



# Characterisations of Bounded Linear and Compact Operators On the Generalised Hahn Space

Diana Dolićanin-Đekić<sup>a</sup>, Ersin Gilić<sup>b</sup>

<sup>a</sup>University of Priština in Kosovska Mitrovica, Faculty of Technical Sciences, Department of Mathematics and Physics, Kneza Miloša 7, 38220 Kosovska Mitrovica, Serbia

<sup>b</sup>State University of Novi Pazar, Department of Mathematics, Vuka Karadžića bb, 36300 Novi Pazar, Serbia

**Abstract.** We establish the characterisations of the classes of bounded linear operators from the generalised Hahn sequence space  $h_d$ , where  $d$  is an unbounded monotone increasing sequence of positive real numbers, into the spaces  $w_0$ ,  $w$  and  $w_\infty$  of sequences that are strongly summable to zero, strongly summable and strongly bounded by the Cesàro method of order one. Furthermore, we prove estimates for the Hausdorff measure of noncompactness of bounded linear operators from  $h_d$  into  $w$ , and identities for the Hausdorff measure of noncompactness of bounded linear operators from  $h_d$  to  $w_0$ , and use these results to characterise the classes of compact operators from  $h_d$  to  $w$  and  $w_0$ . Finally, we provide an example for an application of our results.

## 1. Introduction and notations

The Hahn space  $h$  was originally introduced and studied by Hahn [9] in connection with the theory of singular integrals, and generalised by Goes [8]. More recently, a number of papers have been published on matrix transformations on the Hahn space. Many of these results were used in the determination of various types of spectra of operators from the Hahn space into itself, for instance in [5, 13].

Here, we characterise, for the first time, the classes of bounded linear operators from the generalised Hahn space  $h_d$  into the spaces  $w_0$ ,  $w$  and  $w_\infty$ . Since each one of these operators can be represented by an infinite matrix of complex numbers, these characterisations are achieved by establishing necessary and sufficient conditions on the entries of the matrices to map between the respective spaces. We also derive a formula for the operator norm in each case.

We also establish, for the first time, formulas for the Hausdorff measure of noncompactness of bounded linear operators from  $h_d$  into  $w$  and  $w_0$ , and use these results to characterise the respective subclasses of compact operators. Finally, we present an example of our results, in which we show that the operator of the arithmetic means maps  $h_d$  into  $w_0$ , determine its operator norm, and show that the operator is compact.

Measures of noncompactness play an important role in fixed point theory; they are extensively used in the study of differential and integral equations. We refer the interested reader to [1–3, 17, 18, 28]. Our results

---

2020 Mathematics Subject Classification. 46A45, 40C05, 46B45, 47H08

Keywords. Generalised Hahn sequence space; bounded linear operators; Hausdorff measure of noncompactness; compact operators

Received: 01 May 2021; Accepted: 31 May 2021

Communicated by Vladimir Rakočević

The Ministry of Education, Science and Technological Development of the Republic of Serbia supported the work of first author under contracts 174005 and 174024.

Email addresses: [diana.dolicanin@pr.ac.rs](mailto:diana.dolicanin@pr.ac.rs) (Diana Dolićanin-Đekić), [egilic@np.ac.rs](mailto:egilic@np.ac.rs) (Ersin Gilić)

could also be used in the study of sequence spaces equations and sequence spaces inclusion relations; for related results we refer to [7].

We use the standard notations  $\omega$  for the set of all complex sequences  $x = (x_k)_{k=1}^\infty$ , and  $\ell_\infty, c, c_0$  and  $\phi$  for the sets of all bounded, convergent, null and finite sequences, that is, sequences terminating in zeros. We also write  $e = (e_k)_{k=1}^\infty$  and  $e^{(n)} = (e_k^{(n)})_{k=1}^\infty$  ( $n \in \mathbb{N}$ ) for the sequences with  $e_k = 1$  for all  $k$ , and  $e_n^{(n)} = 1$  and  $e_k^{(n)} = 0$  for  $k \neq n$ .

We recall that a BK space  $X$  is a Banach sequence space with continuous coordinates  $P_n : X \rightarrow \mathbb{C}$  ( $n \in \mathbb{N}$ ), where  $P_n(x) = x_n$  for all  $x = (x_k)_{k=1}^\infty \in X$ . A BK space  $X \supset \phi$  is said to have AK if  $x = \lim_{m \rightarrow \infty} x^{[m]}$  for all  $x = (x_k)_{k=1}^\infty \in X$ , where  $x^{[m]} = \sum_{k=1}^m x_k e^{(k)}$  denotes the  $m$ -section of the sequence  $x$ . It is well known that the sets  $\ell_\infty, c$ , and  $c_0$  are BK spaces with their natural norms  $\|x\|_\infty = \sup_k |x_k|$ ,  $c_0$  has AK, every sequence  $x = (x_k)_{k=1}^\infty \in c$  has a unique representation  $x = \xi e + \sum_{k=1}^\infty (x_k - \xi) e^{(k)}$ , where  $\xi = \lim_{k \rightarrow \infty} x_k$ , and finally,  $\ell_\infty$  is not separable and consequently has no Schauder basis.

