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Statistical Convergence of Bivariate Generalized Bernstein Operators via Four-Dimensional Infinite Matrices

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Abstract. Our main aim in this work is to construct an original extension of bivariate Bernstein type operators based on multiple shape parameters to give an application of four-dimensional infinite matrices to approximation theory, and prove some Korovkin theorems using two summability methods: a statistical convergence method which is stronger than the classical case and a power series method. We obtain the rate of generalized statistical convergence, and the rate of convergence for the power series method. Moreover, we provide some computer graphics to numerically analyze the efficiency and accuracy of convergence of our operators and obtain corresponding error plots. All the results that have been obtained in the present paper can be extended to the case of n-variate functions.

1. Introduction and Preliminaries

Bernstein opened a new way [1, 2] by giving the most well-known proof of Weierstrass approximation theorem (see [42]). He constructed a sequence of approximating polynomials and many researchers have successfully extended this idea to approximate functions (see [22–24, 32, 33]). Korovkin-type theorems provide a process to decide whether a given sequence of positive linear operators converges strongly. Using certain types of statistical convergences instead of the classical convergence in Korovkin type approximation theory gives us many advantages. Applications of Korovkin type approximation on positive linear operators can be seen in [6–8]. We also note that classical convergence is also used in Korovkin type approximation theory in many papers [10–12, 32–39].

In this study, we construct an original extension of bivariate Bernstein type operators based on multiple shape parameters and prove certain Korovkin theorems using a four-dimensional summability method, and a power series method. We obtain the rate of D-statistical convergence, and the rate of convergence for the power series method (PSM) with the help of the modulus of continuity. Finally, we demonstrate some computer graphics to numerically see the efficiency and accuracy of convergence of proposed operators, and obtain corresponding error plots.

First we provide standard notations, notions and auxiliary results.

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Assume that there is $N = N(\tau) \in \mathbb{N}$ for each $\tau > 0$, so that $\left|\varrho_{u,v} - Q\right| < \tau$ whenever $u, v > N$, in this case double sequence $\varrho = (\varrho_{u,v})$ is said to be convergent to Q in Pringsheim's sense (or simply Π-convergent), and it is denoted by $\Pi - \lim_{u,v} \varrho_{u,v} = Q$ (see [17]). When there is a positive number E so that $\left|\varrho_{u,v}\right| \leq E$ for all $(u, v) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, double sequence is said to be bounded. As it is well known, a convergent single sequence is bounded whereas a convergent double sequence need not to be bounded.

Assume that $D = (d_{l,o,u,v})$ is a four-dimensional summability method. Given a double sequence $\varrho = (\varrho_{u,v})$, D transform of ϱ, denoted by $D\varrho := ((D\varrho)_{l,o})$, is defined as

$$(D\varrho)_{l,o} = \sum_{u,v=1}^{\infty} d_{l,o,u,v}\varrho_{u,v},$$

and the double series is Π-convergent for $(l, o) \in \mathbb{N}^2$. When four-dimensional matrix $D = (d_{l,o,u,v})$ maps every bounded $\Pi-$convergent sequence into a $\Pi-$convergent sequence with the same $\Pi-$limit, it is called *RH−regular* (shortly *RHR*). A four-dimensional matrix $D = (d_{l,o,u,v})$ is *RHR* if and only if

(a) $\Pi - \lim_{l,o} d_{l,o,u,v} = 0$,

(b) $\Pi - \lim_{l,o} \sum_{u,v=1}^{\infty} d_{l,o,u,v} = 1$,

(c) $\Pi - \lim_{l,o} \sum_{u=1}^{\infty} \left|d_{l,o,u,v}\right| = 0$ $(\forall v \in \mathbb{N})$,

(d) $\Pi - \lim_{l,o} \sum_{v=1}^{\infty} \left|d_{l,o,u,v}\right| = 0$ $(\forall u \in \mathbb{N})$,

(e) $\sum_{u,v=1}^{\infty} \left|d_{l,o,u,v}\right|$ is $\Pi-$convergent,

(f) The inequality $\sum_{u,v>E_2} \left|d_{l,o,u,v}\right| < E_1$ is satisfied for finite positive integers E_1 and E_2 and for each $(l, o) \in \mathbb{N}^2$.

These conditions are called Robison-Hamilton conditions [18]. Assume that $D = (d_{l,o,u,v})$ is a nonnegative *RHR* matrix, and $S \subset \mathbb{N}^2$, then $D-$density of S is defined as

$$\delta_D^2(S) := \Pi - \lim_{l,o} \sum_{(u,v)\in S} d_{l,o,u,v}$$

provided that the limit on the right-hand side exists in the Pringsheim sense. A real double sequence $\varrho = (\varrho_{u,v})$ is called *D−statistically convergent* to Q and denoted by $st_D^2 - \lim_{u,v} \varrho_{u,v} = Q$ if, for every $\tau > 0$,

$$\delta_D^2(\{(u, v) \in \mathbb{N}^2 : \left|\varrho_{u,v} - Q\right| \geq \tau\}) = 0$$

(see also [16, 19]). A $\Pi-$convergent double sequence is $D-$statistically convergent to the same number even if converse statement may not be true. When $D = C(1, 1)$, $C(1, 1)-$statistical convergence becomes statistical convergence for double sequences (see also [15]), where $C(1, 1) = (c_{l,o,u,v})$ is double Cesàro matrix, defined by $c_{l,o,u,v} = 1/lo$ if $1 \leq u \leq o$, $1 \leq v \leq l$, and $c_{l,o,u,v} = 0$ otherwise. Suppose that $(\xi_{u,v})$ is a double sequence of nonnegative numbers with condition $\xi_{0,0} > 0$, then power series

$$\xi(a, b) := \sum_{u,v=0}^{\infty} \xi_{u,v} a^u b^v$$

has radius of convergence Θ, where $\Theta \in (0, \infty]$ and $a, b \in (0, \Theta)$. When following equality is satisfied

$$\lim_{a,b\to\Theta^-} \frac{1}{\xi(a, b)} \sum_{u,v=0}^{\infty} \xi_{u,v} a^u b^v \varrho_{u,v} = Q$$

for each $a, b \in (0, \Theta)$, then double sequence $\varrho = (\varrho_{u,v})$ is said to be convergent to Q in the sense of PSM [27]. PSM for double sequences is regular if and only if

$$\lim_{a,b\to\Theta^-} \frac{\sum_{r=0}^{\infty} \xi_{r,v} a^u}{\xi(a,b)} = 0; \qquad \lim_{a,b\to\Theta^-} \frac{\sum_{s=0}^{\infty} \xi_{\mu,s} b^v}{\xi(a,b)} = 0$$

are satisfied for any μ, v [27]. In this work, we assume that PSM is regular. When $\Theta = 1$ and $\xi_{u,v} = 1$ PSM becomes Abel summability method, and it becomes logarithmic summability method if $\xi_{u,v} = \frac{1}{(u+1)(v+1)}$. Also, PSM becomes Borel summability method when $\Theta = \infty$ and $\xi_{u,v} = \frac{1}{u!v!}$. Some properties of modified Szász-Mirakyan, Baskakov-Schurer-Szasz, and generalized Szasz operators in polynomial weight spaces were studied by power summability methods in [3–5]. We also note that applications of various statistical summability methods in approximation theory can be seen in the papers [13, 14, 29–31]. Finally, one can see more information about double sequences in [15, 16, 20, 21, 40, 41], and application of double sequences in approximation theory in [6, 27].

