A})},$$

which means $f^{\diamond}_{\mathcal{MN}}(\mathcal{A})$ is also centrohermitian. Skew-centrohermitian can be obtained similarly.

(2) Similar to the matrix case, the result could be achieved if the condition $f^{\diamond}_{\mathcal{MN}}(\mathcal{A}^{\#}) * f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) = f^{\diamond}_{\mathcal{MN}}(\mathcal{A}^{\#} * \mathcal{A})$ holds. Note that

$$\begin{aligned} f^{\diamond}_{\mathcal{MN}}(\mathcal{S}) * f^{\diamond}_{\mathcal{MN}}(\mathcal{S}^H) &= \mathrm{bcirc}^{-1}\left((F_{p_3} \otimes I_{p_1}) \begin{pmatrix} f(\Sigma_1) & & & \\ & f(\Sigma_2) & & \\ & & \ddots & \\ & & & f(\Sigma_{p_3}) \end{pmatrix} (F^H_{p_3} \otimes I_{p_2})\right) \\ & * \mathrm{bcirc}^{-1}\left((F_{p_3} \otimes I_{p_2}) \begin{pmatrix} f(\Sigma_1)^H & & & \\ & f(\Sigma_2)^H & & \\ & & \ddots & \\ & & & f(\Sigma_{p_3})^H \end{pmatrix} (F^H_{p_3} \otimes I_{p_1})\right) \\ &= f^{\diamond}_{\mathcal{MN}}(\mathcal{S} * \mathcal{S}^H). \end{aligned}$$

Since the T-MN-SVD of $\mathcal{A}$,

$$\begin{aligned} f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) * f^{\diamond}_{\mathcal{MN}}(\mathcal{A}^{\#}) &= \mathcal{U} * f^{\diamond}_{\mathcal{MN}}(\mathcal{S}) * \mathcal{V}^H * \mathcal{N}^{-1} * \mathcal{V} * f^{\diamond}_{\mathcal{MN}}(\mathcal{S}^H) * \mathcal{U}^H * \mathcal{M} \\ &= \mathcal{U} * f^{\diamond}_{\mathcal{MN}}(\mathcal{S} * \mathcal{S}^H) * \mathcal{U}^H * \mathcal{M} = f^{\diamond}_{\mathcal{MN}}(\mathcal{A}^{\#} * \mathcal{A}) = f^{\diamond}_{\mathcal{MN}}(\mathcal{A}^{\#}) * f^{\diamond}_{\mathcal{MN}}(\mathcal{A}), \end{aligned}$$

Hence, the second result holds.

(3) If $\mathcal{A} \in \mathbb{C}^{p_2 \times p_2 \times p_3}$, since $\mathcal{A}$ is an F-circulant tensor, then

$$\begin{aligned} \mathrm{bcirc}(\mathcal{A}) &= \begin{pmatrix} A^{(1)} & A^{(p_3)} & \cdots & A^{(2)} \\ A^{(2)} & A^{(1)} & \cdots & A^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ A^{(p_3)} & A^{(p_3-1)} & \cdots & A^{(1)} \end{pmatrix} = \begin{pmatrix} F_{p_2}\Lambda^{(1)}F^H_{p_2} & F_{p_2}\Lambda^{(p_3)}F^H_{p_2} & \cdots & F_{p_2}\Lambda^{(2)}F^H_{p_2} \\ F_{p_2}\Lambda^{(2)}F^H_{p_2} & F_{p_2}\Lambda^{(1)}F^H_{p_2} & \cdots & F_{p_2}\Lambda^{(3)}F^H_{p_2} \\ \vdots & \vdots & \ddots & \vdots \\ F_{p_2}\Lambda^{(p_3)}F^H_{p_2} & F_{p_2}\Lambda^{(p_3-1)}F^H_{p_2} & \cdots & F_{p_2}\Lambda^{(1)}F^H_{p_2} \end{pmatrix} \\ &= (I_{p_3} \otimes F_{p_2})(F_{p_3} \otimes I_{p_2}) \begin{pmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_{p_3} \end{pmatrix} (F^H_{p_3} \otimes I_{p_2})(I_{p_3} \otimes F^H_{p_2}), \end{aligned}$$

