Filomat 36:18 (2022), 6173–6185
https://doi.org/10.2298/FIL2218173H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On ⋆-Metric Spaces

S.Y. He[a], L.H. Xie[a], P.F. Yan[a]

[a] *School of Mathematics and Computational Science, Wuyi University, Jiangmen 529020, P.R. China*

Abstract. Metric spaces are generalized by many scholars. Recently, Khatami and Mirzavaziri use a mapping called t-definer to popularize the triangle inequality and give a generalization of the notion of a metric, which is called a ⋆-metric. In this paper, we prove that every ⋆-metric space is metrizable. Also, we study the total boundedness and completeness of ⋆-metric spaces.

1. Introduction

It is well known that metric spaces are widely used in analysis. There are several common metric spaces, such as the numerical straight line $\mathbb{R}$, the n-dimensional Euclidean space $\mathbb{R}^n$, the continuous functions space and the Hilbert space. Therefore, every metric space is an important kind of topological space. A function $d : X \times X \to \mathbb{R}^+$ is called a *metric* on a set X if d satisfies the following conditions for every $x, y, z \in X$:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leqslant d(x, z) + d(z, y)$.

Then set X with d is called a *metric space*, denoted by (X, d).

We can obtain the generalizations of metric spaces when we weaken or modify the conditions of metric axiom. Pseudo-metrics are obtained when we change that condition '$\rho(x, y) = 0$ if and only if $x = y$' into '$\rho(x, y) = 0$ if $x = y$' [3]. Quasi-metrics are defined by omitting the condition (2) [12]. Symmetrics are defined by omitting the triangle inequality [1]. The ultrametric is a metric with the strong triangle inequality $d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$, for $x, y, z \in X$ [10]. There are many generalizations of metric spaces which have appeared in literatures (e.g. see [4–6, 8, 11, 13])

Recently, Khatami and Mirzavaziri popularized the concept of metric. By extending the famous function which is called t-conorm, a new operation called t-definer is obtained. It is defined as:

Definition 1.1. ([7, Definition 2.1]) A *t-definer* is a function $\star : [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying the following conditions for each $a, b, c \in [0, \infty)$:

(T1) $a \star b = b \star a$;

2020 *Mathematics Subject Classification*. Primary 54E35; Secondary 54E45, 54E50, 54E52
Keywords. ⋆-metric spaces, total boundedness, completeness
Received: 06 December 2021; Revised: 13 December 2022; Accepted: 14 December 2022
Communicated by Ljubiša D.R. Kočinac
Corresponding author: L.H. Xie
Research supported by the Natural Science Foundation of Guangdong Province under Grant (No. 2021A1515010381) and the Innovation Project of Department of Education of Guangdong Province (No. 2022KTSCX145).
Email addresses: 2644481035@qq.com (S.Y. He), yunli198282@126.com (L.H. Xie), ypengfei@sina.com (P.F. Yan)

(T2) $a \star (b \star c) = (a \star b) \star c$;
(T3) if $a \leqslant b$, then $a \star c \leqslant b \star c$;
(T4) $a \star 0 = a$;
(T5) $\star$ is continuous in its first component with respect to the Euclidean topology.

When the condition (3) in the metric axiom is extended to the $\star$-triangle inequality, the following definition of $\star$-metrics can be obtained.

Definition 1.2. ([7, Definition 2.2]) Let X be a nonempty set and $\star$ is a t-definer. If for every $x, y, z \in X$, a function $d^\star : X \times X \to [0, \infty)$ satisfies the following conditions:

(M1) $d^\star(x, y) = 0$ if and only if $x = y$;
(M2) $d^\star(x, y) = d^\star(y, x)$;
(M3) $d^\star(x, y) \leqslant d^\star(x, z) \star d^\star(z, y)$,

then $d^\star$ is called a $\star$-*metric* on X. The set X with a $\star$-metric is called a $\star$-*metric space*, denoted by $(X, d^\star)$.

The following example shows that there are $\star$-metrics which are not metrics.

Example 1.3. ([7, Example 2.4.]) Clearly, $a \star b = (\sqrt{a} + \sqrt{b})^2$ is a t-definer. The function $d^\star(a, b) = (\sqrt{a} - \sqrt{b})^2$ forms an $\star$-metric which is not a metric. In fact,

$$\begin{aligned} d^\star(a, b) &= (\sqrt{a} - \sqrt{b})^2 = (\sqrt{a} - \sqrt{c} + \sqrt{c} - \sqrt{b})^2 \\ &\leqslant \left[\sqrt{(\sqrt{a} - \sqrt{c})^2} + \sqrt{(\sqrt{c} - \sqrt{b})^2} \right]^2 \\ &= \left[\sqrt{d^\star(a, c)} + \sqrt{d^\star(c, b)} \right]^2 \\ &= d^\star(a, c) \star d^\star(c, b) \end{aligned}$$

while $d^\star(1, 25) = 16 \nleqslant d^\star(1, 16) + d^\star(16, 25) = 10$.

Remark 1.4. There are two most important t-definers:
• Lukasiewicz t-definer: $a \star_L b = a + b$;
• Maximum t-definer: $a \star_m b = \max\{a, b\}$.
Obviously, a $\star_L$-metric is actually a metric and a $\star_m$-metric is an ultrametric.

Assume that $(M, d^\star)$ is a $\star$-metric space. For any $a \in M$ and $r > 0$, denote by

$$B_{d^\star}(a, r) = \{x \in M : d^\star(a, x) < r\}$$

and

$$\mathcal{T}_{d^\star} = \{U \subseteq M : \text{ for each } a \in U \text{ there is an } r > 0 \text{ such that } B_{d^\star}(a, r) \subseteq U\}.$$

Let $\mathcal{B} = \{B_{d^\star}(x, \epsilon) \mid x \in X, \epsilon > 0\}$ be a family of open balls on a $\star$-metric space $(X, d^\star)$.
Khatami and Mirzavaziri proved the following results:

Theorem 1.5. ([7, Theorems 3.2, 3.4, 3.5]) *For every $\star$-metric space $(X, d^\star)$, the $\mathcal{T}_{d^\star}$ forms a Hausdorff topology on X and the topological space $(X, \mathcal{T}_{d^\star})$ is first countable and satisfies the normal separation axiom.*

The following problem is posed naturally.

Problem 1.6. *Is the topological space $(X, \mathcal{T}_{d^\star})$ metrizable for every $\star$-metric space $(X, d^\star)$?*

In this paper, we give a positive answer to Problem 1.6. Also, we extend the concepts of the totally boundedness and completeness in metric spaces into $\star$-metric spaces and discuss their properties.

This paper is organized as follows. Section 2 is given a positive answer to Problem 1.6. We obtain that let $(X, d^\star)$ be a $\star$-metric space, then the set X with the topology $\mathscr{T}_{d^\star}$ induced by $d^\star$ is metrizable (see Theorem 2.4). In Section 3, total boundedness of $\star$-metric spaces are studied. We prove that: (1) let $(X, d^\star)$ be a totally bounded $\star$-metric space, then for every subset M of X the $\star$-metric space $(M, d^\star)$ is totally bounded (see Theorem 3.4); (2) let $(X, d^\star)$ be a $\star$-metric space; then for every subset M of X the space $(M, d^\star)$ is totally bounded if and only if $(\overline{M}, d^\star)$ is totally bounded (see Theorem 3.5). We show that the Cartesian product and disjoint union of finite totally bounded $\star$-metric spaces are totally bounded under specific $\star$-metrics (see Theorems 3.7 and 3.9). In Section 4, the completeness of $\star$-metric spaces are explored. We obtain that: (1) a $\star$-metric space $(X, d^\star)$ is compact if and only if $(X, d^\star)$ is complete and totally bounded (see Theorem 4.6); (2) a $\star$-metric space is complete if and only if for every decreasing sequence $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ of non-empty closed subsets of space X, such that $\lim_{n\to\infty} \delta(F_n) = 0$, the intersection $\bigcap_{n=1}^{\infty} F_n$ is a one-point set (see Theorem 4.9); (3) the completeness of the Cartesian product and disjoint union of complete $\star$-metric spaces are studied under specific $\star$-metrics (see Theorems 4.12 and 4.13); (4) in a complete $\star$-metric space $(X, d^\star)$ the intersection $A = \bigcap_{n=1}^{\infty} A_n$ of a sequence $A_1, A_2, \ldots$ of dense open subsets is a dense set (see Theorem 4.14).