2. Bivariate Operators and Statistical Convergence

In this part, we construct an original extension of bivariate Bernstein type operators based on multiple shape parameters and prove some Korovkin theorems using statistical convergence four-dimensional matrices and power series method. The following polynomial functions

$$\begin{aligned}
a_{u,0}(\rho;x) &= (1-x)^u\,(1-\rho_1 x),\\
a_{u,i}(\rho;x) &= x^i(1-x)^{u-i}\left(\binom{u}{v} + \rho_i - \rho_i x - \rho_{i+1}x\right),\ \ i = 1,2\ldots,[\frac{u}{2}]-1,\\
a_{u,[\frac{u}{2}]}(\rho;x) &= x^{[\frac{u}{2}]}(1-x)^{u-[\frac{u}{2}]}\left(\binom{u}{[\frac{u}{2}]} + \rho^{[\frac{u}{2}]} - \rho^{[\frac{u}{2}]}x + \rho^{[\frac{u}{2}]+1}x\right),\\
a_{u,i}(\rho;x) &= x^i(1-x)^{u-i}\left(\binom{u}{v} - \rho_i + \rho_i x + \rho_{i+1}x\right),\ \ i = [\frac{u}{2}]+1,\ldots,u-1,\\
a_{u,u}(\rho;x) &= x^u(1-\rho_u+\rho_u x)
\end{aligned} \tag{1}$$

are called generalized Bernstein polynomials of degree u ($u \geq 2$) and for $x \in [0,1]$ with shape parameters ρ_i, $i = 1,2,\ldots,u$, where

$$\begin{cases} \rho_i \in [-\binom{u}{i}, \binom{u}{i-1}] & ; i = 1,2,\ldots,[\frac{u}{2}] \\ \rho_i \in [-\binom{u}{i-1}, \binom{u}{i}] & ; i = [\frac{u}{2}]+1,\ldots,u \end{cases} \quad \text{with} \quad \begin{cases} [\frac{u}{2}] = \frac{u}{2} & ; \text{if u is even} \\ [\frac{u}{2}] = \frac{u-1}{2} & ; \text{if u is odd.} \end{cases} \tag{2}$$

These polynomials were introduced by Han et al. in. [26] and they are reduced to classical Bernstein basis functions $b_{u,i}(x)$ of degree u on $x \in [0,1]$ which is defined as

$$b_{u,i}(x) = \binom{u}{v} x^i\,(1-x)^{u-i}, \qquad\qquad i = 0,\ldots,u$$

when $\rho_i = 0$ ($i = 1,2,\ldots,u$). Generalized Bernstein basis functions with parameters ρ_i ($i = 1,2,\ldots,u$) are linearly independent (see [25]) and these basis functions are effectively and flexibly used in designing parametric curves and surfaces (see [25, 26]). These functions also have partition of unity, symmetry and nonnegativity properties (see [26]). In 2017, Hu et al. [25] have obtained the following equations to convert classical Bernstein polynomials of degree u to generalized Bernstein polynomials of degree u associated with shape parameters ρ_i:

$$\begin{aligned}
a_{u,0}(\rho;x) &= b_{u+1,0}(x) + \frac{\binom{u}{0}-\rho_1}{\binom{u+1}{1}} b_{u+1,1}(x),\\
a_{u,i}(\rho;x) &= \frac{\binom{u}{i}+\rho_i}{\binom{u+1}{i}} b_{u+1,i}(x) + \frac{\binom{u}{i}-\rho_{i+1}}{\binom{u+1}{i+1}} b_{u+1,i+1}(x),\ \ i=1,2\ldots,\left[\frac{u}{2}\right]-1,\\
a_{u,i}(\rho;x) &= \frac{\binom{u}{i}+\rho_v}{\binom{u+1}{i}} b_{u+1,i}(x) + \frac{\binom{u}{i}+\rho_{i+1}}{\binom{u+1}{i+1}} b_{u+1,i+1}(x),\ i=\left[\frac{u}{2}\right],\\
a_{u,i}(\rho;x) &= \frac{\binom{u}{i}-\rho_i}{\binom{u+1}{i}} b_{u+1,i}(x) + \frac{\binom{u}{i}+\rho_{i+1}}{\binom{u+1}{i+1}} b_{u+1,i+1}(x),\ \ i=\left[\frac{u}{2}\right]+1,\ldots,u-1,\\
a_{u,u}(\rho;x) &= \frac{\binom{u}{u}-\rho_u}{\binom{u+1}{u}} b_{u+1,u}(x) + b_{u+1,u+1}(x). \qquad (3)
\end{aligned}$$

Let $C[0,1]=\mathbf{C}$ be the space of all continuous functions on unit interval $[0,1]$ and $C([0,1]\times[0,1])=\bar{\mathbf{C}}$. The operators $\mathcal{B}_u^{\nu}, \mathcal{B}_v^{\mu} : \mathbf{C} \longrightarrow \mathbf{C}$ for any $u,v\in\mathbb{N}$ are given as follows, respectively,

$$\mathcal{B}_u^{\nu}(f;y) = \sum_{i=0}^{u} f\left(\frac{i}{u}\right) a_{u,i}(\nu;y), \qquad (4)$$

$$\mathcal{B}_v^{\mu}(g;z) = \sum_{j=0}^{v} g\left(\frac{j}{v}\right) a_{v,j}(\mu;z), \qquad (5)$$

where polynomials $a_{u,i}(\nu;y)$ and $a_{v,j}(\mu;z)$ are given in (3). The parametric extension of (4) and (5) for $u,v\in\mathbb{N}$ and $h\in\bar{\mathbf{C}}$ are the operators

$$\mathcal{B}_u^{\nu,y}, \mathcal{B}_v^{\mu,z} : \bar{\mathbf{C}} \longrightarrow \bar{\mathbf{C}},$$

where

$$\mathcal{B}_u^{\nu,y}(h;y,z) = \sum_{i=0}^{u} a_{u,i}(\nu;y) h\left(\frac{i}{u}, \frac{i}{u}\right), \qquad (6)$$

$$\mathcal{B}_v^{\mu,z}(h;y,z) = \sum_{j=0}^{v} a_{v,j}(\mu;z) h\left(\frac{j}{v}, \frac{j}{v}\right). \qquad (7)$$