where each $\Lambda^{(i)}$ is a diagonal matrix composed of eigenvalues of $A^{(i)}$, $i = 1, 2, \cdots, p_3$. In this way, the T-WGTF of $\mathcal{A}$ can be factorized as

$$f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) = \mathrm{bcirc}^{-1}\left((I_{p_3} \otimes F_{p_2})(F_{p_3} \otimes I_{p_2}) \begin{pmatrix} f^{\diamond}_{MN}(\Lambda_1) & & \\ & \ddots & \\ & & f^{\diamond}_{MN}(\Lambda_{p_3}) \end{pmatrix} (F^H_{p_3} \otimes I_{p_2})(I_{p_3} \otimes F^H_{p_2})\right),$$

where $f^{\diamond}_{MN}(\Lambda_i) = \begin{pmatrix} f(\lambda^i_1) & & \\ & \ddots & \\ & & f(\lambda^i_{r_i}) \end{pmatrix}$ and $r_i = \mathrm{rank}(\Lambda_i) = \mathrm{rank}(A^{(i)})$, $i = 1, 2, \cdots, p_3$.

The proof is completed. □

In order to get the structural invariance of block diagonal tensor under the T-WGTF, the below lemma is necessary.

Lemma 4.10. *Suppose that $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$ with $(\mathcal{M}, \mathcal{N})$ coefficient tensors, if*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & & \\ & \ddots & \\ & & \mathcal{A}_k \end{pmatrix}, \tag{18}$$

where $\mathcal{A}_i \in \mathbb{C}^{s_i \times t_i \times p_3}$ *and*

$$\sum_{i=1}^{k} s_i = p_1 \text{ and } \sum_{i=1}^{k} t_i = p_2,$$

then

$$f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) = \begin{pmatrix} f^{\diamond}_{\mathcal{MN}}(\mathcal{A}_1) & & \\ & \ddots & \\ & & f^{\diamond}_{\mathcal{MN}}(\mathcal{A}_k) \end{pmatrix}.$$

Proof. According to Lemma 2.15 and the T-MN-CSVD of $\mathcal{A}_i$, then

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & & \\ & \ddots & \\ & & \mathcal{A}_k \end{pmatrix} = \begin{pmatrix} \mathcal{U}_{r_1} & & \\ & \ddots & \\ & & \mathcal{U}_{r_k} \end{pmatrix} * \begin{pmatrix} \mathcal{S}_{r_1} & & \\ & \ddots & \\ & & \mathcal{S}_{r_k} \end{pmatrix} * \begin{pmatrix} \mathcal{V}^H_{r_1} & & \\ & \ddots & \\ & & \mathcal{V}^H_{r_k} \end{pmatrix},$$

where $\mathcal{U}_{r_i} \in \mathbb{C}^{s_i \times r_i \times p_3}$, $\mathcal{S}_{r_i} \in \mathbb{C}^{r_i \times r_i \times p_3}$ and $\mathcal{V}_i \in \mathbb{C}^{t_i \times r_i \times p_3}$. For convenience, the three F-block diagonal tensors of the above equation are denoted as $\widehat{\mathcal{U}_r}$, $\widehat{\mathcal{D}_r}$ and $\widehat{\mathcal{V}^H_r}$. It is noticed that bcirc($\mathcal{S}_r$) is composed of the weighted singular values of $\mathcal{A}$ by the T-MN-CSVD of $\mathcal{A}$, while bcirc($\widehat{\mathcal{D}_r}$) is composed of the weighted T-singular values of $\mathcal{A}_i$. Then there exists permutation tensor $\mathcal{P} \in \mathbb{R}^{r \times r \times p_3}$ such that $\mathcal{S}_r = \mathcal{P} * \widehat{\mathcal{D}_r} * \mathcal{P}^T$. Hence, $\mathcal{A}$ could be expressed by