Let  $X \subset \omega$ . Then the set  $X^\beta = \{a \in \omega : \sum_{k=1}^\infty a_k x_k \text{ converges for all } x \in X\}$  is the  $\beta$ -dual of  $X$ . Let  $A = (a_{nk})_{n,k=1}^\infty$  be an infinite matrix of complex numbers,  $A_n = (a_{nk})_{k=1}^\infty$  and  $A^k = (a_{nk})_{n=1}^\infty$  be the sequences in the  $n^{\text{th}}$  row and the  $k^{\text{th}}$  column of  $A$ , and  $X$  and  $Y$  be subsets of  $\omega$ . Then we write  $A_n x = \sum_{k=1}^\infty a_{nk} x_k$  and  $Ax = (A_n x)_{n=1}^\infty$  for  $x = (x_k)_{k=1}^\infty$  provided all the series converge. The set  $X_A = \{x \in \omega : Ax \in X\}$  is called the *matrix domain of  $A$  in  $X$* , and  $(X, Y)$  denotes the class of all matrix transformations from  $X$  into  $Y$ , that is,  $A \in (X, Y)$  if and only if  $X \subset Y_A$ .

The reader interested in the theory of sequence spaces and matrix transformations is referred to the monographs [4, 11, 19, 27, 29, 30].

If  $X$  and  $Y$  are Banach spaces, we use the standard notation  $\mathcal{B}(X, Y)$  for the Banach space of all bounded linear operators  $L : X \rightarrow Y$  with the operator norm  $\|L\| = \sup\{\|L(x)\| : \|x\| = 1\}$ ; the space  $X^* = \mathcal{B}(X, \mathbb{C})$  is called the continuous dual of  $X$ ; its norm is  $\|f\| = \sup\{|f(x)| : \|x\| = 1\}$  for all  $f \in X^*$ . Also  $\mathcal{K}(X, Y)$  denotes the class of all compact operators in  $\mathcal{B}(X, Y)$ .

The following well-known result gives the relation between  $(X, Y)$  and  $\mathcal{B}(X, Y)$ .

**Proposition 1.1.** *Let  $X$  and  $Y$  be BK spaces.*

(a) *If  $A \in (X, Y)$ , then  $L_A \in \mathcal{B}(X, Y)$ , where  $L_A(x) = Ax$  for all  $x \in X$ , that is, matrix maps between BK spaces are continuous ([29, Theorem 4.2.8]).*

(b) *If  $X$  has AK, then every operator  $L \in \mathcal{B}(X, Y)$  can be represented by a matrix  $A \in (X, Y)$  such that*

$$Ax = L(x) \text{ for all } x \in X \text{ ([10, Theorem 1.9]).} \tag{1.1}$$

The operator  $\Delta : \omega \rightarrow \omega$  of the so-called forward differences is defined by  $\Delta x_k = x_k - x_{k+1}$  ( $k = 1, 2, \dots$ ). The set  $h = \{x \in \omega : \sum_{k=1}^\infty k|\Delta x_k| < \infty\} \cap c_0$  was defined by Hahn in 1922 ([9]) in connection with the theory of singular integrals; Hahn showed that  $h$  is a BK space with  $\|x\|' = \sum_{k=1}^\infty k|\Delta x_k| + \sup_k |x_k|$  for all  $x = (x_k)_{k=1}^\infty \in h$ . Rao [24] showed that the Hahn space is a BK space with AK with the norm  $\|x\| = \sum_{k=1}^\infty k|\Delta x_k|$  for all  $x = (x_k)_{k=1}^\infty \in h$ .

Goes [8] introduced and studied the generalised Hahn space  $h_d$  for arbitrary complex sequences  $d = (d_k)_{k=1}^\infty$  with  $d_k \neq 0$  for all  $k$  by  $h_d = \{x \in \omega : \sum_{k=1}^\infty |d_k| \cdot |\Delta x_k| < \infty\} \cap c_0$  with the norm

$$\|x\|_d = \sum_{k=1}^\infty |d_k| \cdot |\Delta x_k| \text{ for all } x = (x_k)_{k=1}^\infty \in h_d. \tag{1.2}$$

The following result is known.

**Proposition 1.2.** *Let  $d$  be a increasing unbounded sequence of positive reals.*

(a) *Then  $h_d$  with the norm in (1.2) is a BK space with AK ([20, Proposition 2.1]).*

(b) *We write*

$$bs_d = \left\{ a \in \omega : \sup_n \frac{1}{d_n} \left| \sum_{k=1}^n a_k \right| \right\} \text{ and } \|a\|_{bs_d} = \sup_n \frac{1}{d_n} \left| \sum_{k=1}^n a_k \right| \text{ for all } a \in bs_d.$$

*Then  $h_d^\beta = bs_d$  and  $h_d^\beta$  and  $h_d^*$  are norm isomorphic ([20, Proposition 2.3]).*

Recent research on the Hahn space and its generalisations can be found, for instance, in [6, 12, 23, 25, 26] and the survey paper [13].

Let

$$w_0 = \left\{ x \in \omega : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |x_k| = 0 \right\},$$

$$w = w_0 \oplus e = \{x \in \omega : x - \xi e \in w_0 \text{ for some } \xi \in \mathbb{C}\}$$

and

$$w_\infty = \left\{ x \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^n |x_k| < \infty \right\}$$

denote the sets of sequences that are strongly summable  $C_1$  to zero, strongly summable  $C_1$  and strongly bounded  $C_1$ , first introduced by Maddox in [14], and later studied in [16, 21, 22].