Lemma 2.1. *The parametric extension of operators defined in (6) and (7) are linear and positive.*

Proof. The assertion follows from the definitions of $\mathcal{B}_u^{\nu,y}$ and $\mathcal{B}_v^{\mu,z}$. □

Lemma 2.2. *The parametric extensions of bivariate operators commute on $\bar{\mathbf{C}}$. Their product establishes bivariate operators $\mathcal{B}_{u,v}^{\nu,\mu} : \bar{\mathbf{C}} \longrightarrow \bar{\mathbf{C}}$ defined for any $u,v\in\mathbb{N}$ and any $h\in\bar{\mathbf{C}}$ by the relation*

$$\mathcal{B}_{u,v}^{\nu,\mu}(h;y,z) \quad = \quad \sum_{i=0}^{u}\sum_{j=0}^{v} a_{u,i}(\nu;y) a_{v,j}(\mu;z) h\left(\frac{i}{u}, \frac{j}{v}\right). \qquad (8)$$

Proof. We get the desired result by direct computation, taking into account definitions (6), (7) and Lemma 2.1. □

Lemma 2.3. *The bivariate operators (8) are linear and positive.*

Proof. Using the fact that the product of linear and positive operators are also linear and positive, and applying Lemma 2.1 we obtain desired result. □

Lemma 2.4. *[13] For any natural number u ($u \geq 2$) the following equalities hold:*

$$\begin{aligned}
&\mathcal{B}_u^{\rho}(1;x) = 1,\ \mathcal{B}_u^{\rho}(t;x) = x + \frac{1-x}{u}\mathcal{R}_0(x),\ \mathcal{B}_u^{\rho}(t^2;x) = \frac{u-1}{u}x^2 + \frac{x}{u} + \frac{1-x}{u^2}\Big[2\mathcal{R}_1(x) - \mathcal{R}_0(x)\Big],\\
&\mathcal{B}_u^{\rho}(t^3;x) = \frac{(u-1)(u-2)}{u^2}x^3 + \frac{3(u-1)}{u^2}x^2 + \frac{x}{u^2} + \frac{1-x}{u^3}\Big[3\mathcal{R}_2(x) - 3\mathcal{R}_1(x) + \mathcal{R}_0(x)\Big],\\
&\mathcal{B}_u^{\rho}(t^4;x) = \frac{(u-1)(u-2)(u-3)}{u^3}x^4 + \frac{6(u-1)(u-2)}{u^3}x^3 + \frac{7(u-1)}{u^3}x^2 + \frac{x}{u^3}\\
&\qquad + \frac{1-x}{u^4}\Big[4\mathcal{R}_3(x) - 6\mathcal{R}_2(x) + 4\mathcal{R}_1(x) - \mathcal{R}_0(x)\Big],
\end{aligned}$$

where

$$\mathcal{R}_p(x) = \sum_{i=1}^{[\frac{u}{2}]} \frac{i^p \rho_i}{\binom{u}{v}} b_{u,i}(x) - \sum_{i=[\frac{u}{2}]+1}^{u} \frac{i^p \rho_i}{\binom{u}{v}} b_{u,i}(x),\ p = 0,1,2,\dots u,\ x \in [0,1].$$

Lemma 2.5. *The parametric extension $\mathcal{B}_u^{\nu,y}$ satisfies the identities in Lemma 2.1.*

Proof. By using the definition (6) of $\mathcal{B}_u^{\nu,y}$ and Lemma 2.1, we get the result. □

Remark 2.6. *The parametric extension $\mathcal{B}_v^{\mu,z}$ satisfies identities similar to the identities in Lemma 2.1.*

Lemma 2.7. *Let $e_{uv}(y,z) = y^u z^v$, $u, v \in \mathbb{N}$, $y, z \in \mathbb{R}$ be the two-dimensional test functions. The bivariate operators defined in (8) satisfy*

$$\begin{aligned}
&\mathcal{B}_{u,v}^{\nu,\mu}(e_{00};y,z) = 1,\ \mathcal{B}_{u,v}^{\nu,\mu}(e_{10};y,z) = y + \frac{1-y}{u}\mathcal{R}_0(y),\ \mathcal{B}_{u,v}^{\nu,\mu}(e_{01};y,z) = z + \frac{1-z}{v}\mathcal{R}_0(z),\\
&\mathcal{B}_{u,v}^{\nu,\mu}(e_{20};y,z) = \frac{u-1}{u}y^2 + \frac{y}{u} + \frac{1-y}{u^2}\Big[2\mathcal{R}_1(y) - \mathcal{R}_0(y)\Big],\\
&\mathcal{B}_{u,v}^{\nu,\mu}(e_{02};y,z) = \frac{v-1}{v}z^2 + \frac{z}{v} + \frac{1-z}{v^2}\Big[2\mathcal{R}_1(z) - \mathcal{R}_0(z)\Big],\\
&\mathcal{B}_{u,v}^{\nu,\mu}(e_{30};y,z) = \frac{(u-1)(u-2)(u-3)}{u^3}y^4 + \frac{6(u-1)(u-2)}{u^3}y^3 + \frac{7(u-1)}{u^3}y^2 + \frac{y}{u^3}\\
&\quad + \frac{1-y}{u^4}\Big[4\mathcal{R}_3(y) - 6\mathcal{R}_2(x) + 4\mathcal{R}_1(y) - \mathcal{R}_0(y)\Big],\\
&\mathcal{B}_{u,v}^{\nu,\mu}(e_{03};y,z) = \frac{(v-1)(v-2)(v-3)}{v^3}z^4 + \frac{6(v-1)(v-2)}{v^3}z^3 + \frac{7(v-1)}{v^3}z^2 + \frac{z}{v^3}\\
&\quad + \frac{1-z}{v^4}\Big[4\mathcal{R}_3(z) - 6\mathcal{R}_2(z) + 4\mathcal{R}_1(z) - \mathcal{R}_0(z)\Big],\\
&\mathcal{B}_{u,v}^{\nu,\mu}(e_{40};y,z) = \frac{(u-1)(u-2)(u-3)}{u^3}y^4 + \frac{6(u-1)(u-2)}{u^3}y^3 + \frac{7(u-1)}{u^3}y^2 + \frac{y}{u^3}\\
&\quad + \frac{1-y}{u^4}\Big[4\mathcal{R}_3(y) - 6\mathcal{R}_2(y) + 4\mathcal{R}_1(y) - \mathcal{R}_0(y)\Big],\\
&\mathcal{B}_{u,v}^{\nu,\mu}(e_{04};y,z) = \frac{(v-1)(v-2)(v-3)}{v^3}z^4 + \frac{6(v-1)(v-2)}{v^3}z^3 + \frac{7(v-1)}{v^3}z^2 + \frac{z}{v^3}\\
&\quad + \frac{1-z}{v^4}\Big[4\mathcal{R}_3(z) - 6\mathcal{R}_2(z) + 4\mathcal{R}_1(z) - \mathcal{R}_0(z)\Big].
\end{aligned}$$