$$\mathcal{A} = \mathcal{U}_r * \mathcal{S}_r * \mathcal{V}^H_r = \mathcal{U}_r * \mathcal{P} * \widehat{\mathcal{D}_r} * \mathcal{P}^T * \mathcal{V}^H_r = \widehat{\mathcal{U}_r} * \widehat{\mathcal{D}_r} * \widehat{\mathcal{V}^H_r},$$

which means that $\widehat{\mathcal{U}_r} = \mathcal{U}_r * \mathcal{P}$ and $\widehat{\mathcal{V}_r} = \mathcal{V}_r * \mathcal{P}$. Therefore, the T-WGTF could be expressed as followed

$$\begin{aligned} f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) &= \mathcal{U}_r * f^{\diamond}_{\mathcal{MN}}(\mathcal{S}_r) * \mathcal{V}^H_r = \mathcal{U}_r * \mathcal{P} * f^{\diamond}_{\mathcal{MN}}(\widehat{\mathcal{D}_r}) * \mathcal{P}^T * \mathcal{V}^H_r = \widehat{\mathcal{U}_r} * f^{\diamond}_{\mathcal{MN}}(\widehat{\mathcal{D}_r}) * \widehat{\mathcal{V}^H_r} \\ &= \begin{pmatrix} f^{\diamond}_{\mathcal{MN}}(\mathcal{A}_1) & & \\ & \ddots & \\ & & f^{\diamond}_{\mathcal{MN}}(\mathcal{A}_k) \end{pmatrix}. \end{aligned}$$

The proof is completed. □

As a consequence, if a third-order tensor could be expressed by the "direct sum" of the other third-order tensors, then it is invariant under the T-WGTF.

Theorem 4.11. *Let* $\mathcal{A} \in \mathbb{C}^{p_1 \times p_2 \times p_3}$ *with* $(\mathcal{M}, \mathcal{N})$ *coefficient tensors,* $f : \mathbb{C} \to \mathbb{C}$, *and* $f^{\diamond}_{\mathcal{MN}} : \mathbb{C}^{p_1 \times p_2 \times p_3} \to \mathbb{C}^{p_1 \times p_2 \times p_3}$. *If there have two permutation tensors*

$$\mathcal{P} = \text{fold}\begin{pmatrix} P \\ O \\ \vdots \\ O \end{pmatrix} \in \mathbb{R}^{p_1 \times p_1 \times p_3} \text{ and } \mathcal{Q} = \text{fold}\begin{pmatrix} Q \\ O \\ \vdots \\ O \end{pmatrix} \in \mathbb{R}^{p_2 \times p_2 \times p_3}$$

make $\mathcal{P} * \mathcal{A} * \mathcal{Q}$ *be an F-block diagonal tensor, where* $P \in \mathbb{R}^{p_1 \times p_1}$ *and* $Q \in \mathbb{R}^{p_2 \times p_2}$ *are permutation matrices. Then* $\mathcal{P} * f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) * \mathcal{Q}$ *is also F-block diagonal.*

Proof. Suppose $\mathcal{P} * \mathcal{A} * \mathcal{Q} = \mathcal{B}$, then $\mathcal{B}$ can be factorized as (18). In this case, since $\mathcal{P}$ and $\mathcal{Q}$ are unitary tensors,

$$\mathcal{P} * f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) * \mathcal{Q} = f^{\diamond}_{\mathcal{MN}}(\mathcal{P} * \mathcal{A} * \mathcal{Q}) = f^{\diamond}_{\mathcal{MN}}(\mathcal{B}).$$

According to the result of Lemma 4.10, $\mathcal{P} * f^{\diamond}_{\mathcal{MN}}(\mathcal{A}) * \mathcal{Q}$ is also F-block diagonal. □

The following theorem states that tensors preserved nonnegativity under the T-WGTF.

Theorem 4.12. *If $\mathcal{A} \in \mathbb{R}^{p_1\times p_2\times p_3}$ is a nonnegative tensor with $(\mathcal{M},\mathcal{N})$ coefficient tensors, $f:\mathbb{R}\to\mathbb{R}$ is the odd part of an analytic function, and its Maclaurin expansion is*

$$f(z)=\sum_{k=0}^{\infty}c_k z^k$$

with $c_{2k+1}\geq 0$, suppose that it is convergent for $|z|<R$. Then the corresponding T-WGTF $f^{\diamond}_{\mathcal{M}\mathcal{N}}:\mathbb{C}^{p_1\times p_2\times p_3}\to\mathbb{C}^{p_1\times p_2\times p_3}$ is nonnegative for $|c^i_j|<R$.