=

2. Metrizability of $\star$-metric spaces

In this section, we shall give a positive answer to Question 1.6. Let $(X, d^\star)$ be a $\star$-metric space. We shall prove that the set X with the topology $\mathscr{T}_{d^\star}$ induced by $d^\star$ is metrizable. We need to use some related symbols, terms, and preliminary facts.

Let $\mathscr{U}$ be a cover of a set X. For $x \in X$, denote by $\mathrm{st}(x, \mathscr{U}) = \bigcup\{U : U \in \mathscr{U}, x \in U\}$ and $\mathrm{st}(A, \mathscr{U}) = \bigcup_{x\in A} \mathrm{st}(x, \mathscr{U})$ for $A \subseteq X$. Let $\mathscr{U}, \mathscr{V}$ be two covers of a set X. We say that the cover $\mathscr{V}$ *refines* $\mathscr{U}$, if for every $V \in \mathscr{V}$, there exists a $U \in \mathscr{U}$ such that $V \subset U$, denoted by $\mathscr{V} < \mathscr{U}$; if $\{st(V, \mathscr{V}) : V \in \mathscr{V}\}$ refines $\mathscr{U}$, then we said that $\mathscr{V}$ *star refines* $\mathscr{U}$, denoted by $\mathscr{V} \overset{*}{<} \mathscr{U}$.

Definition 2.1. ([3, Definition 4.5.1]) Let X be a set and $\Phi = \{\mathscr{U}_\alpha : \alpha \in A\}$ a non-empty collection of covers of X which satisfies:

(U1) if $\mathscr{U}$ is a cover of X such that $\mathscr{U}_\alpha < \mathscr{U}$ for some $\alpha \in A$, then $\mathscr{U} \in \Phi$;

(U2) for any $\alpha, \beta \in A$, there exists a $\gamma \in A$ such that $\mathscr{U}_\gamma < \mathscr{U}_\alpha$, $\mathscr{U}_\gamma < \mathscr{U}_\beta$;

(U3) for every $\alpha \in A$, there exists a $\beta \in A$ such that $\mathscr{U}_\beta \overset{*}{<} \mathscr{U}_\alpha$;

(U4) if x, y are different elements of X, then $x \notin \mathrm{st}(y, \mathscr{U}_\alpha)$ for some $\alpha \in A$.

Then X is called a *uniform space* with the uniformity Φ, denoted by (X, Φ). Let X be a uniform space with the uniformity $\Phi = \{\mathscr{U}_\alpha : \alpha \in A\}$ and let $\Phi' = \{\mathscr{U}_\beta : \beta \in B\}$ be a subcollection of Φ. If for every $\alpha \in A$, there exists a $\beta \in B$ such that $\mathscr{U}_\beta < \mathscr{U}_\alpha$, then the collection Φ' is called *a base of the uniformity*.

Let X be a uniform space with the uniformity $\Phi = \{\mathscr{U}_\alpha : \alpha \in A\}$ and

$$\mathscr{T}_\Phi = \{U : U \subseteq X, \text{ for each } x \in U, \text{ there is a } \alpha \in A \text{ such that } \mathrm{st}(x, \mathscr{U}_\alpha) \subset U\}.$$

Then $\mathscr{T}_\Phi$ is a topology on the X.

Recalled that a topological space X is said to be *metrizable* if there exists a metric d on the set X that induces the topology of X.

Lemma 2.2. ([3, Theorem 4.5.9]) *Let (X, Φ) be a uniform space. Then the set with the topology $\mathscr{T}_\Phi$ induced by Φ is metrizable if and only if there is a base of the uniformity consisting of countably many covers.*

The following theorem shows that $\star$ is continuous at the point $(0, 0)$.

Lemma 2.3. *For $r > 0$, there exists an $r_1 > 0$ such that $[0, r_1) \star [0, r_1) \subseteq [0, r)$.*

Proof. For $r > 0$, we have $0 \star \frac{1}{2}r \in [0, r)$ by [Definition 1.1, (T4)]. According to [Definition 1.1, (T5)], there exists an $r_0 > 0$ such that $[0, r_0) \star \frac{1}{2}r \subseteq [0, r)$. Without loss of generality, let $0 \leqslant r_0 \leqslant r$ by [Definition1.1, (T3)]. Take $r_1 = \frac{1}{2}r_0$. Then, we can claim that $[0, r_1) \star [0, r_1) \subseteq [0, r)$. For every $x, y \in [0, r_1)$, we have that

$$x \star y \leqslant x \star \frac{1}{2}r.$$

Noting that $x \star \frac{1}{2}r \in [0, r_1) \star \frac{1}{2}r \subseteq [0, r_0) \star \frac{1}{2}r \subseteq [0, r)$, we have that $x \star \frac{1}{2}r \in [0, r)$, which means $x \star \frac{1}{2}r < r$. Since $x \star y \leqslant x \star \frac{1}{2}r$, $x \star y < r$, we have that $x, y \in [0, r)$. □

The following theorem gives a positive answer to Question 1.6.

Theorem 2.4. *Let $(X, d^\star)$ be a $\star$-metric space. Then the set X with the topology $\mathscr{T}_{d^\star}$ induced by $d^\star$ is metrizable.*

Proof. First we shall show that $\mathscr{B} = \{\mathscr{B}_{\frac{1}{n}} : n \in \mathbb{N}\}$ is a base of a uniformity on set X, where $\mathscr{B}_{\frac{1}{n}} = \{B_{d^\star}(x, \frac{1}{n}) \mid x \in X\}$. Indeed, it is enough to show that $\mathscr{B}$ satisfies the conditions (U2)~(U4) in Definition 2.1.

(U2). For each $x \in X$ and arbitrary $n_1, n_2 \in \mathbb{N}$, take an $n_0 \in \mathbb{N}$ such that $n_0 > n_1$ and $n_0 > n_2$. Take a $y \in B_{d^\star}(x, \frac{1}{n_0})$. Then we have that $d^\star(x, y) < \frac{1}{n_0} < \frac{1}{n_1}$, which implies that $y \in B_{d^\star}(x, \frac{1}{n_1})$. Thus we have that $B_{d^\star}(x, \frac{1}{n_0}) \subset B_{d^\star}(x, \frac{1}{n_1})$. Therefore $\mathscr{B}_{\frac{1}{n_0}} < \mathscr{B}_{\frac{1}{n_1}}$. Similarly, we can prove that $\mathscr{B}_{\frac{1}{n_0}} < \mathscr{B}_{\frac{1}{n_2}}$. So, $\mathscr{B}$ satisfies the condition (U2).

For (U3). For any $n_0 \in \mathbb{N}$, by Lemma 2.3, there exists an $r_1 \in \mathbb{N}$ such that $r_1 \star r_1 \star r_1 < \frac{1}{n_0}$. Take $n_1 \in \mathbb{N}$ such that $\frac{1}{n_1} < r_1$. Now we shall prove that $\mathscr{B}_{\frac{1}{n_1}} \overset{*}{<} \mathscr{B}_{\frac{1}{n_0}}$. Hence the proof is completed once we show that $\mathrm{st}(B_{d^\star}(x, \frac{1}{n_1}), \mathscr{B}_{\frac{1}{n_1}}) \subseteq B_{d^\star}(x, \frac{1}{n_0})$, for any $x \in X$. Take any $y \in X$ such that $B_{d^\star}(y, \frac{1}{n_1}) \cap B_{d^\star}(x, \frac{1}{n_1}) \neq \emptyset$. Then, there exists a $z_1 \in B_{d^\star}(y, \frac{1}{n_1}) \cap B_{d^\star}(x, \frac{1}{n_1})$. For each $z_2 \in B_{d^\star}(y, \frac{1}{n_1})$, we have that

$$\begin{aligned} d^\star(z_2, x) &\leqslant d^\star(z_2, y) \star d^\star(y, z_1) \star d^\star(z_1, x) \\ &< \frac{1}{n_1} \star \frac{1}{n_1} \star \frac{1}{n_1} < r_1 \star r_1 \star r_1 \\ &< \frac{1}{n_0}. \end{aligned}$$

Therefore, $z_2 \in B_{d^\star}(x, \frac{1}{n_0})$. This implies $\mathscr{B}$ satisfies the condition (U3).