Proof. Taking into account definition (8) and Lemma 2.4, we complete the proof. □

Lemma 2.8. *The bivariate operators (8) satisfy the relations*

$$\mathcal{B}_{u,v}^{\nu,\mu}((e_{10}-y)^2\,;y,z) = \frac{y(1-y)}{u}(1-2\mathcal{R}_0(y)) + \frac{1-y}{u^2}\Big[2\mathcal{R}_1(y)-\mathcal{R}_0(y)\Big], \tag{9}$$

$$\mathcal{B}_{u,v}^{\nu,\mu}((e_{01}-z)^2\,;y,z) = \frac{z(1-z)}{v}(1-2\mathcal{R}_0(z)) + \frac{1-z}{v^2}\Big[2\mathcal{R}_1(z)-\mathcal{R}_0(z)\Big]. \tag{10}$$

Proof. Since $\mathcal{B}_{u,v}^{\nu,\mu}$ is linear, we have

$$\begin{aligned}\mathcal{B}_{u,v}^{\nu,\mu}((e_{10}-y)^2\,;y,z) &= \mathcal{B}_{u,v}^{\nu,\mu}(e_{20};y,z)\\ &\quad -2y\mathcal{B}_{u,v}^{\nu,\mu}(e_{10};y,z)+y^2\mathcal{B}_{u,v}^{\nu,\mu}(e_{10};y,z).\end{aligned}$$

By applying Lemma 2.7, we get relation (9). Similary we have the equality (10). □

Following theorem gives Korovkin type approximation for statistical convergence by four-dimensional matrices:

Theorem 2.9. *[19] Let $D = (d_{l,o,u,v})$ be a nonnegative RHR matrix. Let $(Q_{u,v})$ be a double sequence of operators acting from $C([a,b]\times[c,d])$ into itself. So, for each $h \in C([a,b]\times[c,d])$,*

$$st_D^2 - \lim_{u,v}\left\|Q_{u,v}(h)-h\right\|_{C([a,b]\times[c,d])} = 0$$

if and only if

$$st_D^2 - \lim_{u,v}\left\|Q_{u,v}(h_u)-h_u\right\|_{C([a,b]\times[c,d])} = 0,$$

where $h_0(y,z)=1$, $h_1(y,z)=y$, $h_2(y,z)=z$ and $h_3(y,z)=y^2+z^2$.

Theorem 2.9 provides next result.

Theorem 2.10. *Let $h \in \bar{C}$, then*

$$st_D^2 - \lim_{u,v}\left\|\mathcal{B}_{u,v}^{\nu,\mu}(h)-h\right\|_{\bar{C}} = 0.$$

Proof. Assume that

$$st_D^2 - \lim_{u,v}\left\|\mathcal{B}_{u,v}^{\nu,\mu}(h_u)-h_u\right\|_{\bar{C}} = 0. \tag{11}$$

We have the following result for h_0 using Lemma 2.7 (a):

$$st_D^2 - \lim_{u,v}\left\|\mathcal{B}_{u,v}^{\nu,\mu}(h_0)-h_0\right\|_{\bar{C}} = 0.$$

By Lemma 2.7, we obtain

$$\left\|\mathcal{B}_{u,v}^{\nu,\mu}(h_1)-h_1\right\|_{\bar{C}} = \sup_{(y,z)\in[0,1]\times[0,1]}\left|y+\frac{1-y}{u}\mathcal{R}_0(y)-y\right| \le \frac{1}{u}. \tag{12}$$

For a given $\epsilon' > 0$, we choose a number $\epsilon > 0$ such that $\epsilon < \epsilon'$. Let us define the following sets:

$$\begin{aligned}\mathcal{S} \;&:= \left\{(u,v) : \left\|\mathcal{B}_{u,v}^{\nu,\mu}(h_1)-h_1\right\|_{C([0,1]\times[0,1])} \ge \epsilon'\right\},\\ \mathcal{S}_1 \;&:= \left\{(u,v) : \frac{1}{u} \ge \epsilon-\epsilon'\right\}.\end{aligned}$$

We observe that $\mathcal{S} \subseteq \mathcal{S}_1$. So $\delta_D^2(\mathcal{S}) \le \delta_D^2(\mathcal{S}_1)$ is satisfied and the following relation is satisfied for h_1 :

$$st_D^2 - \lim_{u,v} \left\| \mathcal{B}_{u,v}^{\nu,\mu}(h_1) - h_1 \right\|_{\bar{C}} = 0.$$

Using the same manner we have

$$st_D^2 - \lim_{u,v} \left\| \mathcal{B}_{u,v}^{\nu,\mu}(h_2) - h_2 \right\|_{\bar{C}} = 0,$$

that is (11) is satisfied for h_2. Considering the following relations

$$\begin{aligned}
& \left\| \mathcal{B}_{u,v}^{\nu,\mu}(h_3) - h_3 \right\|_{\bar{C}} && (13)\\
\le\ & \left\| \mathcal{B}_{u,v}^{\nu,\mu}(e_{20}) - e_{20} \right\|_{\bar{C}} + \left\| \mathcal{B}_{u,v}^{\nu,\mu}(e_{02}) - e_{02} \right\|_{\bar{C}} \\
=\ & \sup_{(y,z)\in[0,1]\times[0,1]} \left| \frac{y(1-y)}{u} + \frac{1-y}{u^2}\Big[2\mathcal{R}_1(y) - \mathcal{R}_0(y)\Big] \right| \\
+\ & \sup_{(y,z)\in[0,1]\times[0,1]} \left| \frac{z(1-z)}{v} + \frac{1-z}{v^2}\Big[2\mathcal{R}_1(z) - \mathcal{R}_0(z)\Big] \right| \\
\le\ & \sup_{(y,z)\in[0,1]\times[0,1]} \left| \frac{y(1-y)}{u} + \frac{2(1+u)}{u^2}\Big[y^2 + \frac{y(1-y)}{u}\Big] \right| && (14)\\
+\ & \sup_{(y,z)\in[0,1]\times[0,1]} \left| \frac{z(1-z)}{v} + \frac{2(1+v)}{v^2}\Big[z^2 + \frac{z(1-z)}{v}\Big] \right|
\end{aligned}$$

and taking D-statistical limit in both-sides of last inequality, we obtain

$$st_D^2 - \lim_{u,v} \left\| \mathcal{B}_{u,v}^{\nu,\mu}(h_3) - h_3 \right\|_{\bar{C}} = 0,$$

that is (11) is satisfied for h_3. □

The following result is obtained by replacing the matrix D in Theorem 2.9 with double identity matrix.