Proof. Assume the complex-value function f is

$$f(z)=\sum_{k=0}^{\infty}c_{2k+1}z^{2k+1},$$

where $c_{2k+1}\geq 0$. Since $\mathcal{A}^{2k+1}=(\mathcal{A}*\mathcal{A}^{\#})^{2k}*\mathcal{A}$, it is known that

$$(\mathcal{A}*\mathcal{A}^{\#})^{2k}*\mathcal{A}=(\mathcal{U}_r*\mathcal{S}_r*\mathcal{V}_r^T*\mathcal{N}^{-1}*\mathcal{V}_r*\mathcal{S}_r^T*\mathcal{U}_r^T*\mathcal{M})^k*\mathcal{U}_r*\mathcal{S}_r*\mathcal{V}_r^T=\mathcal{U}_r*\mathcal{S}_r^{2k+1}*\mathcal{V}_r^T,$$

According to $\mathcal{A}$ is nonnegative, we have

$$f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{A})=\mathcal{U}_r*(\sum_{k=0}^{\infty}c_{2k+1}\mathcal{S}_r^{2k+1})*\mathcal{V}_r^T\geq 0.$$

Therefore, $f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{A})$ is also nonnegative. □

The following result may transform the calculation of complex tensors into real tensors.

Theorem 4.13. *If $\mathcal{A}\in\mathbb{C}^{p_1\times p_2\times p_3}$ with $(\mathcal{M},\mathcal{N})$ coefficient tensors, $f:\mathbb{C}\to\mathbb{C}$ and $f(0)=0$, $f^{\diamond}_{\mathcal{M}\mathcal{N}}:\mathbb{C}^{p_1\times p_2\times p_3}\to\mathbb{C}^{p_1\times p_2\times p_3}$ is T-WGTF, and $\phi:\mathbb{C}^{p_1\times p_2\times p_3}\to\mathbb{R}^{(2p_1)\times(2p_2)\times p_3}$ is a mapping satisfies $\phi(\mathcal{A})=\begin{pmatrix}\mathcal{B} & -\mathcal{C}\\ \mathcal{C} & \mathcal{B}\end{pmatrix}$ with $\mathcal{B},\mathcal{C}\in\mathbb{C}^{p_1\times p_2\times p_3}$. Then*

$$f^{\diamond}_{(2\mathcal{M})(2\mathcal{N})}(\phi(\mathcal{A}))=\phi(f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{A}))\tag{19}$$

Proof. Suppose $\mathcal{U}=\mathcal{U}_1+i\mathcal{U}_2$ and $\mathcal{V}=(\mathcal{V}_1+i\mathcal{V}_2)^H$, where $\mathcal{U}_1,\mathcal{U}_2\in\mathbb{R}^{p_1\times p_1\times p_3}$ and $\mathcal{V}_1,\mathcal{V}_2\in\mathbb{R}^{p_2\times p_2\times p_3}$. It follows from T-MN-SVD that

$$\mathcal{A}=\mathcal{U}*\mathcal{S}*\mathcal{V}^H=(\mathcal{U}_1+i\mathcal{U}_2)*\mathcal{S}*(\mathcal{V}_1+i\mathcal{V}_2)^H=\mathcal{B}+i\mathcal{C},$$

then $\mathcal{B}=\mathcal{U}_1*\mathcal{S}*\mathcal{V}_1^T+\mathcal{U}_2*\mathcal{S}*\mathcal{V}_2^T$ and $\mathcal{C}=\mathcal{U}_2*\mathcal{S}*\mathcal{V}_1^T-\mathcal{U}_1*\mathcal{S}*\mathcal{V}_2^T$. The T-WGTF is factorized as

$$\begin{aligned}f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{A})&=\mathcal{U}*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*\mathcal{V}^H=(\mathcal{U}_1+i\mathcal{U}_2)*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*(\mathcal{V}_1+i\mathcal{V}_2)^H\\&=\mathcal{U}_1*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*\mathcal{V}_1^T+\mathcal{U}_2*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*\mathcal{V}_2^T+i(\mathcal{U}_2*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*\mathcal{V}_1^T-\mathcal{U}_1*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*\mathcal{V}_2^T).\end{aligned}$$