For (U4). For $x, y \in X$ with $x \neq y$, put $d^\star(x, y) = r$. Then $r > 0$. Take an $n_1 \in \mathbb{N}$ such that $\frac{1}{n_1} < r$. Then we have that $y \notin B_{d^\star}(x, \frac{1}{n_1})$. By Lemma 2.3, there exists an $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} \star \frac{1}{n_0} < \frac{1}{n_1}$. Then, we claim that for each $B_{d^\star}(z_0, \frac{1}{n_0}) \in \mathscr{B}_{\frac{1}{n_0}}$ $(z_0 \in X)$, $B_{d^\star}(z_0, \frac{1}{n_0})$ can not contain both x and y. If not, $x, y \in B_{d^\star}(z_0, \frac{1}{n_0})$, then $d^\star(x, y) \leqslant d^\star(x, z_0) \star d^\star(z_0, y) < \frac{1}{n_0} \star \frac{1}{n_0} < \frac{1}{n_1} < r$, which is a contradiction with $d^\star(x, y) = r$. Thus $\mathscr{B}$ satisfies the condition (U4).

Thus we have proved that $\mathscr{B}$ is a base of a uniformity on X, denoted by $\Phi_{d^\star}$. According to Lemma 2.2, the set X with the topology $\mathscr{T}_{\Phi_{d^\star}}$ induced by the uniformity $\Phi_{d^\star}$ is metrizable. Therefore, to complete the proof, it is enough to prove that the topology $\mathscr{T}_{d^\star}$ induced by $d^\star$ is the same as the topology $\mathscr{T}_{\Phi_{d^\star}}$.

(1) For each $U \in \mathscr{T}_{d^\star}$ and $x \in U$, there exists an $n \in \mathbb{N}$ such that $B_{d^\star}(x, \frac{1}{n}) \subset U$. By Lemma 2.3, there exists an $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} \star \frac{1}{n_0} < \frac{1}{n}$. Then we shall prove that $\mathrm{st}(x, \mathscr{B}_{\frac{1}{n_0}}) \subset B_{d^\star}(x, \frac{1}{n}) \subset U$. Take each $z \in \mathrm{st}(x, \mathscr{B}_{\frac{1}{n_0}})$. Then there exists a $y \in X$ such that $x \in \mathrm{st}(y, \mathscr{B}_{\frac{1}{n_0}})$ and $z \in \mathrm{st}(y, \mathscr{B}_{\frac{1}{n_0}})$. From the fact that

$$d^\star(x, z) \leqslant d^\star(x, y) \star d^\star(y, z) < \frac{1}{n_0} \star \frac{1}{n_0} < \frac{1}{n},$$

it follows that $z \in \mathrm{st}(x, \mathscr{B}_{\frac{1}{n}})$, i.e. $\mathrm{st}(x, \mathscr{B}_{\frac{1}{n_0}}) \subset U$, which implies that $U \in \mathscr{T}_{\Phi_{d^\star}}$. Therefore $\mathscr{T}_{d^\star} \subseteq \mathscr{T}_{\Phi_{d^\star}}$.

(2) For each $U \in \mathscr{T}_{\Phi_{d^\star}}$ and $x \in U$, there exists an $n \in \mathbb{N}$ such that $\mathrm{st}(x, \mathscr{B}_{\frac{1}{n}}) \subset U$. Since $B_{d^\star}(x, \frac{1}{n}) \subset \mathrm{st}(x, \mathscr{B}_{\frac{1}{n}}) \subset U$, we have $U \in \mathscr{T}_{d^\star}$, i.e. $\mathscr{T}_{\Phi_{d^\star}} \subseteq \mathscr{T}_{d^\star}$. Then we get that $\mathscr{T}_{\Phi_{d^\star}} = \mathscr{T}_{d^\star}$. □

From Theorem 2.4 it follows the following two corollaries.

Corollary 2.5. *Let $(X, d^\star)$ be a $\star$-metric space and X with the topology $\mathscr{T}_{d^\star}$ induced by $d^\star$. Then the following statements are equivalent:*

(1) X has countable base;
(2) X is Lindelöf;
(3) every closed discrete subspace is a countable set;
(4) every discrete subspace is a countable set;
(5) every collection of disjoint open sets in X is countable;
(6) X has a countable dense subset.

Recalled that a topological space X is called *pseudo-compact* if every real valued continuous function on X is bounded. X is called *countably compact* if every countable open cover of X has a finite subcover. X is called *sequentially compact* if every sequence of points of X has a convergent subsequence. A point x is called *ω-accumulation point* of X if arbitrary neighborhood of the point x contains infinite points of X.

Corollary 2.6. *Let $(X, d^\star)$ be a $\star$-metric space and X with the topology $\mathscr{T}_{d^\star}$ induced by $d^\star$. Then the following statements are equivalent:*

(1) X is pseudo-compact;
(2) every infinite subset of X has a cluster point;
(3) every infinite subset of X has an ω-accumulation point;
(4) every sequence of X has a cluster point;
(5) X is countably compact;
(6) X is sequentially compact;
(7) X is compact.

3. Total boundedness of $\star$-metric spaces

Total boundedness is an important property in metric spaces. We generalize the concept of totally boundedness into $\star$-metric spaces and study their properties. Now we need to give some related definitions.

Definition 3.1. A $\star$-metric space $(X, d^\star)$ is *totally bounded*, if for any $\epsilon > 0$ there exists a finite set $F(\epsilon) \subseteq X$ such that $X = \bigcup_{x \in F(\epsilon)} B_{d^\star}(x, \epsilon)$. We also said that the finite set $F(\epsilon)$ is *ϵ-dense* in X.

Theorem 3.2. *Let $(X, d^\star)$ be a $\star$-metric space and every infinite subset of X have an ω-accumulation point in the topological space X with the topology induced by $d^\star$. Then $(X, d^\star)$ is totally bounded.*

Proof. Suppose the contrary that there exists ϵ_0, but there is no finite set $F(\epsilon_0)$ such that $X = \bigcup_{x \in F(\epsilon_0)} B_{d^\star}(x, \epsilon_0)$. Take an $x_1 \in X$. Since $X \neq B_{d^\star}(x_1, \epsilon_0)$, we can take an $x_2 \in X - B_{d^\star}(x_1, \epsilon_0)$. Next, since $X \neq \bigcup_{i=1}^{2} B_{d^\star}(x_i, \epsilon_0)$, we can take an $x_3 \in X - \bigcup_{i=1}^{2} B_{d^\star}(x_i, \epsilon_0)$. Repeat this procedure, and we obtain an infinite set $\{x_1, x_2, \ldots, x_n, \ldots\}$ such that $d^\star(x_i, x_j) \geqslant \epsilon_0$ $(i \neq j)$. By assumption, this infinite set has an ω-accumulation point $x_0 \in X$. Thus, by Lemma 2.3, take an $\epsilon_1 > 0$ such that $\epsilon_1 \star \epsilon_1 < \epsilon_0$. Then the open-ball $B_{d^\star}(x_0, \epsilon_1)$ should contain infinite points of $\{x_1, x_2, \ldots, x_n, \ldots\}$. Let $x_n, x_m \in B_{d^\star}(x_0, \epsilon_1)$, then

$$d^\star(x_m, x_n) \leqslant d^\star(x_m, x_0) \star d^\star(x_0, x_n) < \epsilon_1 \star \epsilon_1 < \epsilon_0.$$

This contradicts $d^\star(x_i, y_j) \geqslant \epsilon_0$. Therefore $(X, d^\star)$ is totally bounded. □

By Corollary 2.6 and Theorem 3.2, we have the following corollary:

Corollary 3.3. *Let $(X, d^\star)$ be a $\star$-metric space. If X with the topology induced by $d^\star$ is countably compact, then $(X, d^\star)$ is a totally bounded $\star$-metric space.*

One can easily verify that for every subset $M \subseteq X$ of a $\star$-metric space $(X, d^\star)$, $(M, d^\star |_{M\times M})$ is a $\star$-metric space, where $d^\star |_{M\times M}$ is the restriction of the $\star$-metric $d^\star$ on X to the subset M. The $\star$-metric space $(M, d^\star |_{M\times M})$ will also be denoted by $(M, d^\star)$.