Corollary 2.11. *Assume that $h \in \bar{C}$, then*

$$\Pi - \lim_{u,v} \left\| \mathcal{B}_{u,v}^{\nu,\mu}(h) - h \right\|_{\bar{C}} = 0.$$

The $C(1,1)$-statistical convergence becomes statistical convergence for double sequences if $D = C(1,1)$ is chosen, and following result is satisfied:

Corollary 2.12. *Assume that $h \in \bar{C}$, then*

$$st_{C(1,1)}^2 - \lim_{u,v} \left\| \mathcal{B}_{u,v}^{\nu,\mu}(h) - h \right\|_{\bar{C}} = 0.$$

3. Power series method for operators $\mathcal{B}_{u,v}^{\nu,\mu}$

Assume throughout the section, $\Psi := [a,b] \times [c,d]$ and $(Q_{u,v})$ be a double sequence of positive linear operators acting from $C(\Psi)$ into itself such that

$$\sup_{0<a,b<\Theta} \frac{1}{\varrho(a,b)} \sum_{u,v=0}^{\infty} \xi_{u,v} a^u b^v \left\| Q_{u,v}(1) \right\|_{C(\Psi)} < \infty. \tag{15}$$

Set

$$S_{a,b}(h; y, z) = \frac{1}{\varrho(a,b)} \sum_{u,v=0}^{\infty} \xi_{u,v} a^u b^v Q_{u,v}(h; y, z), \ a, b \in (0, \Theta)$$

and

$$T_{a,b}(h;y,z)=\frac{1}{\varrho(a,b)}\sum_{u,v=0}^{\infty}\xi_{u,v}a^u b^v \mathcal{B}_{u,v}^{\nu,\mu}(h;y,z),\ a,b\in(0,\Theta).$$

The proof of following theorem was given in [23], for the readers convince we also give it here.

Theorem 3.1. *Let $h\in C(\Psi)$, then*

$$\lim_{a,b\to\Theta^-}\left\|S_{a,b}(h)-h\right\|_{C(\Psi)}=0 \tag{16}$$

if and only if

$$\lim_{a,b\to\Theta^-}\left\|S_{a,b}(h_u)-h_u\right\|_{C(\Psi)}=0, \tag{17}$$

where $h_0(y,z)=1$, $h_1(y,z)=y$, $h_2(y,z)=z$ and $h_3(y,z)=y^2+z^2$.

Proof. The implication (16)⇒(17) is clear since $h_u\in C(\Psi)$ for each $u=0,1,2,3$. Let $h\in C(\Psi)$ and $(y,z)\in\Psi$ be fixed. Since function h is continuous on Ψ, following inequality is satisfied:

$$\left|h(y,z)\right|\le M_h.$$

Therefore

$$\left|h(s,t)-h(y,z)\right|\le 2M_h.$$

Also, since h is continuous on Ψ, there is a number $\rho>0$ so that $\left|h(s,t)-h(y,z)\right|<\tau$ holds for each $\tau>0$ and $(s,t)\in\Psi$ satisfying $\left|s-y\right|<\rho$ and $|t-z|<\rho$. Hence, we get

$$\left|h(s,t)-h(y,z)\right|<\tau+\frac{2M_h}{\rho^2}\left\{(s-y)^2+(t-z)^2\right\}.$$

This means

$$-\tau-\frac{2M_h}{\rho^2}\left\{(s-y)^2+(t-z)^2\right\}<h(s,t)-h(y,z)<\tau+\frac{2M_h}{\rho^2}\left\{(s-y)^2+(t-z)^2\right\}.$$

So, we can write

$$\begin{aligned}
&\left|S_{a,b}(h;y,z)-h(y,z)\right|\\
=\ &\left|\frac{1}{\varrho(a,b)}\sum_{u,v=0}^{\infty}\xi_{u,v}a^u b^v Q_{u,v}(h;y,z)-h(y,z)\right|\\
\le\ &\frac{1}{\varrho(a,b)}\sum_{u,v=0}^{\infty}\xi_{u,v}a^u b^v Q_{u,v}\left(\left|h(s,t)-h(y,z)\right|\right)\\
&+\left|h(y,z)\right|\left|\frac{1}{\varrho(a,b)}\sum_{u,v=0}^{\infty}\xi_{u,v}a^u b^v Q_{u,v}(h_0;y,z)-h_0(y,z)\right|\\
\le\ &\tau+\left(\tau+M_h+\frac{2M_h\left\|h_3\right\|_{C(\Psi)}}{\rho^2}\right)\left|S_{a,b}(h_0;y,z)-h_0(y,z)\right|\\
&+\frac{4M_h\left\|h_1\right\|_{C(\Psi)}}{\rho^2}\left|S_{a,b}(h_1;y,z)-h_1(y,z)\right|\\
&+\frac{4M_h\left\|h_2\right\|_{C(\Psi)}}{\rho^2}\left|S_{a,b}(h_2;y,z)-h_2(y,z)\right|\\
&+\frac{2M_h}{\rho^2}\left|S_{a,b}(h_3;y,z)-h_3(y,z)\right|.
\end{aligned}$$

Then taking supremum over $(y,z)\in\Psi$, we have

$$\begin{aligned}&\left\|S_{a,b}(h)-h\right\|_{C(\Psi)}\\ \leq\ &\tau+N\left\{\sum_{u=0}^{3}\left\|S_{a,b}(h_u;y,z)-h_u(y,z)\right\|_{C(\Psi)}\right\},\end{aligned}$$

where

$$N:=\max\left\{\tau+M_h+\frac{2M_h\|h_3\|_{C(\Psi)}}{\rho^2},\frac{4M_h\|h_1\|_{C(\Psi)}}{\rho^2},\frac{4M_h\|h_2\|_{C(\Psi)}}{\rho^2},\frac{2M_h}{\rho^2}\right\}.$$

By the aid of relation (17), following result, which completes the proof, is obtained:

$$\lim_{a,b\to\Theta^-}\left\|S_{a,b}(h)-h\right\|_{C(\Psi)}=0.$$

□

Theorem 3.2. *Let $h\in\bar{\mathbf{C}}$, then*

$$\lim_{a,b\to\Theta^-}\left\|T_{a,b}(h)-h\right\|_{\bar{C}}=0.$$

Proof. *Since $\mathcal{B}^{\nu,\mu}_{u,v}(e_{00};y,z)=1$, we see that (15) holds. Considering Lemma 2.7 and the inequalities (12) and (13), the proof is completed.* □

4. Rate of statistical convergence, and rate of convergence for PSM

In this section, with the aid of modulus of continuity, we calculate the rate of statistical convergence by four-dimensional matrices and the rate of convergence for PSM, here modulus of continuity is expressed as

$$\omega(h,\rho)=\sup_{\sqrt{(s-y)^2+(t-z)^2}\leq\rho}\left|h(s,t)-h(y,z)\right|\quad(\rho>0),\ h\in\bar{\mathbf{C}}.$$

We know that, for any $\rho>0$ and for all $h\in\bar{\mathbf{C}}$

$$\omega(h,\rho\rho)\leq(1+[\rho])\,\omega(h,\rho),$$

where $[\rho]$ is greatest integer less than or equal to ρ.