Then $\phi(f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{A}))$ may be expressed as

$$\phi(f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{A}))=\begin{pmatrix}\mathcal{U}_1*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*\mathcal{V}_1^T+\mathcal{U}_2*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*\mathcal{V}_2^T & -\mathcal{U}_2*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*\mathcal{V}_1^T+\mathcal{U}_1*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*\mathcal{V}_2^T\\ \mathcal{U}_2*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*\mathcal{V}_1^T-\mathcal{U}_1*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*\mathcal{V}_2^T & \mathcal{U}_1*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*\mathcal{V}_1^T+\mathcal{U}_2*f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S})*\mathcal{V}_2^T\end{pmatrix}.$$

Consider the right side of (19), according to expressions of $\mathcal{B}$ and $\mathcal{C}$, we have

$$\phi(\mathcal{A})=\begin{pmatrix}\mathcal{B} & -\mathcal{C}\\ \mathcal{C} & \mathcal{B}\end{pmatrix}=\begin{pmatrix}\mathcal{U}_1 & -\mathcal{U}_2\\ \mathcal{U}_2 & \mathcal{U}_1\end{pmatrix}*\begin{pmatrix}\mathcal{S} & \mathcal{O}\\ \mathcal{O} & \mathcal{S}\end{pmatrix}*\begin{pmatrix}\mathcal{V}_1 & -\mathcal{V}_2\\ \mathcal{V}_2 & \mathcal{V}_1\end{pmatrix}^H.$$

It follows from $\mathcal{U}^H * \mathcal{M} * \mathcal{U} = \mathcal{I}_{p_1p_1p_3}$ and $\mathcal{V}^H * \mathcal{N}^{-1} * \mathcal{V} = \mathcal{I}_{p_2p_2p_3}$ that

$$\mathcal{U}_1^T * \mathcal{M} * \mathcal{U}_1 + \mathcal{U}_2^T * \mathcal{M} * \mathcal{U}_2 = \mathcal{I}_{p_1p_1p_3}, \mathcal{U}_1^T * \mathcal{M} * \mathcal{U}_2 - \mathcal{U}_2^T * \mathcal{M} * \mathcal{U}_1 = \mathcal{O},$$

$$\mathcal{V}_1^T * \mathcal{N}^{-1} * \mathcal{V}_1 + \mathcal{V}_2^T * \mathcal{N}^{-1} * \mathcal{V}_2 = \mathcal{I}_{p_2p_2p_3}, \mathcal{V}_2^T * \mathcal{N}^{-1} * \mathcal{V}_1 - \mathcal{V}_1^T * \mathcal{N}^{-1} * \mathcal{V}_2 = \mathcal{O}.$$

Therefore,

$$\begin{pmatrix} \mathcal{U}_1 & -\mathcal{U}_2 \\ \mathcal{U}_2 & \mathcal{U}_1 \end{pmatrix}^T * \begin{pmatrix} \mathcal{M} & \mathcal{O} \\ \mathcal{O} & \mathcal{M} \end{pmatrix} * \begin{pmatrix} \mathcal{U}_1 & -\mathcal{U}_2 \\ \mathcal{U}_2 & \mathcal{U}_1 \end{pmatrix} = \mathcal{I}_{(2p_1)(2p_1)p_3},$$

$$\begin{pmatrix} \mathcal{V}_1 & -\mathcal{V}_2 \\ \mathcal{V}_2 & \mathcal{V}_1 \end{pmatrix}^T * \begin{pmatrix} \mathcal{N}^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{N}^{-1} \end{pmatrix} * \begin{pmatrix} \mathcal{V}_1 & -\mathcal{V}_2 \\ \mathcal{V}_2 & \mathcal{V}_1 \end{pmatrix} = \mathcal{I}_{(2p_2)(2p_2)p_3},$$