Theorem 3.4. *Let $(X, d^\star)$ be a totally bounded $\star$-metric space. Then for every subset M of X the $\star$-metric space $(M, d^\star)$ is totally bounded.*

Proof. Let $(X, d^\star)$ be a $\star$-metric space and $M \subset X$. For every $\epsilon > 0$, by Lemma 2.3, take an $\epsilon_1 > 0$ such that $\epsilon_1 \star \epsilon_1 < \epsilon$. Take a finite set $F(\epsilon_1) = \{x_1, x_2, \dots, x_k\}$ which is ϵ_1-dense in X. Let $\{x_{m_1}, x_{m_2}, \dots, x_{m_l}\}$ be the subset of $F(\epsilon_1)$ consisting of all points which satisfy that $d^\star(x, x_i) < \epsilon_1$, for each $x \in M$ and $x_i \in F(\epsilon_1)$. Let $F' = \{x'_1, x'_2, \dots, x'_l\}$ be a finite set satisfying $d^\star(x'_j, x_{m_j}) < \epsilon_1$ for $j \leqslant l$. We shall show that the set F' is ϵ-dense in M. Let $x \in M$, there exists an $x_i \in F(\epsilon_1)$ such that $d^\star(x, x_i) < \epsilon_1$. Hence $x_i = x_{m_j}$ for some $j \leqslant l$, then we have

$$d^\star(x, x'_j) \leqslant d^\star(x, x_{m_j}) \star d^\star(x_{m_j}, x'_j) < \epsilon_1 \star \epsilon_1 < \epsilon.$$

□

Theorem 3.5. *Let $(X, d^\star)$ be a $\star$-metric space and for every subset M of X the space $(M, d^\star)$ is totally bounded if and only if $(\overline{M}, d^\star)$ is totally bounded.*

Proof. Assume that $(M, d^\star)$ is totally bounded. For $\epsilon > 0$, by Lemma 2.3, take an $\epsilon_1 > 0$ such that $\epsilon_1 \star \epsilon_1 < \epsilon$, and take a finite set $F(\epsilon_1) = \{x_1, x_2, \dots, x_k\}$ which is ϵ_1-dense in M. For each $x \in \overline{M}$, we have $B_{d^\star}(x, \epsilon_1) \cap M \neq \emptyset$. Take a $y \in B_{d^\star}(x, \epsilon_1) \cap M$. Then there exists an $x_i \in F(\epsilon_1)$ such that $d^\star(y, x_i) < \epsilon_1$. Then we have $d^\star(x, x_i) \leqslant d^\star(x, y) \star d^\star(y, x_i) < \epsilon_1 \star \epsilon_1 < \epsilon$.

On the other hand, assume that $(\overline{M}, d^\star)$ is totally bounded. One can easily obtain that $(M, d^\star)$ is totally bounded by Theorem 3.4, because M is a subset of $\overline{M}$. □

Corollary 3.6. *If a $\star$-metric space $(X, d^\star)$ has a dense totally bounded subspace, then the space $(X, d^\star)$ is totally bounded.*

Let $\{(X_i, d_i^\star)\}_{i=1}^n$ be a family of finite nonempty $\star$-metric spaces. Consider the Cartesian product $X = \prod_{i=1}^n X_i$ and for every pair $x = (x_i)_{i\leq 1\leq n}$, $y = (y_i)_{i\leq 1\leq n}$ of points of X let

$$d_T^\star(x, y) = d_1^\star(x_1, y_1) \star d_2^\star(x_2, y_2) \star \cdots \star d_n^\star(x_n, y_n) \tag{3.1}$$

and

$$d_{\max}^\star(x, y) = \max_{1\leqslant i\leqslant n} d_i^\star(x_i, y_i) \qquad (3.2).$$

In [7, Theorem 4.3], Khatami and Mirzavaziri proved that the formulas (3.1) and (3.2) define two $\star$-metrics on the Cartesian product $X = \prod_{i=1}^n X_i$. Furthermore the topology induced by these two $\star$-metrics on X is the same as the product topology on X.

Theorem 3.7. *Let $\{(X_i, d_i^\star)\}_{i=1}^n$ be a family of finite nonempty $\star$-metric spaces and $X = \prod_{i=1}^n X_i$ the Cartesian product. Then:*

(1) X with the $\star$-metric $d_T^\star$ defined by formula (3.1) is totally bounded if and only if all $\star$-metric spaces $(X_i, d_i^\star)$ are totally bounded;

(2) X with the $\star$-metric $d_{max}^\star$ defined by formula (3.2) is totally bounded if and only if all $\star$-metric spaces $(X_i, d_i^\star)$ are totally bounded.

Proof. (1) Necessity. Assume that the $\star$-metric space $(X, d_T^\star)$ is totally bounded. The subset $X_m^* = \prod_{i=1}^n A_i$ of X , where $A_m = X_m$ and $A_i = \{x_i^*\}$ is a one-point subset of X_i for $i \neq m$. Then the subspace X_m^* is totally bounded by Theorem 3.4. One can easily verify that $p_m^* = p_m |_{X_m^*} \colon X_m^* \to X_m$ is a isometric isomorphism and according to the definition of $d_T^\star$, for $x^*, y^* \in X_m^* \subset X$, $d_T^\star(x^*, y^*) = d_m^\star(p_m(x^*), p_m(y^*))$. Therefore, if a finite set F is ϵ-dense in $(X_m^*, d_T^\star)$, then $p_m(F)$ is ϵ-dense in $(X_m, d_T^\star)$, and from this it follows further that $(X_m, d_T^\star)$ is totally bounded.

Sufficiency. Let every $(X_i, d_i^\star)$ be totally bounded. For $\epsilon > 0$, by Lemma 2.3, take an $\epsilon_1 > 0$ such that $\overbrace{\epsilon_1 \star \epsilon_1 \star \cdots \star \epsilon_1}^{n \text{ times}} < \epsilon$. For every $i \leqslant n$ take a finite set F_i which is ϵ_1-dense in X_i. We define that

$$F = \prod_{i=1}^n F_i,$$

then F is a finite set. To conclude the proof it suffices to show that F is ϵ-dense in the space $(X, d_T^\star)$. Let $x = (x_1, x_2, \ldots, x_n)$ be an arbitrary point of X. For every $i \leqslant n$, since F_i is ϵ-dense in X_i, there exists a $y_i \in F_i$ such that $d_i^\star(x_i, y_i) < \epsilon_1$ and take a point $y = (y_1, y_2, \ldots, y_n) \in F$ we have

$$d_T^\star(x, y) = d_1^\star(x_1, y_1) \star d_2^\star(x_2, y_2) \star \cdots \star d_n^\star(x_n, y_n) < \overbrace{\epsilon_1 \star \epsilon_1 \star \cdots \star \epsilon_1}^{n \text{ times}} < \epsilon.$$

By the foregoing, F is ϵ-dense in $(X, d_T^\star)$.

(2) Necessity. Assume that the $\star$-metric space $(X, d_{max}^\star)$ is totally bounded. Then the method of the proof is the same as that of necessity in (1).

Sufficiency. Let every $(X_i, d_i^\star)$ be totally bounded. For $\epsilon > 0$, take a finite set F_i which is ϵ-dense in X_i, for every $i \leqslant n$. We define that

$$F = \prod_{i=1}^n F_i.$$

Clearly, F is a finite set. To conclude the proof it suffices to show that F is ϵ-dense in the space $(X, d_{\max}^\star)$. Let $x = (x_1, x_2, \ldots, x_n)$ be an arbitrary point of X. For every $i \leqslant n$, there exists a $y_i \in F_i$ such that $d_i^\star(x_i, y_i) < \epsilon$ and take a point $y = (y_1, y_2, \ldots, y_n) \in F$. Without loss of generality, choose $\max_{1 \leqslant i \leqslant n} d_i^\star(x_i, y_i) = d_k^\star(x_k, y_k)$, then we have

$$d_{\max}^\star(x, y) = \max_{1 \leqslant i \leqslant n} d_i^\star(x_i, y_i) = d_k^\star(x_k, y_k) < \epsilon.$$

By the foregoing, F is ϵ-dense in $(X, d_{\max}^\star)$.

This completes the proof. □

Let (X, d) be a metric space. Define $\tilde{d}(x, y) = \min\{1, d(x, y)\}$ for each $x, y \in X$. It is well known that $\tilde{d}$ is a metric on X such that the topology induced by $\tilde{d}$ is the same as induced by d. For $\star$-metric space, we have the following result.