Theorem 4.1. *Assume that $h\in\bar{\mathbf{C}}$, $u,v\in\mathbb{Z}_+$, $D=(d_{l,o,u,v})$ is a nonnegative RHR matrix and $(\alpha_{u,v})$ is a positive non-increasing double sequence so that $\omega(h,\rho_{u,v})=st^2_D-o(\alpha_{u,v})$, then*

$$\left\|\mathcal{B}^{\nu,\mu}_{u,v}(h)-h\right\|_{\bar{C}}=st^2_D-o(\alpha_{u,v}),$$

where

$$\rho_{u,v}:=\left\{\frac{2}{u}+\frac{3}{u^2}+\frac{4(u+1)}{u^3}+\frac{2}{v}+\frac{3}{v^2}+\frac{4(v+1)}{v^3}\right\}^{\frac{1}{2}}.$$

Proof. Assume that the hypotheses are satisfied. Using positivity and monotonicity of operators $\mathcal{B}^{\nu,\mu}_{u,v}$ we get

$$\begin{aligned}\left|\mathcal{B}^{\nu,\mu}_{u,v}(h;y,z)-h(y,z)\right| &\leq \mathcal{B}^{\nu,\mu}_{u,v}\left(\left|h(s,t)-h(y,z)\right|;y,z\right)\\ &\leq \mathcal{B}^{\nu,\mu}_{u,v}\left(\left(1+\frac{(s-y)^2+(t-z)^2}{\rho^2}\right)\omega(h,\rho);y,z\right)\\ &= \omega(h,\rho)+\frac{\omega(h,\rho)}{\rho^2}\mathcal{B}^{\nu,\mu}_{u,v}\left((s-y)^2+(t-z)^2;y,z\right).\end{aligned}$$

Then taking supremum over $(y,z)\in[0,1]\times[0,1]$, we have

$$\begin{aligned}
&\left\|\mathcal{B}_{u,v}^{\nu,\mu}(h)-h\right\|_{\bar{C}}\\
\leq\ & \omega(h,\rho)+\frac{\omega(h,\rho)}{\rho^2}\left\{\left\|\mathcal{B}_{u,v}^{\nu,\mu}\left((s-.)^2\right)\right\|_{\bar{C}}+\left\|\mathcal{B}_{u,v}^{\nu,\mu}\left((t-.)^2\right)\right\|_{\bar{C}}\right\}\\
\leq\ & \omega(h,\rho)+\frac{\omega(h,\rho)}{\rho^2}\left\{\frac{y(1-y)}{u}+\frac{2y^2-y}{u^2}+\frac{2(u+1)y(1-y)}{u^3}\right\}\\
+\ & \frac{\omega(h,\rho)}{\rho^2}\left\{\frac{z(1-z)}{v}+\frac{2z^2-z}{v^2}+\frac{2(v+1)z(1-z)}{v^3}\right\}\\
\leq\ & \omega(h,\rho)+\frac{\omega(h,\rho)}{\rho^2}\left\{\frac{2}{u}+\frac{3}{u^2}+\frac{4(u+1)}{u^3}+\frac{2}{v}+\frac{3}{v^2}+\frac{4(v+1)}{v^3}\right\}.
\end{aligned}$$

By the following chose of ρ

$$\rho=\rho_{u,v}:=\left\{\frac{2}{u}+\frac{3}{u^2}+\frac{4(u+1)}{u^3}+\frac{2}{v}+\frac{3}{v^2}+\frac{4(v+1)}{v^3}\right\}^{\frac{1}{2}}$$

the following inequality is satisfied for any positive integers u,v :

$$\left\|\mathcal{B}_{u,v}^{\nu,\mu}(h)-h\right\|_{\bar{C}}\leq 2\omega(h,\rho_{u,v}).$$

Hence the following relation is satisfied for any $\tau>0$:

$$\frac{1}{\alpha_{u,v}}\sum_{\left\|\mathcal{B}_{u,v}^{\nu,\mu}(h)-h\right\|_{\bar{C}}\geq\tau}d_{l,o,u,v}\leq\frac{1}{\alpha_{u,v}}\sum_{\omega(h,\rho_{u,v})\geq\frac{\tau}{2}}d_{l,o,u,v}$$

and the following final step is obtained from the hypothesis

$$\left\|\mathcal{B}_{u,v}^{\nu,\mu}(h)-h\right\|_{\bar{C}}=st_D^2-o\left(\alpha_{u,v}\right).$$

□

We give rate of convergence for PSM by the following theorem.

Theorem 4.2. *Let $h\in\bar{C}$ and ζ be a positive real function defined on $(0,\Theta)\times(0,\Theta)$. If $\omega(h,\psi)=o(\zeta)$, as $a,b\to\Theta^-$, then*

$$\left\|T_{a,b}(h)-h\right\|_{\bar{C}}=o(\zeta)$$

as $a,b\to\Theta^-$, where $\psi:(0,\Theta)\times(0,\Theta)\to\mathbb{R}$ is given as

$$\begin{aligned}
&\psi(a,b)\\
:\ =&\left\{\frac{1}{\varrho(a,b)}\sum_{u,v=0}^{\infty}\xi_{u,v}a^ub^v\left\|\mathcal{B}_{u,v}^{\nu,\mu}\left((s-.)^2+(t-.)^2\right)\right\|_{\bar{C}}\right\}^{\frac{1}{2}}.
\end{aligned}$$