hence, $\begin{pmatrix} \mathcal{M} & \mathcal{O} \\ \mathcal{O} & \mathcal{M} \end{pmatrix}$ and $\begin{pmatrix} \mathcal{N}^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{N}^{-1} \end{pmatrix}$ are Hermite positive definite tensors. As a result,

$$f^{\diamond}_{(2\mathcal{M})(2\mathcal{N})}(\phi(\mathcal{A})) = \begin{pmatrix} \mathcal{U}_1 & -\mathcal{U}_2 \\ \mathcal{U}_2 & \mathcal{U}_1 \end{pmatrix} * \begin{pmatrix} f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S}) & \mathcal{O} \\ \mathcal{O} & f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{S}) \end{pmatrix} * \begin{pmatrix} \mathcal{V}_1 & -\mathcal{V}_2 \\ \mathcal{V}_2 & \mathcal{V}_1 \end{pmatrix}^T = \phi(f^{\diamond}_{\mathcal{M}\mathcal{N}}(\mathcal{A})).$$

The proof is completed. □

References

[1] R. Behera, J. Sahoo, R. Mohapatra and M. Nashed, Computation of generalized inverses of tensors via t-product, Numerical Linear Algebra with Applications, 29 (2022) e2416.
[2] A. Ben-Israel, T. Greville, Generalized Inverses Theory and Applications, (2nd edition), Springer, New York, 2003.
[3] M. Benzi, R. Huang, Some matrix properties preserved by generalized matrix functions, Special Matrices, 7 (2019) 27–37.
[4] K. Braman, Third-order tensors as linear operators on a space of matrices, Linear Algebra and its Applications, 433 (2010) 1241–1253.
[5] R. Chan, X. Jin, An Introducton to Iterative Toeplitz Solvers, SIAM, Philadel-phia, 2007.
[6] M. Che, Y. Wei, Theory and Computation of Complex Tensors and its Applications, Springer, Singapore, 2020.
[7] A. Cichocki, N. Lee, L. Oseledets, A. Phan, Q. Zhao, M. Sugiyama and D. Mandic, Tensor Networks for Dimensionality Reduction and Large-scale Optimization, Part 2 Applications and Future Perspectives, Foundations and Trends in Machine Learning, 9 (2017) 431–673.
[8] W. Ding, Y. Wei, Theory and Computation of Tensors: Multi-Dimensional Arrays, Academic Press, New York, 2016.
[9] N. Hao, M. Kilmer, K. Braman and R. Hoover, Facial recognition using tensor-tensor decompositions, SIAM Journal on Imaging Sciences, 6 (2013) 437–463.
[10] J. Hawkins, A. Ben-Israel, On generalized matrix functions, Linear Multilinear Algebra, 1 (1973) 163–171.
[11] N. Higham, Functions of Matrices: Theory and Computation, SIAM, Philadelphia, 2008.
[12] B. Khoromskij, Tensor numerical methods for multidimensional PDES: Theoretical analysis and initial applications, ESAIM: Proceedings and Surveys, 48 (2015) 1–28.
[13] M. Kilmer, K. Braman, N. Hao and R. Hoover, Third-order tensors as operators on matrices: a theoretical and computational framework with applications in imaging, SIAM Journal on Matrix Analysis and Applications, 34 (2013) 148–172.
[14] M. Kilmer, C. Martin, Factorization strategies for third-order tensors, Linear Algebra and its Applications, 435 (2011) 641–658.
[15] H. Kong, X. Xie and Z. Lin, T-Schatten-p norm for low-rank tensor recovery, IEEE Journal of Selected Topics in Signal Processing, 12 (2018) 1405–1419.
[16] A. Lee, Centrohermitian and skew-centrohermitian matrices, Linear Algebra and its Applications, 29 (1980) 205–210.
[17] C. Ling, L. Qi, Some results on l^k eigenvalues of tensor and related radius, Numerical Algebra, Control and Optimization, 1 (2011) 381–388.
[18] W. Liu, X. Jin, A Study on T-eigenvalues of Third-order Tensors, Linear Algebra and its Applications, 612 (2021) 357–374.
[19] T. Liu, L. Chen and C. Zhu, Improved robust tensor principal component analysis via low-rank core matrix, IEEE Journal of Selected Topics in Signal Processing, 12 (2018) 1378-1389.
[20] K. Lund, The tensor t-function: a definition for functions of third-order tensors, Numerical Linear Algebra with Applications, 27 (2020) e2288.
[21] H. Ma, N. Li, P. Stanimirović and V. Katsikis, Perturbation theory for Moore-Penrose inverse of tensor via Einstein product, Computational and Applied Mathematics, 8 (2019) article 111.
[22] Y. Miao, L. Qi and Y. Wei, T-Jordan canonical form and T-Drazin inverse based on the T-product, Communications on Applied Mathematics and Computation, 3 (2021) 201–220.