The following result is well-known.

**Proposition 1.3.** ([17, Proposition 3.44]) *The sets  $w_0$ ,  $w$  and  $w_\infty$  are BK spaces with*

$$\|x\|_{w_\infty} = \sup_n \frac{1}{n} \sum_{k=1}^n |x_k|;$$

$w_0$  is a closed subspace of  $w$ , and  $w$  is a closed subspace of  $w_\infty$ ;  $w_0$  has AK and every sequence  $x = (x_k)_{k=1}^\infty \in w$  has a representation

$$x = \xi e + \sum_{k=1}^\infty (x_k - \xi) e^{(k)}, \tag{1.3}$$

where  $\xi$  is the unique complex number such that  $x - \xi e \in w_0$ , the so-called  $w$ -limit of  $x$ .

In this paper, we characterise the classes  $\mathcal{B}(h_d, w_0)$ ,  $\mathcal{B}(h_d, w)$  and  $\mathcal{B}(h_d, w_\infty)$ , when  $d$  is monotone increasing unbounded sequence of positive real numbers. Furthermore, we establish estimates for the Hausdorff measure of noncompactness of operators in the class  $\mathcal{B}(h_d, w)$ , and identities for the Hausdorff measure of noncompactness of operators in the class  $\mathcal{B}(h_d, w_0)$ . Finally, we characterise the classes  $\mathcal{K}(h_d, w)$  and  $\mathcal{K}(h_d, w_0)$ .

## 2. The classes $\mathcal{B}(h_d, Y)$ for $Y \in \{w_\infty, w, w_0\}$

Throughout let  $d$  be an unbounded increasing sequence of positive real numbers.

We are going to characterise the classes  $\mathcal{B}(h_d, Y)$  and compute the operator norm of  $L \in \mathcal{B}(h_d, Y)$  for  $Y \in \{w_\infty, w, w_0\}$ . Since  $h_d$  is a BK space with AK by Proposition 1.2 (a), and each space  $Y$  is a BK space by Proposition 1.3, each operator  $L \in \mathcal{B}(h_d, Y)$  can be represented by a matrix  $A \in (h_d, Y)$  as in (1.1) by Proposition 1.1 (b). We will use this fact and notation throughout the paper.

We need the following definition and results which we state here for the reader's convenience.

**Definition 2.1.** ([29, Definition 7.4.2])

Let  $X$  be a BK space and  $\bar{B}_X$  denote the closed unit ball in  $X$ . A subset  $E$  of the set  $\phi$  called a determining set for  $X$  if  $D(X) = \bar{B}_X \cap \phi$  is the absolutely convex hull of  $E$ .

**Proposition 2.2.** ([29, Theorem 8.3.4]) *Let  $X$  be a BK space with AK,  $E$  be a determining set for  $X$ , and  $Y$  be a BK space. Then  $A \in (X, Y)$  if and only if:*

(i) The columns of  $A$  belong to  $Y$ , that is,  $A^k = (a_{nk})_{n=1}^\infty \in Y$  for all  $k$ ,

and

(ii)  $L(E)$  is a bounded subset of  $Y$ , where  $L(x) = Ax$  for all  $x \in X$ .

**Proposition 2.3.** ([20, Proposition 3.2]) *The set*

$$E = \left\{ \frac{1}{d_m} \cdot e^{[m]} : m \in \mathbb{N} \right\} \tag{2.1}$$

is a determining set for  $h_d$ .

**Theorem 2.4.** *We have*

(a)  $L \in \mathcal{B}(h_d, w_\infty)$  if and only if

$$\|A\|_{(h_d, w_\infty)} = \sup_{l, m} \frac{1}{l} \sum_{n=1}^l \frac{1}{d_m} \left| \sum_{k=1}^m a_{nk} \right| < \infty; \tag{2.2}$$

(b)  $L \in \mathcal{B}(h_d, w)$  if and only if (2.2) holds and

$$\left\{ \begin{array}{l} \text{for each } k \in \mathbb{N}, \text{ there exists } \alpha_k \in \mathbb{C} \text{ such that} \\ \lim_{l \rightarrow \infty} \frac{1}{l} \sum_{n=1}^l |a_{nk} - \alpha_k| = 0; \end{array} \right\} \tag{2.3}$$

(c)  $L \in \mathcal{B}(h_d, w_0)$  if and only if (2.2) holds and

$$\lim_{l \rightarrow \infty} \frac{1}{l} \sum_{n=1}^l |a_{nk}| = 0 \text{ for each } k. \tag{2.4}$$

(d) If  $L \in \mathcal{B}(h_d, Y)$  for  $Y \in \{w_\infty, w, w_0\}$ , then

$$\|L\| = \|A\|_{(h_d, w_\infty)}. \tag{2.5}$$

*Proof.* (a) Let  $L \in \mathcal{B}(h_d, w_\infty)$ .

Since the set  $E$  in (2.1) is a determining set for  $h_d$  by Proposition 2.3, we apply Proposition 2.2, and show that the matrix  $A$  that represents  $L$  satisfies the conditions in (i) and (ii) of Proposition 2.2.