Proposition 3.8. *Let $(X, d^\star)$ be a $\star$-metric space and define $\tilde{d^\star}(x, y) = min\{1, d^\star(x, y)\}$ for each $x, y \in X$. Then $(X, \tilde{d^\star})$ is also a $\star$-metric space on X. Furthermore the topology induced by $\tilde{d^\star}$ on X is the same as induced by $d^\star$.*

Proof. We shall verify that $\tilde{d^\star}$ is a $\star$-metric. Clearly, $\tilde{d^\star}$ satisfies (M1) and (M2) in the definition 1.2. Suppose the contrary that there exist points $x, y, z \in X$ such that

$$1 \geqslant \tilde{d^\star}(x, z) > \tilde{d^\star}(x, y) \star \tilde{d^\star}(y, z).$$

Then, $\tilde{d^\star}(x, y) < 1$, $\tilde{d^\star}(y, z) < 1$ (since, if $\tilde{d^\star}(x, y) \geqslant 1$, then $\tilde{d^\star}(x, y) \star \tilde{d^\star}(y, z) \geqslant 1 \star 0 = 1$, i.e $\tilde{d^\star}(x, z) > 1$). Therefore

$$\tilde{d^\star}(x, y) \star \tilde{d^\star}(y, z) = d^\star(x, y) \star d^\star(y, z) \geqslant d^\star(x, z).$$

This implies that $\tilde{d^\star}(x,z) > d^\star(x,z)$, which is a contradiction with $\tilde{d^\star}(x,z) \leqslant d^\star(x,z)$. Thus $(X, \tilde{d^\star})$ is a $\star$-metric space.

For any $\epsilon > 0$, $x \in X$ we define

$$B_{d^\star}(x,\epsilon) = \{y \in X : d^\star(x,y) < \epsilon\}$$

and

$$B_{\tilde{d^\star}}(x,\epsilon) = \{y \in X : d^\star(x,y) < \epsilon\}.$$

Clearly, $B_{d^\star}(x,\epsilon) = B_{\tilde{d^\star}}(x,\epsilon)$ whenever $0 < \epsilon < 1$ for each $x \in X$. Thus the topology induced by $\tilde{d^\star}$ on X is the same as induced by $d^\star$. This completes the proof. $\square$

Let $\{(X_\alpha, d^\star_\alpha)\}_{\alpha\in A}$ be a family of $\star$-metric spaces and $X = \bigoplus_{\alpha\in A} X_\alpha$ be the disjoint union of $\{X_\alpha\}_{\alpha\in A}$. By Proposition 3.8, one can suppose that $d^\star_\alpha(x,y) \leqslant 1$ for $x, y \in X_\alpha$ and $\alpha \in A$. For every $x, y \in X$, we define

$$d^\star_q(x,y) = \begin{cases} d^\star_\alpha(x,y), & \text{if } x, y \in X_\alpha \text{ for some } \alpha \in A, \\ 1, & \text{otherwise.} \end{cases} \tag{3.3}$$

Then $(X, d^\star_q)$ is a $\star$-metric space.

Obviously, $d^\star_q$ satisfies conditions (M1) and (M2). It remains to show that condition (M3) $d^\star_q(x,z) \leqslant d^\star_q(x,y) \star d^\star_q(y,z)$ is also satisfied. Otherwise, if there exist points $x, y, z \in X$, such that $1 \geqslant d^\star_q(x,z) > d^\star_q(x,y) \star d^\star_q(y,z)$, then $d^\star_q(x,y) < 1$, $d^\star_q(y,z) < 1$. Since, if $d^\star_q(x,y) \geqslant 1$, then $d^\star_q(x,y) \star d^\star_q(y,z) \geqslant 1 \star 0 = 1$, i.e. $d^\star_q(x,z) > 1$. This implies that $d^\star_q(x,z) > d^\star_\alpha(x,z)$, which is a contradiction with $d^\star_q(x,z) \leqslant d^\star_\alpha(x,z)$. Thus, there exists an $\alpha \in A$ such that $x, y, z \in X_\alpha$, then we have

$$d^\star_q(x,y) \star d^\star_q(y,z) = d^\star_\alpha(x,y) \star d^\star_\alpha(y,z) \geqslant d^\star_\alpha(x,z) = d^\star_q(x,z),$$

which is a contradiction with $d^\star_q(x,z) > d^\star_q(x,y) \star d^\star_q(y,z)$.

One can easily show that for every $\alpha \in A$, the set X_α is open in the space X with the topology induced by $d^\star_q$. Since $d^\star_\alpha$ induces the topology on X_α, $d^\star_q$ induces the topology of the disjoint union of topological spaces $\{X_\alpha\}_{\alpha\in A}$ on X.

Theorem 3.9. *Let $\{(X_i, d^\star_i)\}_{i=1}^n$ be a family of $\star$-metric spaces such that the metric $d^\star_i$ is bounded by 1 for $1 \leqslant i \leqslant n$, and $X = \bigoplus_{1\leqslant i\leqslant n} X_i$ the disjoint union of $\{X_i\}_{i\leq n}$. Then $(X, d^\star_q)$ is totally bounded if and only if all spaces $(X_i, d^\star_i)$ are totally bounded, where $d^\star_q$ is defined as the formula* (3.3)*.*

Proof. Necessity. Assume that $(X, d^\star_q)$ is totally bounded. One can easily show that $(X_i, d^\star_i)$ is a subspace of $(X, d^\star_q)$. So, all spaces $(X_i, d^\star_i)$ are totally bounded by Theorem 3.4.

Sufficiency. Assume that all spaces $(X_i, d^\star_i)$ are totally bounded. Then, for $\epsilon > 0$, and for each $x \in X_i$, there exists a $y_0 \in F_i(\epsilon)$ such that $d^\star_i(x, y_0) < \epsilon$. Put $F(\epsilon) = \bigcup_{i=1}^n F_i(\epsilon)$ and let x be an arbitrary point of X. Obviously, x is also a point on some X_i. Thus, we can find a $y_0 \in F_i(\epsilon)$, such that $d^\star_q(x, y_0) = d^\star_i(x, y_0) < \epsilon$. So, $(X, d^\star_q)$ is totally bounded. $\square$

4. The completeness of $\star$-metric spaces

Completeness is an important property in metric spaces. The completeness of metric spaces depend on the convergence of Cauchy sequences. Therefore, we extend the definition of Cauchy sequences and completeness in metric spaces to $\star$-metric spaces. Further, we study complete properties of $\star$-metric spaces and give a characterization.

Definition 4.1. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of a $\star$-metric space $(X, d^\star)$, and $x \in X$. If for every $\epsilon > 0$, there exists a $k \in \mathbb{N}$ such that $d^\star(x, x_n) < \epsilon$ whenever $n \geqslant k$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to *converge to x under $d^\star$*, and we write $x_n \xrightarrow{d^\star} x$.

Proposition 4.2. *Let $(X, d^\star)$ be a $\star$-metric space. Then the following statements are equivalent:*

(1) $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0 under $\mathscr{T}_{d^\star}$;
(2) $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0 under $d^\star$.

Proof. (1) ⇒ (2) For every $\epsilon > 0$, clearly, $B_{d^\star}(x_0, \epsilon)$ is a neighborhood of x_0. Since $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0 under $\mathscr{T}_{d^\star}$, there exists a $k \in \mathbb{N}$ such that $x_n \in B_{d^\star}(x_0, \epsilon)$ whenever $n \geqslant k$, i.e. $d^\star(x_n, x_0) < \epsilon$. Therefore $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0 under $d^\star$.

(2) ⇒ (1) For any neighborhood U of the point x_0, there exists $\epsilon > 0$ such that $B_{d^\star}(x_0, \epsilon) \subset U$. Since $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0 under $d^\star$, there exists a $k \in \mathbb{N}$ such that $d^\star(x_n, x_0) < \epsilon$ whenever $n \geqslant k$, i.e. $x_n \in B_{d^\star}(x_0, \epsilon)$. Thus $x_n \in U$ whenever $n \geqslant k$. Therefore $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0 under $\mathscr{T}_{d^\star}$. □

Definition 4.3. Let $(X, d^\star)$ be a $\star$-metric space, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is called *Cauchy sequence* in $(X, d^\star)$ if for every $\epsilon > 0$ there exists a $k \in \mathbb{N}$ such that $d^\star(x_n, x_m) < \epsilon$ whenever $m, n \geqslant k$.