[23] Y. Miao, L. Qi and Y. Wei, Generalized tensor function via the tensor singular value decomposition based on the T-product, Linear Algebra and its Applications, 590 (2020) 258–303.
[24] M. Ng, L. Qi and G. Zhou, Finding the largest eigenvalue of a nonnegative tensor, SIAM Journal on Matrix Analysis and Applications, 31 (2010) 1090–1099.
[25] L. Qi, Eigenvalues of a real supersymmetric tensor, Journal of Symbolic Computation, 40 (2005) 1302–1324.
[26] L. Qi, H. Chen and Y. Chen, Tensor Eigenvalues and their Applications, Springer, Singapore, 2018.
[27] L. Qi, Z. Luo, Tensor Analysis Spectral Theory and Special Tensors, SIAM, Philadelphia, 2017.
[28] C. Rao, S. Mitra, Generalized Inverses of Matrices and its Applications, Wiley, New York, 1971.
[29] G. Song, M. Ng and X. Zhang, Robust tensor completion using transformed tensor singular value decomposition, Numerical Linear Algebra with Applications, 27 (2020) e2299.
[30] P. Stanimirović, M. Ćirić, V. Katsikis, C. Li and H. Ma, Outer and (b, c) inverses of tensors, Linear Multilinear Algebra, 68 (2020) 940–971.
[31] L. Sun, B. Zheng, C. Bu and Y. Wei, Moore-Penrose inverse of tensors via Einstein product, Linear Multilinear Algebra, 64 (2016) 686–698.
[32] L. Sun, B. Zheng, Y. Wei and C. Bu, Generalized inverses of tensors via a general product of tensors, Frontiers of Mathematics in China, 13 (2018) 893–911.
[33] C. Van Loan, Generalizing the singular value decomposition, SIAM Journal on Numerical Analysis, 13 (1976) 76–83.
[34] A. Wang, Z. Lai and Z. Jin, Noisy low-tubal-rank tensor completion, Neurocomputing, 330 (2019) 267–279.
[35] B. Wang, H. Du, and H. Ma, Perturbation bounds for DMP and CMP inverses of tensors via Einstein product, Computational and Applied Mathematics, 39 (2020) 28–45.
[36] G. Wang, Y. Wei and S. Qiao, Generalized Inverses: Theory and Computations, Science Press, Beijing, 2018.
[37] X. Wang, M. Che and Y. Wei, Tensor neural network models for tensor singular value decompositions, Computational optimization and applications, 75 (2020) 753–777.
[38] H. Yang, H. Li, Generalized UDV*-decomposition and weighted spectual decomposition for rectangular matrices and their applications, Applied Mathematics and Computation, 198 (2008) 150–162.
[39] L. Yang, Z. Huang, S. Hu and J. Han, An iterative algorithm for third-order tensor multi-rank minimization, Computational optimization and applications, 63 (2016) 169–202.
[40] J. Zhang, A. Saibaba, M. Kilmer and S. Aeron, A randomized tensor singular value decomposition based on the t-product, Numerical Linear Algebra with Applications, 25 (2018) e2179.