Let  $m \in \mathbb{N}$  be given and  $y^{(m)} = (1/d_m)e^{[m]} \in E$ . Then we have

$$A_n y^{(m)} = \sum_{k=1}^\infty a_{nk} y_k^{(m)} = \frac{1}{d_m} \sum_{k=1}^m a_{nk},$$

hence

$$\|A y^{(m)}\|_{w_\infty} = \sup_l \frac{1}{l} \sum_{n=1}^l |A_n y^{(m)}| = \sup_l \frac{1}{l} \sum_{n=1}^l \frac{1}{d_m} \left| \sum_{k=1}^m a_{nk} \right|.$$

So (2.2) is the condition in (ii) of Proposition 2.2.

It remains to show that the condition in (i) of Proposition 2.2 is redundant.

We have  $|a_{nk}| = |d_k A_n y^{(k)} - d_{k-1} A_n y^{(k-1)}|$  for all  $n$  and  $k$ , hence

$$\|A^k\|_{w_\infty} = \sup_l \frac{1}{l} \sum_{n=1}^l |a_{nk}| \leq \sup_l \frac{1}{l} \sum_{n=1}^l d_k |A_n y^{(k)}| + \sup_l \frac{1}{l} \sum_{n=1}^l d_{k-1} |A_n y^{(k-1)}|$$

$$\leq 2d_k \cdot \|A\|_{(h_d, w_\infty)} < \infty \text{ for all } k.$$

This completes the proof of Part (a).

(b) and (c) Since  $h_d$  is a BK space with AK and  $w_0$  and  $w$  are closed subspace of the BK space  $w_\infty$  by Proposition 1.3, Parts (b) and (c) follow by [29, Theorem 8.3.6].

(d) Finally we assume that  $L \in \mathcal{B}(h_d, Y)$ , where  $Y \in \{w_0, w, w_\infty\}$ . Then  $A_n \in h_d^\beta$  for all  $n$  and  $h_d^\beta = bs_d$  by Proposition 1.2 (b).

First we show that we obtain for  $A_n = L_n(x)$  ( $x \in h_d$ )

$$|A_n x| \leq \sum_{k=1}^{\infty} d_k |\Delta x_k| \frac{1}{d_k} \left| \sum_{j=1}^k a_{nj} \right| \text{ for all } n \in \mathbb{N} \text{ and all } x \in h_d. \tag{2.6}$$

To prove (2.6), let  $m \in \mathbb{N}$  be given. Then Abel’s summation by parts yields

$$\begin{aligned} L_n(x^{[m]}) &= A_n x^{[m]} = \sum_{k=1}^m a_{nk} x_k = \sum_{k=1}^{m-1} \Delta x_k \sum_{j=1}^k a_{nj} + x_m \sum_{j=1}^m a_{nj} \\ &= \sum_{k=1}^{m-1} d_k \Delta x_k \frac{1}{d_k} \sum_{j=1}^k a_{nj} + d_m x_m \frac{1}{d_m} \sum_{j=1}^m a_{nj} \end{aligned}$$

Since  $h_d$  has AK and  $x \in h_d$ , it follows that

$$\begin{aligned} 0 \leq |d_m x_m| &= \sum_{k=m}^{\infty} d_k |\Delta x_k^{[m]}| \leq \sum_{k=1}^{\infty} d_k \left| \Delta (x_k^{[m]} - x_k) \right| + \sum_{k=m}^{\infty} d_k |\Delta x_k| \\ &= \|x^{[m]} - x\|_{h_d} + \sum_{k=m}^{\infty} d_k |\Delta x_k| \rightarrow 0 \text{ (} m \rightarrow \infty \text{)}. \end{aligned}$$

Since  $A_n \in bs_d$ , the continuity of  $L_n$  yields

$$|A_n x| = |L_n(x)| = \lim_{m \rightarrow \infty} |L_n(x^{[m]})| \leq \sum_{k=1}^{\infty} d_k |\Delta x_k| \frac{1}{d_k} \left| \sum_{j=1}^k a_{nj} \right| < \infty,$$

which is (2.6).

Now (2.6) yields for all  $l \in \mathbb{N}$  and all  $x \in h_d$

$$\begin{aligned} \frac{1}{l} \sum_{n=1}^l |A_n x| &\leq \frac{1}{l} \sum_{n=1}^l \sum_{k=1}^{\infty} d_k |\Delta x_k| \frac{1}{d_k} \left| \sum_{j=1}^k a_{nj} \right| = \sum_{k=1}^{\infty} d_k |\Delta x_k| \left( \frac{1}{ld_k} \sum_{n=1}^l \left| \sum_{j=1}^k a_{nj} \right| \right) \\ &\leq \sup_{l,k} \left( \frac{1}{ld_k} \sum_{n=1}^l \left| \sum_{j=1}^k a_{nj} \right| \right) \|x\|_{h_d} = \|A\|_{(h_d, w_\infty)} \|x\|_d, \end{aligned}$$

that is,

$$\|L\| \leq \|A\|_{(h_d, w_\infty)}. \tag{2.7}$$

Now let  $m \in \mathbb{N}$  be given and  $x^{(m)} = (1/d_m)e^{[m]}$ . Then we have

$$\|x^{(m)}\|_{h_d} = \frac{1}{d_m} \sum_{k=1}^{\infty} d_k |\Delta x_k^{(m)}| = \frac{d_m}{d_m} = 1,$$

and

$$\|L(x^{(m)})\|_{w_\infty} = \sup_l \frac{1}{l} \sum_{n=1}^l |A_n x^{(m)}| = \sup_l \frac{1}{l} \sum_{n=1}^l \frac{1}{d_m} \left| \sum_{k=1}^m a_{nk} \right| \leq \|L\|.$$