Proof. Since the operators are linear and the positive the following relations are satisfied

$$\begin{aligned}
&\left|T_{a,b}(h;y,z)-h(y,z)\right|\\
=\ & \left|\frac{1}{\varrho(a,b)}\sum_{u,v=0}^{\infty}\xi_{u,v}a^ub^v\mathcal{B}_{u,v}^{\nu,\mu}(h;y,z)-h(y,z)\right|\\
\leq\ & \frac{1}{\varrho(a,b)}\sum_{u,v=0}^{\infty}\xi_{u,v}a^ub^v\mathcal{B}_{u,v}^{\nu,\mu}\left(\left|h(s,t)-h(y,z)\right|;y,z\right)\\
\leq\ & \omega(h,\rho)+\frac{\omega(h,\rho)}{\rho^2}\left\{\frac{1}{\varrho(a,b)}\sum_{u,v=0}^{\infty}\xi_{u,v}a^ub^v\mathcal{B}_{u,v}^{\nu,\mu}\left((s-y)^2+(t-z)^2;y,z\right)\right\}
\end{aligned}$$

for any $a, b \in (0, \Theta)$ and $(y, z) \in [0,1] \times [0,1]$. Then taking supremum, we have

$$\left\| \mathcal{T}_{u,v}^{\nu,\mu}(h) - h \right\|_{\bar{\mathrm{C}}} \leq 2\, \omega(h, \psi),$$

where

$$\begin{aligned} \rho &= \psi(a,b) \\ : &= \left\{ \frac{1}{\varrho(a,b)} \sum_{u,v=0}^{\infty} \xi_{u,v} a^u b^v \left\| \mathcal{B}_{u,v}^{\nu,\mu}\left((s-.)^2 + (t-.)^2 \right) \right\|_{\bar{\mathrm{C}}} \right\}^{\frac{1}{2}}. \end{aligned}$$

□

5. Convergence by graphics

In this part, we first focus on the behavior of generalized basis polynomials $a_{u,v}(\nu;y)a_{u,v}(\mu;z)$ and give several illustrative examples with the help of Mathematica to verify the convergence behavior, computational efficiency and consistency of the generalized bivariate Bernstein operators.

In order to see the effect of shape parameters ν and μ to behaviour of polynomials $a_{u,v}(\nu;y)a_{u,v}(\mu;z)$ we choose certain parameters ν and μ. In Figure 1 (**a**), we present behaviour of polynomials $b_{u,v}(y)b_{u,v}(z)$ to see the difference. In Figure 1 (**b**)-(**f**), we choose the following shape parameters, respectively:

(**b**) $\nu_i = \binom{u}{i-1} - \binom{u}{i},\ i = 1,2,\ldots,\left[\frac{u}{2}\right],\ \mu_j = \binom{v}{j-1} - \binom{v}{j},\ j = 1,2,\ldots,\left[\frac{v}{2}\right],$

$\nu_i = \binom{u}{i} - \binom{u}{i-1},\ i = \left[\frac{u}{2}\right]+1,\ldots,u, \quad \mu_j = \binom{v}{j} - \binom{v}{j-1},\ j = \left[\frac{v}{2}\right]+1,\ldots,v;$

(**c**) $\nu_i = \binom{u}{i-1},\ i = 1,2,\ldots,\left[\frac{u}{2}\right],\ \mu_j = \binom{v}{j-1},\ j = 1,2,\ldots,\left[\frac{v}{2}\right],$

$\nu_i = \binom{u}{i},\ i = \left[\frac{u}{2}\right]+1,\ldots,u,\ \mu_j = \binom{v}{j},\ j = \left[\frac{v}{2}\right]+1,\ldots,v;$

(**d**) $\nu_i = -\binom{u}{i},\ i = 1,2,\ldots,\left[\frac{u}{2}\right],\ \mu_j = -\binom{v}{j},\ j = 1,2,\ldots,\left[\frac{v}{2}\right],$

$\nu_i = \binom{u}{i},\ i = \left[\frac{u}{2}\right]+1,\ldots,u, \quad \mu_j = \binom{v}{j},\ j = \left[\frac{v}{2}\right]+1,\ldots,v;$

(**e**) $\nu_i = \binom{u}{i-1},\ i = 1,2,\ldots,\left[\frac{u}{2}\right],\ \mu_j = \binom{v}{j-1},\ j = 1,2,\ldots,\left[\frac{v}{2}\right],$

$\nu_i, \mu_j = -\binom{u}{v-1},\ i = \left[\frac{u}{2}\right]+1,\ldots,u,\ j = \left[\frac{v}{2}\right]+1,\ldots,v;$

(**f**) $\nu_i = -\binom{u}{i},\ i = 1,2,\ldots,\left[\frac{u}{2}\right],\ \mu_j = -\binom{v}{j},\ j = 1,2,\ldots,\left[\frac{v}{2}\right],$

$\nu_i = -\binom{u}{i-1},\ i = \left[\frac{u}{2}\right]+1,\ldots,u,\ \mu_j = -\binom{v}{j-1},\ j = \left[\frac{v}{2}\right]+1,\ldots,v.$

Example 5.1. *Consider the following function*

$$h(y,z) = \frac{\left(z^3 - \frac{1}{2}\right)\sin(\pi y)\cos(\pi y)}{(y^7+2)\left(z^7 + \frac{1}{2}\right)}$$

u	v	Absolute Error	y	z
4	4	1 .4204	1	0.8635
10	10	0.7281	1	0.8617
20	20	0.3976	1	0.8589
30	30	0.2731	1	0.8579
40	40	0.2079	1	0.8575
50	50	0.1679	1	0.8573
5	50	0.5915	0.8349	0.8490
5	100	0.5291	0.8346	0.8476
100	100	0.0855	1	0.8569

Table 1: Absolute errors of approximation for certain u and v values and related occurring points (y, z) for Example 5.1

u	v	GB	B
4	4	0.1507	0.1596
6	6	0.1148	0.1216
8	8	0.0936	0.0984
10	10	0.0793	0.0826
12	12	0.0688	0.0712
20	4	0.0552	0.0658
4	20	0.1482	0.1565
50	10	0.0261	0.0313

Table 2: Comparison of bivarite Bernstein and $\mathcal{B}_{u,v}^{\nu,\mu}$ operators by absolute errors for Example 5.3

on $(y, z) \in [0, 1] \times [0, 1]$. *We choose the following shape parameters for this example:*

$$\nu_i = \frac{1}{10000}\binom{u}{i-1}, \quad i = 1, 2, \ldots, \left[\frac{u}{2}\right], \quad \mu_j = \frac{1}{10000}\binom{v}{j-1}, \quad j = 1, 2, \ldots, \left[\frac{v}{2}\right],$$
$$\nu_i = \binom{u}{i}, \quad i = \left[\frac{u}{2}\right] + 1, \ldots, u, \quad \mu_j = \binom{v}{j}, \quad j = \left[\frac{v}{2}\right] + 1, \ldots, v.$$

In Table 1, we give maximum error of approximation for $\mathcal{B}_{u,v}^{\nu,\mu}$ *for certain u and v values and related occurring* (y, z) *points.*