Proposition 4.4. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\star$-metric space $(X, d^\star)$. If $\{x_n\}_{n\in\mathbb{N}}$ has an accumulation point x_0, then $x_n \xrightarrow{d^\star} x$.*

Proof. For every $\epsilon > 0$, by Lemma 2.3, take an $\epsilon_1 > 0$ such that $\epsilon_1 \star \epsilon_1 < \epsilon$. Since $\{x_n\}$ is a Cauchy sequence, there exists a $k_1 \in \mathbb{N}$ such that $d^\star(x_n, x_m) < \epsilon_1$ whenever $m, n \geqslant k_1$. Noting that x_0 is an accumulation point of $\{x_n\}$, there exists a $k_2 \in \mathbb{N}$ such that $d^\star(x_0, x_{k_2}) < \epsilon_1$ and $k_2 \geq k_1$. Therefore, while $m \geqslant k_1$, we have

$$d^\star(x_0, x_m) \leqslant d^\star(x_0, x_{k_2}) \star d^\star(x_{k_2}, x_m) < \epsilon_1 \star \epsilon_1 < \epsilon.$$

This shows that $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0. □

Definition 4.5. A $\star$-metric space $(X, d^\star)$ is *complete* if every Cauchy sequence in $(X, d^\star)$ is convergent to a point of X.

Theorem 4.6. *A $\star$-metric space $(X, d^\star)$ is compact if and only if $(X, d^\star)$ is complete and totally bounded.*

Proof. Necessity. Let $(X, d^\star)$ be a compact $\star$-metric space. According to Corollary 3.3, $(X, d^\star)$ is totally bounded. According to Proposition 4.4, if a Cauchy sequence in space $(X, d^\star)$ has convergent subsequences, then this Cauchy sequence converges. Since compact $\star$-metric space is sequentially compact, which means that every sequence of points of X has a convergent subsequence. By Corollary 2.6, every Cauchy sequence in $(X, d^\star)$ is convergent to a point of X. This implies that $(X, d^\star)$ is complete.

Sufficiency. Let $(X, d^\star)$ be a complete and totally bounded $\star$-metric space. To conclude the proof it suffices to show that X is sequentially compact which implies that X is compact, by Corollary 2.6.

Let $\{x_n\}$ be any sequence in the $\star$-metric space $(X, d^\star)$. From the total boundedness of space X, there exists finite open-balls cover X with radius 1. At least one of the finite open-ball $B^1_{d^\star}$ contains infinite points x_n in sequence $\{x_n\}$. Let the set formed by the subscript n of x_n contained in $B^1_{d^\star}$ be N_1. Then N_1 is an infinite set, such that $x_n \in B^1_{d^\star}$ whenever $n \in N_1$. Then use finite open-balls to cover X with radius 1/2. Among these finite open-balls, there must be at least one open-ball $B^2_{d^\star}$ and an infinite subset N_2 of N_1, such that $x_n \in B^2_{d^\star}$ whenever $n \in N_2$. Generally speaking, taking the infinite subset N_k of the positive integer set, we can select an open-ball $B^{k+1}_{d^\star}$ with radius of 1/(k+1) and an infinite set $N_{k+1} \subset N_k$, such that $x_n \in B^{k+1}_{d^\star}$ whenever $n \in N_{k+1}$.

Take $n_1 \in N_1$, $n_2 \in N_2$, which $n_2 > n_1$. Generally speaking, when n_k has been taken, we can choose $n_{k+1} \in N_{k+1}$ such that $n_{k+1} > n_k$. Since for every N_k is an infinite set, the above method can be completed. For $i, j \geqslant k$, we have $n_i, n_j \in N_k$ such that $x_{n_i}, x_{n_j} \in B^k_{d^\star}$. This implies that $\{x_{n_k}\}$ is a Cauchy sequence, and it is convergent by completeness of $(X, d^\star)$. □

The *distance $D(x, A)$ from a point to a set A* in a $\star$-metric space $(X, d^\star)$ is defined as

$$D(A, x) = D(x, A) = \inf_{y\in A}\{d^\star(x, y)\},\ if\ A \neq \emptyset,\ and\ D(x, \emptyset) = D(\emptyset, x) = 1.$$

Proposition 4.7. *Let $(X, d^\star)$ be a $\star$-metric space, and $A \subset X$. Then $\overline{A} = \{x : D(A, x) = 0\}$.*

Proof. For any $x_0 \in \overline{A}$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that $x_n \xrightarrow{d^\star} x_0$. This implies that $d^\star(x_n, x_0) \to 0$. Since $0 \leqslant d^\star(x_0, A) \leqslant d^\star(x_n, x_0) \to 0$, we have that $d^\star(x_0, A) = 0$, which implies that $x_0 \in \{x : D(A, x) = 0\}$. Therefore $\overline{A} \subseteq \{x : D(A, x) = 0\}$.

Suppose the contrary that take $y \in \{x : D(A, x) = 0\}$ which satisfies $d^\star(y, A) = 0$ and $y \notin \overline{A}$. Then there exists $\epsilon_0 > 0$ such that $B_{d^\star}(y, \epsilon_0) \cap A = \emptyset$, which implies that $d^\star(y, A) \geqslant \epsilon_0$. This is a contradiction with $d^\star(y, A) = 0$. Thus $\overline{A} \supseteq \{x : D(A, x) = 0\}$.

This shows that $\overline{A} = \{x : D(A, x) = 0\}$. □

Definition 4.8. Let A be a subset of $\star$-metric space $(X, d^\star)$. We define $\delta(A) = \sup_{x,y\in A}\{d^\star(x, y)\}$ as the *diameter* of the set A; it can be finite or equal to ∞. We also define $\delta(\emptyset) = 0$.

Then Cantor theorem is an important characterization of complete metric spaces. Similarly, we extend the Cantor theorem in metric spaces into $\star$-metric spaces.

Theorem 4.9. *A $\star$-metric space is complete if and only if for every decreasing sequence $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ of non-empty closed subsets of space X, such that $\lim_{n\to\infty} \delta(F_n) = 0$, the intersection $\bigcap_{n=1}^\infty F_n$ is a one-point set.*

Proof. Necessity. Let $(X, d^\star)$ be a complete $\star$-metric space, and $F_1, F_2, \dots$ a sequence of non-empty closed subsets of X such that

$$\lim_{n\to\infty} \delta(F_n) = 0 \textit{ and } F_{n+1} \subset F_n \textit{ for } n = 1, 2, \dots$$

Choose $x_n \in F_n$, for every $n \in \mathbb{N}$. Now we shall prove that $\{x_n\}$ is a Cauchy sequence. According to $\lim_{n\to\infty} \delta(F_n) = 0$, for $\epsilon > 0$, there exists a $k \in \mathbb{N}$ such that $\delta(F_n) < \epsilon$ when $n > k$. Whenever $n \geqslant m > k$, we have $x_n \in F_n \subset F_m$, because $\{F_n\}$ is a decreasing sequence. Furthermore $x_m \in F_m$, so that

$$d^\star(x_n, x_m) < \delta(F_m) < \epsilon.$$

So, $\{x_n\}$ is a Cauchy sequence and thus is convergent to a point $x_0 \in X$. Thus, any neighborhood of x_0 intersects F_n $(n = 1, 2, \dots)$. The sets F_n being closed, we have $x_0 \in \bigcap_{n=1}^\infty F_n$

Now, we need prove $\bigcap_{n=1}^\infty F_n = \{x_0\}$. Take arbitrary point $y \in \bigcap_{n=1}^\infty F_n$. By $lim_{n\to\infty}\delta(F_n) = 0$, we can choose a $k \in \mathbb{N}$ such that $\delta(F_n) < \epsilon$ when $n > k$, Thus we have

$$d^\star(x_0, y) < \delta(F_n) < \epsilon, \textit{ where } x_0, y \in F_n.$$

This implies that $d^\star(x_0, y) = 0$. Thus we have $x_0 = y$.