Since  $m \in \mathbb{N}$  was arbitrary, we conclude that  $\|A\|_{(h_d, w_\infty)} \leq \|L\|$ , and this and (2.7) imply (2.5).  $\square$

Now we establish a formula for the  $w$ -limits of  $L(x)$  and  $x \in h_d$ , when  $L \in \mathcal{B}(h_d, w)$ .

**Theorem 2.5.** *Let  $L \in \mathcal{B}(h_d, w)$  and  $\alpha_k$  for  $k \in \mathbb{N}$  be the complex numbers in (2.3). Then the  $w$ -limit  $\eta(x)$  of  $L(x)$  for each sequence  $x \in h_d$  is given by*

$$\eta(x) = \sum_{k=1}^\infty \alpha_k x_k. \tag{2.8}$$

*Proof.* Let  $L \in \mathcal{B}(h_d, w)$ . We define the matrix  $B = (b_{nk})_{n,k=1}^\infty$  by  $b_{nk} = a_{nk} - \alpha_k$  for all  $n$  and  $k$ , and show

$$B \in (h_d, w_0). \tag{2.9}$$

First we show

$$(\alpha_k)_{k=1}^\infty \in bs_d. \tag{2.10}$$

We have for all  $l, m \in \mathbb{N}$

$$\begin{aligned} \frac{1}{d_m} \left| \sum_{k=1}^m \alpha_k \right| &= \frac{1}{d_m} \cdot \frac{1}{l} \sum_{n=1}^l \left| \sum_{k=1}^m \alpha_k \right| \leq \frac{1}{d_m} \cdot \frac{1}{l} \sum_{n=1}^l \left| \sum_{k=1}^m (a_{nk} - \alpha_k) \right| + \frac{1}{d_m} \cdot \frac{1}{l} \sum_{n=1}^l \left| \sum_{k=1}^m a_{nk} \right| \\ &\leq \frac{1}{d_m} \sum_{k=1}^m \frac{1}{l} \sum_{n=1}^l |a_{nk} - \alpha_k| + \|A\|_{(h_d, w_\infty)}. \end{aligned} \tag{2.11}$$

Since the first term in the last inequality above tends to 0 as  $l$  tends to infinity for each fixed  $m$  by (2.3), it follows that  $\sup_m (1/d_m) \sum_{k=1}^m |\alpha_k| \leq \|A\|_{(h_d, w_\infty)} < \infty$ , and so (2.10) is satisfied and  $(\alpha_k)_{k=1}^\infty \in h_d^\beta$  by Proposition 1.2 (b). Also  $A \in (h_d, w)$  implies  $A_n \in h_d^\beta$  for each  $n$ , and consequently  $B_n = A_n - (\alpha_k)_{k=1}^\infty \in h_d^\beta$  for each  $n$ . It follows as in (2.11) and by (2.10) that  $\|B\|_{(h_d, w_\infty)} \leq \|A\|_{(h_d, w_\infty)} + \|(\alpha_k)_{k=1}^\infty\|_{bs_d} \leq 2 \cdot \|A\|_{(h_d, w_\infty)} < \infty$ , hence  $B \in (h_d, w_\infty)$  by Theorem 2.4 (a).

Furthermore,  $\lim_{l \rightarrow \infty} (1/l) \sum_{n=1}^l |b_{nk}| = 0$  for each  $k$ , by definition of the matrix  $B$ , that is, the condition in (2.4) also holds, and so (2.9) is satisfied by Theorem 2.4 (c).

Finally (2.8) is an immediate consequence of (2.9).  $\square$

### 3. The Hausdorff measure of noncompactness of operators

In this section, we establish an identity for the Hausdorff measure on noncompactness of operators in  $\mathcal{B}(h_d, w_0)$  and an estimate for the Hausdorff measure of noncompactness of operators in  $\mathcal{B}(h_d, w)$ . We also characterise the classes  $\mathcal{K}(h_d, w_0)$  and  $\mathcal{K}(h_d, w)$ .

We list the necessary, known concepts and results concerning the Hausdorff measure of noncompactness. First we recall the definition of the Hausdorff measure of noncompactness of bounded sets in complete metric spaces ([28, Definition II.2.1]), and the Hausdorff measure of noncompactness of operators between Banach spaces ([19, Definition 7.11.1]).

Let  $X$  be a complete metric space and  $\mathcal{M}_X$  be the class of bounded subsets of  $X$ . Then the function  $\chi : \mathcal{M}_X \rightarrow [0, \infty)$  with  $\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\}$  is called the *Hausdorff measure of noncompactness on  $X$* .

Let  $\chi_1$  and  $\chi_2$  be Hausdorff measures of noncompactness on the Banach spaces  $X$  and  $Y$ , respectively. Then an operator  $L : X \rightarrow Y$  is said to be  $(\chi_1, \chi_2)$ -bounded, if  $L(Q) \in \mathcal{M}_Y$  for all  $Q \in \mathcal{M}_X$  and there exists a non-negative real number  $c$  such that

$$\chi_2(L(Q)) \leq c \cdot \chi_1(Q) \text{ for all } Q \in \mathcal{M}_X. \tag{3.1}$$

If an operator  $L$  is  $(\chi_1, \chi_2)$ -bounded, then the number

$$\|L\|_{(\chi_1, \chi_2)} = \inf\{c \geq 0 : (3.1) \text{ is satisfied}\}$$

is called the  $(\chi_1, \chi_2)$ -measure of noncompactness of the operator  $L$ . If  $\chi_1 = \chi_2$ , we write  $\|L\|_\chi = \|L\|_{(\chi, \chi)}$ , for short, and refer to  $\|L\|_\chi$  as the Hausdorff measure of noncompactness of the operator  $L$ .

We need the following known results.

**Proposition 3.1.** *Let  $X$  and  $Y$  be Banach spaces and  $L \in \mathcal{B}(X, Y)$  and  $S_X$  denote the unit sphere in  $X$ . Then we have*

$$\|L\|_\chi = \chi(L(S_X)) \text{ ([19, Theorem 7.11.4])} \tag{3.2}$$

and  $L \in \mathcal{K}(X, Y)$  if and only if

$$\|L\|_\chi = 0 \text{ ([19, Theorem 7.11.5]).} \tag{3.3}$$

**Proposition 3.2.** ([15, Proposition 5]) (a) *Let the operators  $\mathcal{R}_n : w \rightarrow w$  for  $n \in \mathbb{N}$  be defined by  $\mathcal{R}_n(x) = \sum_{k=n+1}^\infty (x_k - \xi)e^{(k)}$  for all  $x = (x_k)_{k=1}^\infty \in w$ , where  $\xi$  is the  $w$ -limit of the sequence  $x$ . Then we have for all  $Q \in \mathcal{M}_w$*

$$\frac{1}{2} \cdot \lim_{n \rightarrow \infty} \left( \sup_{x \in Q} \|\mathcal{R}_n(x)\|_{w_\infty} \right) \leq \chi(Q) \leq \lim_{n \rightarrow \infty} \left( \sup_{x \in Q} \|\mathcal{R}_n(x)\|_{w_\infty} \right). \tag{3.4}$$

(b) *Let the operators  $\mathcal{R}_n : w_0 \rightarrow w_0$  for  $n \in \mathbb{N}$  be defined by  $\mathcal{R}_n(x) = \sum_{k=n+1}^\infty x_k e^{(k)}$  for all  $x = (x_k)_{k=1}^\infty \in w_0$ . Then we have for all  $Q \in \mathcal{M}_{w_0}$*

$$\chi(Q) = \lim_{n \rightarrow \infty} \left( \sup_{x \in Q} \|\mathcal{R}_n(x)\|_{w_\infty} \right). \tag{3.5}$$

Now we prove an estimate for  $\|L\|_\chi$ , if  $L \in \mathcal{B}(h_d, w)$ , and an identity  $\|L\|_\chi$ , if  $L \in \mathcal{B}(h_d, w_0)$ .

**Theorem 3.3.** (a) *Let  $L \in \mathcal{B}(h_d, w)$ . Then we have*

$$\frac{1}{2} \cdot \lim_{r \rightarrow \infty} \left( \sup_{m;l \geq r} \frac{1}{ld_m} \sum_{n=r}^l \left| \sum_{k=1}^m (a_{nk} - \alpha_k) \right| \right) \leq \|L\|_\chi \leq \lim_{r \rightarrow \infty} \left( \sup_{m;l \geq r} \frac{1}{ld_m} \sum_{n=r}^l \left| \sum_{k=1}^m (a_{nk} - \alpha_k) \right| \right), \tag{3.6}$$

where the complex numbers  $\alpha_k$  are defined in (2.3).

(b) *Let  $L \in \mathcal{B}(h_d, w_0)$ . Then we have*

$$\|L\|_\chi = \lim_{r \rightarrow \infty} \left( \sup_{m;l \geq r} \frac{1}{ld_m} \sum_{n=r}^l \left| \sum_{k=1}^m a_{nk} \right| \right). \tag{3.7}$$

*Proof.* Let  $A = (a_{nk})_{n,k=1}^\infty$  be any infinite matrix and  $r \in \mathbb{N}$ . We write  $A^{<r>} = (a_{nk}^{<r>})_{n,k=1}^\infty$  for the matrix with the rows  $A_n^{<r>} = 0$  for  $1 \leq n \leq r$  and  $A_n^{<r>} = A_n$  for  $n \geq r + 1$ .