Sufficiency. Let $\{x_n\}$ be a Cauchy sequence of $(X, d^\star)$. For every $k \in \mathbb{N}$, there exist points $l_k,\ r_k \in \mathbb{N}$ such that $d^\star(x_{l_k}, x_n) \leqslant \frac{1}{r_{k+1}}$ where $n > l_k$. Let x_{l_k} be the smallest positive integer with the above properties, so that $l_k \leqslant l_{k+1}, r_k \leqslant r_{k+1} (k = 1, 2, \dots)$. Construct the following sequence of closed subset $\{F_n\}$ which defined by letting

$$F_k = \overline{B_{d^\star}(x_{l_k}, \frac{1}{r_k})},\ (k = 1, 2, \dots),$$

where $B_{d^\star}(x_{l_k}, \frac{1}{r_k}) = \{y \in X : d^\star(x_{l_k}, y) \leqslant \frac{1}{r_k}\}$. By Proposition 4.7, we can get that $d^\star(F_k) \leqslant \frac{2}{r_k}$, which implies that $\lim_{n\to\infty} \delta(F_n) = 0$.

Now define by induction a subfamily $\{H_n\}$ of $\{F_n\}$. We define $H_1 = F_{k_1}, k_1 = 1$. Then take $H_2 = F_{k_2}$, by Lemma 2.3, we can set that $k_2 = \min\{j \geqslant 2 : \frac{1}{r_j} \star \frac{1}{r_j} < \frac{1}{r_{k_1}}\}$. Generally speaking, if we take the positive integer k_n, we can take $k_{n+1} = \min\{j \geqslant k_n + 1 : \frac{1}{r_j} \star \frac{1}{r_j} < \frac{1}{r_{k_n}}\}$, such that $H_{n+1} = F_{k_{n+1}}$. Now we shall show that $\{H_n\}$ satisfies the conditions in our theorem.

Let $y \in H_{n+1}$, then according to the selection method of l_k, we have

$$d^\star(y, x_{l_{k_{n+1}}}) < \frac{1}{r_{k_{n+1}}},\ d^\star(x_{l_{k_{n+1}}}, x_{l_{k_n}}) < \frac{1}{r_{k_n+1}},$$

thus

$$d^\star(y, x_{l_{k_n}}) \leqslant d^\star(y, x_{l_{k_{n+1}}}) \star d^\star(x_{l_{k_{n+1}}}, x_{l_{k_n}}) < \frac{1}{r_{k_{n+1}}} \star \frac{1}{r_{k_n+1}} < \frac{1}{r_{k_{n+1}}} \star \frac{1}{r_{k_{n+1}}} < \frac{1}{r_{k_n}}.$$

This implies that $y \in H_n$, i.e. $H_{n+1} \subset H_n$.

According to the assumption $\bigcap_{n=1}^{\infty} F_n = \{x_0\}$. Now we shall show that a Cauchy sequence $\{x_n\}$ is convergent to the point x_0. For every $\epsilon > 0$, by Lemma 2.3, take a $r_k \in \mathbb{N}$ such that $\frac{1}{r_k} \star \frac{1}{r_k} < \epsilon$, and $d^\star(x_{l_k}, x_n) \leqslant \frac{1}{r_{k+1}}$. In addition, $x_0 \in F_k$, $d^\star(x_{l_k}, x_0) \leqslant \frac{1}{r_k}$, so we have

$$d^\star(x_0, x_n) \leqslant d^\star(x_0, x_{l_k}) \star d^\star(x_{l_k}, x_n) < \frac{1}{r_k} \star \frac{1}{r_{k+1}} < \frac{1}{r_k} \star \frac{1}{r_k} < \epsilon.$$

Thus, $\lim_{n\to\infty} x_n = x_0$, this shows that $(X, d^\star)$ is a complete $\star$-metric space. □

Theorem 4.10. *A $\star$-metric space is complete if and only if every family of closed subsets of X which has the finite intersection property and for every $\epsilon > 0$ contains a set of diameter less than ϵ has non-empty intersection.*

Proof. Sufficiency of the condition in our theorem for completeness of a $\star$-metric space follows from the Theorem 4.9.

We shall show that the condition holds in every complete $\star$-metric space $(X, d^\star)$. Consider a family $\{F_s\}_{s\in S}$ of closed subsets of X which has the finite intersection property and which for every $j \in \mathbb{N}$ contains a set F_{s_j}, such that $\delta(F_{s_j}) < \frac{1}{j}$. Let $F_i = \bigcap_{j\leqslant i} F_{s_j}$. One easily sees that the sequence $F_1, F_2, \dots$ satisfies the condition of the Cantor theorem, since $F_{n+1} \subset F_n$ and $\delta(F_i) \leqslant \delta(F_{s_i}) < \frac{1}{i}$ which means $\lim_{n\to\infty} \delta(F_n) = 0$. So that there exists an $x \in \bigcap_{i=1}^{\infty} F_i$. Clearly, we have $\bigcap_{i=1}^{\infty} F_i = \{x\}$. Now, let us take an arbitrary $s_0 \in S$; letting $F_i' = F_{s_0} \cap F_i$ for $i = 1, 2, \dots$ we obtain again a sequence $F_1', F_2', \dots$ satisfying the conditions of the Theorem 4.9. Since

$$\emptyset \neq \bigcap_{i=1}^{\infty} F_i' = F_{s_0} \cap \bigcap_{i=1}^{\infty} F_i = F_{s_0} \cap \{x\},$$

we have $x \in F_{s_0}$. Hence $x \in \bigcap_{s\in S} F_s$. □

Theorem 4.11. *A subspace $(M, d^\star)$ of a complete $\star$-metric space $(X, d^\star)$ is complete if and only if M is closed in X.*

Proof. Necessity. Let $x \in \overline{M}$, and we define $F_k = M \cap \overline{B_{d^\star}(x, \frac{1}{k})}(k = 1, 2, \dots)$, then sequence $\{F_k\}$ is non-empty closed subsets in subspace M, so one can easily check that $\{F_k\}$ satisfies the conditions (1) *and* (2) in the Theorem 4.9. Since subspace $(M, d^\star)$ is complete, by Theorem 4.9, obviously $\bigcap_{k=1}^{\infty} F_k = \{x\}$, it follows that $x \in M$. Therefore $M = \overline{M}$.

Sufficiency. Let M be a closed set, every Cauchy sequence of $\star$-metric space $(M, d^\star)$ is also a Cauchy sequence of complete $\star$-metric space $(X, d^\star)$, so it converges to a certain point $x \in X$. Since M is closed in X, $x \in M$. This completes the proof. □

The following theorem shows that in a class of $\star$-metric spaces, the completeness is preserved by finite products.

Theorem 4.12. *Let $\{(X_i, d_i^\star)\}_{i=1}^{n}$ be a family of finite nonempty $\star$-metric spaces and $X = \prod_{i=1}^{n} X_i$ the Cartesian product. Then*

(1) X with the $\star$-metric $d_T^\star$ defined by formula (3.1) is complete if and only if all $\star$-metric spaces $(X_i, d_i^\star)$ are complete;

(2) X with the $\star$-metric $d_{max}^\star$ defined by formula (3.2) is complete if and only if all $\star$-metric spaces $(X_i, d_i^\star)$ are complete.

Proof. (1) Assume that the space $(X, d_T^\star)$ is complete. For every subspace $X_m^* = \prod_{i=1}^n A_i$ of X, where $A_m = X_m$ and $A_i = \{x_i^*\}$ is a one-point subset of X_i for $i \neq m$, is closed in $(X, d_T^\star)$. Then the subspace X_m^* is complete by Theorem 4.11. One can easily verify that $p_m^* = p_m \mid_{X_m^*}: X_m^* \to X_m$ is a isometric isomorphism, since $d_T^\star|_{X_m^*}(p_m^*(x), p_m^*(y)) = d_2^\star(x, y)$. Therefore, for every Cauchy sequence $\{x_n\}$ in $(X_m, d_m^\star)$, the sequence $\{{p_m^*}^{-1}(x_n)\}$ is a Cauchy sequence in X_m^*. Then

$$p_m^*(\lim_{n\to\infty} {p_m^*}^{-1}(x_n)) = \lim_{n\to\infty} x_n,$$

so that the space $(X_m, d_m^\star)$ is complete.