(a) Let  $L \in \mathcal{B}(h_d, w)$ ,  $B = (b_{nk})_{n,k=1}^\infty$  be the matrix with  $b_{nk} = a_{nk} - \alpha_k$  for all  $n$  and  $k$ , and  $L^{<r>} \in \mathcal{B}(h_d, w)$  be the operator with  $L^{<r>} = \mathcal{R}_r \circ L$ . We denote the unit sphere in  $h_d$  by  $S_{h_d}$ . Then  $L^{<r>}(x) = B^{<r>}x$  for all  $x \in h_d$  by (1.3) and (2.8) and we obtain by (2.5)

$$\mu(r) = \sup_{x \in S_{h_d}} \|(\mathcal{R}_r \circ L)(x)\|_{w_\infty} = \|B^{<r>}\|_{(h_d, w_\infty)} = \sup_{l,m} \frac{1}{l} \frac{1}{d_m} \sum_{n=1}^l \left| \sum_{k=1}^m b_{nk}^{<r>} \right|$$

$$= \sup_{l,m} \frac{1}{l} \frac{1}{d_m} \sum_{n=r+1}^l \left| \sum_{k=1}^m (a_{nk} - \alpha_k) \right| = \sup_{l \geq r+1; m} \frac{1}{l} \frac{1}{d_m} \sum_{n=r+1}^l \left| \sum_{k=1}^m (a_{nk} - \alpha_k) \right|.$$

Finally we get by (3.2) and (3.4)  $(1/2) \lim_{r \rightarrow \infty} \mu(r) \leq \|L\|_{\chi} \leq \lim_{r \rightarrow \infty} \mu(r)$ , which is (3.6).

(b) The proof is similar to that of Part (a) with  $\alpha_k = 0$  for all  $k$  and (3.5) instead of (3.4).  $\square$

Finally the characterisations of the classes  $\mathcal{K}(h_d, w)$  and  $\mathcal{K}(h_d, w_0)$  are immediate consequences of (3.3) and Theorem 3.3.

**Corollary 3.4.** (a) Let  $L \in \mathcal{B}(h_d, w)$ . Then  $L \in \mathcal{K}(h_d, w)$  if and only if

$$\lim_{r \rightarrow \infty} \left( \sup_{m; l \geq r} \frac{1}{ld_m} \sum_{n=r}^l \left| \sum_{k=1}^m (a_{nk} - \alpha_k) \right| \right) = 0,$$

where the complex numbers  $\alpha_k$  are defined in (2.3).

(b) Let  $L \in \mathcal{B}(h_d, w_0)$ . Then  $L \in \mathcal{K}(h_d, w)$  if and only if

$$\lim_{r \rightarrow \infty} \left( \sup_{m; l \geq r} \frac{1}{ld_m} \sum_{n=r}^l \left| \sum_{k=1}^m a_{nk} \right| \right) = 0.$$

We close with an application of our results.

**Example 3.5.** We consider the classical Hahn space  $h = d_d$ , where  $d_k = k$  for all  $k = 1, 2, \dots$ , and the Cesàro matrix  $C_1 = A = (a_{nk})_{n,k=1}^{\infty}$  of order 1, where  $a_{nk} = 1/n$  for  $1 \leq k \leq n$  and  $a_{nk} = 0$  for  $k > n$  ( $n = 1, 2, \dots$ ).

Then we have  $|\sum_{k=1}^m a_{nk}| \leq m/n$  for all  $m$  and  $n$ , hence

$$c_{lm} = \frac{1}{lm} \sum_{n=1}^l \left| \sum_{k=1}^m a_{nk} \right| \leq \frac{1}{l} \sum_{n=1}^l \frac{1}{n} \leq 1,$$

and so

$$\|A\|_{(h, w_{\infty})} = \sup_{l,m} c_{lm} \leq 1, \tag{3.8}$$

that is, the condition in (2.2) is satisfied. Furthermore, for each  $k \in \mathbb{N}$ ,

$$0 \leq \frac{1}{l} \left| \sum_{n=1}^l a_{nk} \right| = \frac{1}{l} \sum_{n=k}^l \frac{1}{n} \leq \frac{1}{l} \sum_{n=1}^l \frac{1}{n} = A_l \left( \left( \frac{1}{n} \right)_{n=1}^{\infty} \right) \rightarrow 0 \quad (l \rightarrow \infty),$$

since  $A = C_1 \in (c_0, c_0)$ . Thus the condition (2.4) is also satisfied and consequently  $C_1 \in (h, w_0)$  by Theorem 2.4 (c).

Now  $c_{11} = 1$ , and so we have  $\|A\|_{(h, w_{\infty})} = \|L_{C_1}\| = 1$  by Theorem 2.4 (d) and (3.8).

Finally, we have

$$c_{lm}^{(r)} = \frac{1}{lm} \sum_{n=r}^l \left| \sum_{k=1}^m a_{nk} \right| \leq \frac{1}{l} \sum_{n=r}^l \frac{1}{n} \leq \frac{l-r+1}{lr} \leq \frac{1}{r} \text{ for all } l \geq r, m \text{ and } r,$$

hence

$$0 \leq \lim_{r \rightarrow \infty} \left( \sup_{m; l \geq r} c_{lm}^{(r)} \right) = \lim_{r \rightarrow 0} \frac{1}{r} = 0,$$

and so  $L_{C_1} \in \mathcal{K}(h, w_0)$  by Corollary 3.4 (b).



## References

- [1] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskii. *Measures of Noncompactness and Condensing Operators*. Birkhäuser Verlag, Basel, 1992.
- [2] J. Banaś and K. Goebel. *Measures of Noncompactness in Banach Spaces*, volume 60 of *Lecture Notes in Pure and Applied Mathematics*. Marcel Dekker Inc., New York and Basel, 1980.
- [3] J. Banaś and M. Mursaleen. *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*. Springer, New Delhi, Heidelberg, New York, Dordrecht, London, 2014.
- [4] J. Boos. *Classical and Modern Methods in Summability*. Oxford University Press, Oxford, 2000.
- [5] R. Das. On the fine spectrum of the lower triangular matrix  $B(r; s)$  over the Hahn sequence space. *Kyungpook Math. J.*, 57:441–455, 2017.
- [6] R. Das. On the fine spectrum of the lower triangular matrix  $B(r; s)$  over the Hahn sequence space. *Kyungpook Math. J.*, 57:441–455, 2017.
- [7] B. de Malafosse, E. Malkowsky, and V. Rakočević. *Operators Between Sequence Spaces and Applications*. Springer, 2021.
- [8] G. Goes. Sequences of bounded variation and sequences of Fourier coefficients II. *JMAA*, 39:477–494, 1972.
- [9] H. Hahn. Über Folgen linearer Operationen. *Monatsh. Math. Phys.*, 32:3–88, 1922.
- [10] A. M. Jarrah and E. Malkowsky. Ordinary, absolute and strong summability and matrix transformations. *Filomat*, 17:59–78, 2003.
- [11] P. K. Kamthan and M. Gupta. *Sequence Spaces and Series*. Marcel Dekker, New York, 1981.
- [12] M. Kirişçi. The Hahn sequence space defined by the Cesàro mean. *Abstr. Appl. Anal.*, 2013, 2013. Article ID 817659, 6 pages.
- [13] M. Kirişçi. A survey of the Hahn sequence space. *Gen. Math. Notes*, 19(2):37–58, 2013.
- [14] I. J. Maddox. On Kuttner's theorem. *London J. Math. Soc.*, 43:285–298, 1968.
- [15] E. Malkowsky and A. Alotaibi. Measure of noncompactness for compact matrix operators on some BK spaces. *J. Funct. Spaces*, 2014(Article ID 196489):7 pages, <http://dx.doi.org/10.1155/2014/196489>, 2014.
- [16] E. Malkowsky, F. Özger, and V. I. Veličković. Some spaces related to Cesàro sequence spaces and an application to crystallography. *MATCH Commun. Math. Comput. Chem.*, 70:867–884, 2013.
- [17] E. Malkowsky and V. Rakočević. *An Introduction into the Theory of Sequence Spaces and Measures of Noncompactness*, volume 9(17) of *Zbornik radova, Matematički institut SANU*, pages 143–234. Mathematical Institute of SANU, Belgrade, 2000.
- [18] E. Malkowsky and V. Rakočević. *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*, chapter On some results using measures of noncompactness, pages 127–180. Springer Verlag, 2017.
- [19] E. Malkowsky and V. Rakočević. *Advanced Functional Analysis*. CRC Press, Taylor & Francis Group, Boca Raton, London, New York, 2019.
- [20] E. Malkowsky, V. Rakočević, and O. Tuğ. Compact operators on the Hahn space. under review.
- [21] E. Malkowsky and V. I. Veličković. Some new sequence spaces, their duals and a connection with Wulff's crystals. *MATCH Commun. Math. Comput. Chem.*, 67(3):589–607, 2012.
- [22] Eberhard Malkowsky and V. I. Veličković. Topologies of some new sequence spaces, their duals, and the graphical representations of neighbourhoods. *Topology Appl.*, 158(12):1369–1380, 2011.
- [23] K. Raj and A. Kiliçman. On generalized difference Hahn sequence spaces. *Hindawi Publishing Corporation, The Scientific World Journal Volume 2014*, 2014, 2014. Article ID 398203, 7 pages.
- [24] K. C. Rao. The Hahn sequence space. *Bull. Cal. Math. Soc.*, 82:72–78, 1990.
- [25] K. C. Rao and T. G. Srinivasalu. The Hahn sequence space–II. *Y. Y. U. Journal of Faculty of Education*, 1(2):43–45, 1996.
- [26] K. C. Rao and N. Subramanian. The Hahn sequence space–III. *Bull. Malaysian Math. Sc. Soc. (Second Series)*, 25:163–171, 2002.
- [27] W. H. Ruckle. *Sequence Spaces*. Pitman, Boston, London, Melbourne, 1981. Research Notes in Mathematics 49.
- [28] J. M. Ayerbe Toledano, T. Dominguez Benavides, and G. Lopez Acedo. *Measures of Noncompactness in Metric Fixed Point Theory*, volume 99 of *Operator Theory Advances and Applications*. Birkhäuser Verlag, Basel, Boston, Berlin, 1997.
- [29] A. Wilansky. *Summability through Functional Analysis*, volume 85. North–Holland, Amsterdam, 1984. Mathematical Studies.
- [30] K. Zeller and W. Beekmann. *Theorie der Limitierungsverfahren*. Springer Verlag, Heidelberg, Berlin, New York, 1968.