Assume that all spaces $(X_i, d_i^\star)$ are complete. Take any Cauchy sequence $\{y_k\}_{k\in\mathbb{N}}$ in $(X, d_T^\star)$, where $y_k = (x_i^k)$, for $1 \leqslant i \leqslant n$. Then the sequence $\{x_i^k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $(X_i, d_i^\star)$ and thus converges to a point $x_i^0 \in X_i$. Now, we shall show that $\{y_k\}_{k\in\mathbb{N}}$ converges to a point $x^0 = (x_i^0)$. For $\epsilon > 0$, by Lemma 2.3, take an $\epsilon_1 > 0$ such that $\overbrace{\epsilon_1 \star \epsilon_1 \star \cdots \star \epsilon_1}^{n \text{ times}} < \epsilon$. Since $\{x_i^k\}_{k\in\mathbb{N}}$ converges to a point x_i^0, there exists $m_i \in \mathbb{N}$, such that $d_i^\star(x_i^k, x_i^0) < \epsilon_1$, where $k \geqslant m_i$. Thus choose $m = \max_{1\leqslant i\leqslant n}\{m_i\}$, such that

$$d_T^\star(y_k, x^0) = d_1^\star(x_1^k, x_1^0) \star d_2^\star(x_2^k, x_2^0) \star \cdots \star d_n^\star(x_n^k, x_n^0) < \overbrace{\epsilon_1 \star \epsilon_1 \star \cdots \star \epsilon_1}^{n \text{ times}} < \epsilon,$$

whenever $k \geqslant m$. We have shown that $(X, d_T^\star)$ is complete.

(2) Assume that the space $(X, d_T^\star)$ is complete. The method of proof is the same as (1).

Assume that all spaces $(X_i, d_i^\star)$ are complete. Take any Cauchy sequence $\{y_k\}_{k\in\mathbb{N}}$ in $(X, d_{\max}^\star)$, where $y_k = (x_i^k)$, for $1 \leqslant i \leqslant n$. Then the sequence $\{x_i^k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $(X_i, d_i^\star)$ and thus converges to a point $x_i^0 \in X_i$. Now, we shall show that $\{y_k\}_{k\in\mathbb{N}}$ converges to a point $x^0 = (x_i^0)$. Since $\{x_i^k\}_{k\in\mathbb{N}}$ converges to a point x_i^0. For every $\epsilon > 0$ there exists $m_i \in \mathbb{N}$, such that $d_i^\star(x_i^k, x_i^0) < \epsilon$, where $k \geqslant m_i$. Without loss of generality, let $\max_{1\leqslant i\leqslant n} d_i^\star(x_i^k, x_i^0) = d_j^\star(x_j^k, x_j^0)$, then while $k \geqslant m = m_j$, such that

$$d_{\max}^\star(y_k, x^0) = \max_{1\leqslant i\leqslant n} d_i^\star(x_i^k, x_i^0) = d_j^\star(x_j^k, x_j^0) < \epsilon.$$

We have shown that $(X, d_{\max}^\star)$ is complete. $\square$

Theorem 4.13. *If $\{(X_\alpha, d_\alpha^\star)\}_{\alpha\in A}$ is a family of $\star$-metric spaces such that the $\star$-metric $d_\alpha^\star$ is bounded for each $\alpha \in A$, and $X = \bigoplus_{\alpha\in A} X_\alpha$ be the disjoint union of $\{X_\alpha\}$. Then X with the $\star$-metric $d_q^\star$ defined by formula (3.3) is complete if and only if all spaces $(X_\alpha, d_\alpha^\star)$ are complete.*

Proof. Necessity. Assume that $(X, d_q^\star)$ is complete. Then it is easy to see that all sets X_α are open-and-closed in X. So, all spaces $(X_\alpha, d_\alpha^\star)$ are complete by Theorem 4.11.

Sufficiency. Assume that all spaces $(X_\alpha, d_\alpha^\star)$ are complete. Then $(X, d_q^\star)$ is complete, because every Cauchy sequence of $\star$-metric space $(X_\alpha, d_\alpha^\star)$ is also a Cauchy sequence of $(X, d_q^\star)$ and it converges to a certain point $x \in X_\alpha \subseteq X$. $\square$

The Baire theorem is a very important result in complete metric spaces. We shall extend this theorem to complete $\star$-metric spaces.

Theorem 4.14. *In a complete $\star$-metric space $(X, d^\star)$ the intersection $A = \bigcap_{n=1}^\infty A_n$ of a sequence $A_1, A_2, \ldots$ of dense open subsets is a dense set.*

Proof. Let $A = \bigcap_{n=1}^\infty A_n$, for every A_n an open dense subset of complete $\star$-metric space $(X, d^\star)$. Now, construct the sequence of closed subset $\{F_n\}$ which satisfies conditions in Theorem 4.9. Since A_1 is dense in X, and U is a non-empty open set, then $A_1 \cap U \neq \emptyset$. Take $x_1 \in A_1 \cap U$, since $A_1 \cap U$ is an open set, there exists ϵ_1 which satisfies $0 < \epsilon_1 < 1/2^2$, such that $\overline{B_{d^\star}(x_1, \epsilon_1)} \subset A_1 \cap U$. Since A_2 is dense in X, and $B_{d^\star}(x_1, \epsilon_1)$ is an open set, then $A_2 \cap B_{d^\star}(x_1, \epsilon_1) \neq \emptyset$. Take $x_2 \in A_2 \cap B_{d^\star}(x_1, \epsilon_1)$, since $A_2 \cap B_{d^\star}(x_1, \epsilon_1)$ is an open set, there exists

ϵ_2 which satisfies $0 < \epsilon_2 < \epsilon_1/2$, such that $\overline{B_{d^\star}(x_2, \epsilon_2)} \subset A_2 \cap B_{d^\star}(x_1, \epsilon_1)$. Obviously, $\overline{B_{d^\star}(x_2, \epsilon_2)} \subset \overline{B_{d^\star}(x_1, \epsilon_1)}$ and $\overline{B_{d^\star}(x_2, \epsilon_2)} \subset A_2 \cap U$. Going on, one can easily obtain the sequence of closed subset $\{F_n\} = \{\overline{B_{d^\star}(x_n, \epsilon_n)}\}$ which satisfies $F_{n+1} \subset F_n$ and $\delta(F_n) \leqslant 1/2^n$ (n=1,2,...). This implies that $\{F_n\}$ satisfies conditions in Theorem 4.9. Noting that $F_n \subset A_n \cap U$, by Theorem 4.9, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$, then we have

$$A \cap U = (\bigcap_{n=1}^{\infty} A_n) \cap U = \bigcap_{n=1}^{\infty} (A_n \cap U) \supset \bigcap_{n=1}^{\infty} F_n \neq \emptyset,$$

this implies that A is dense in X. □

Every metric space is isometric to a subspace of a complete metric space. It would be interesting to find out whether this result remain valid in the class of $\star$-metric spaces:

Problem 4.15. *Is every $\star$-metric space isometric to a subspace of a complete $\star$-metric space?*

Acknowledgments

We wish to thank the referees for the detailed list of comments, suggestions to the paper, and all their efforts in order to improve the paper.

References

[1] P.S. Alexandroff, V.V. Niemytzki, The condition of metrizability of topological spaces and the axiom of symmetry (in Russian), Sb. Math. 3(45) (1938) 663–672.
[2] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
[3] G.S. Gao, Topological Space Theory (2nd Edition), Science Press, Beijing, 2008.
[4] A. George, P. Veeramani, On some result in fuzzy metric space, Fuzzy Sets Syst. 64 (1994) 395–399.
[5] V. Gregori, R. Salvador, Fuzzy quasi metric spaces, Appl. Gen. Topol. 1 (2004) 129–136.
[6] M. Jleli, B. Samet, On a new generalization of metric spaces, J. Fixed Point Theory Appl. 20 (2018) 1–20.
[7] S.M.A. Khatami, M. Mirzavaziri, Yet another generalization of the notion of a metric space, arXiv:2009.00943v1 [math.GN], 2020.
[8] M. Mirzavaziri, Function valued metric spaces, Surveys Math. Appl. 5 (2010) 321–332.
[9] J.R. Munkres, Topology (2nd Edition), Prentice Hall, New Jersey, 2000.
[10] W.H. Schikhof, Ultrametric Calculus, Cambridge University Press, Cambridge, 1985.
[11] W. Shatanawi, A. Pitea, Some coupled fixed point theorems in quasi-partial metric spaces, Fixed Point Theorey Appl. 153 (2013).
[12] W.A. Wilson, On quasi-metric spaces, Amer. J. Math. 53 (1931) 675–684.
[13] W.A. Wilson, On semi-metric spaces, Amer. J. Math. 53 (1931) 361–373.