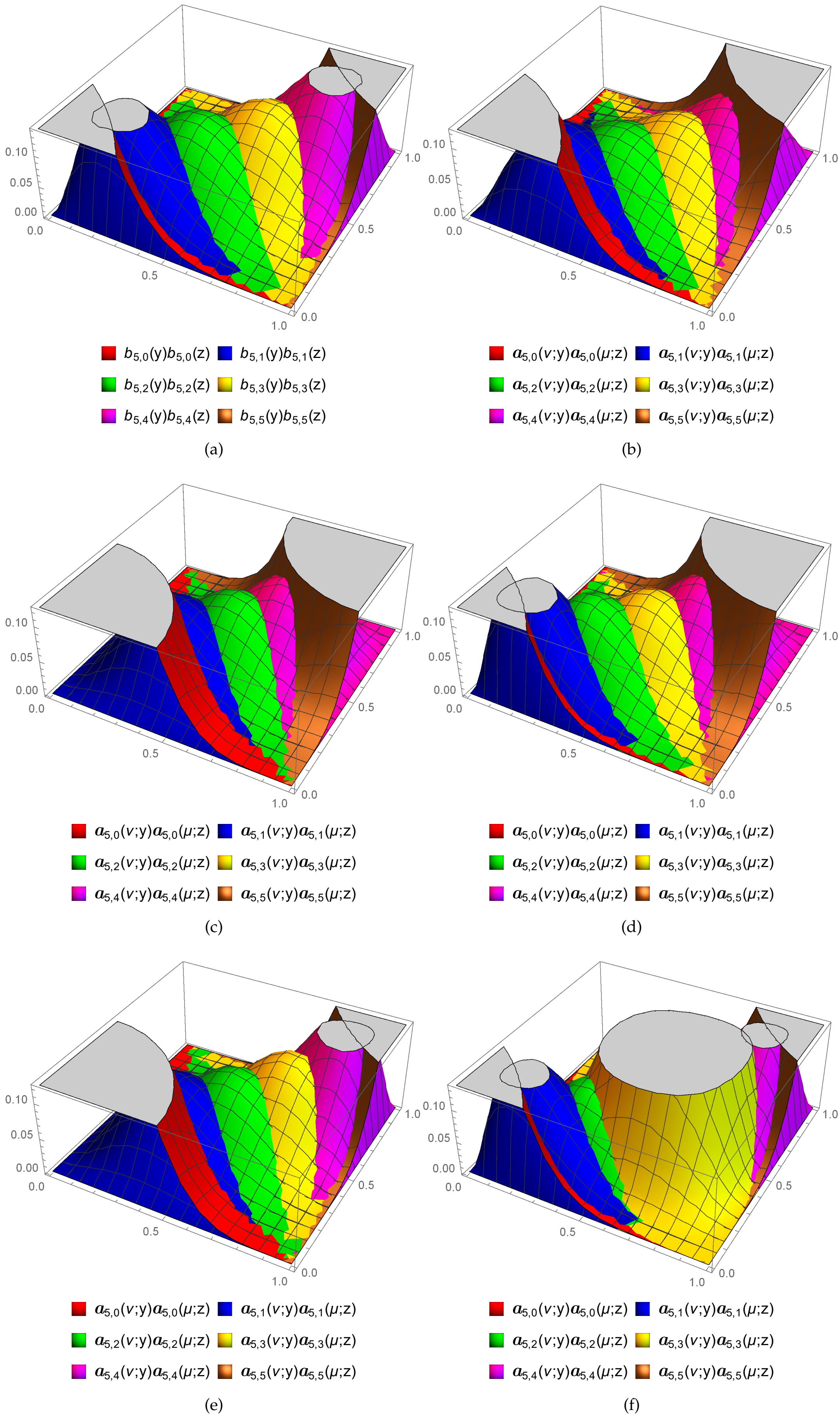


Figure 1: Behaviour of polynomials $a_{u,v}(\nu;y)a_{u,v}(\mu;z)$ and $b_{u,v}(y)b_{u,v}(z)$

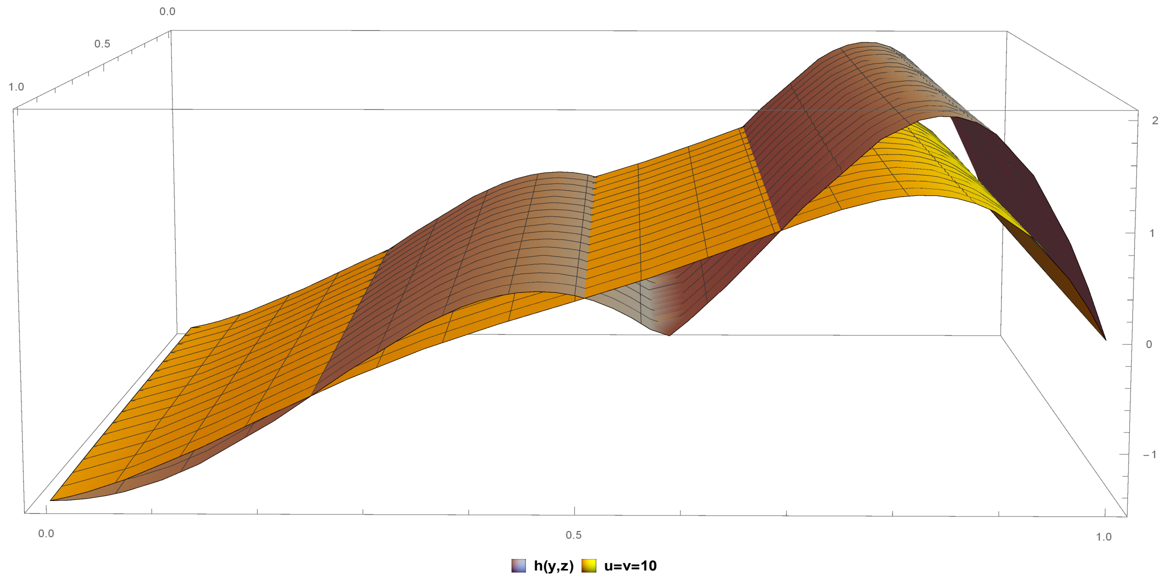


Figure 2: Approximations of $\mathcal{B}_{u,v}^{\nu,\mu}$ operators with different values of u and v for Example 5.2

Example 5.2. *Consider the following function*

$$h(y,z) = \frac{\left|5z^3 - 1\right| \left(y^3 - 22\right)(z+1)\cos(1.5\pi z)}{\left|y^3 - 15\right|}$$

on $(y,z) \in [0,1] \times [0,1]$*. We choose the following shape parameters for this example:*

$$\nu_i = \frac{7}{ui}\binom{u}{i-1},\ i = 1,2,\dots,\left[\frac{u}{2}\right],\ \mu_j = \frac{7}{vj}\binom{v}{j-1},\ j = 1,2,\dots,\left[\frac{v}{2}\right],$$
$$\nu_i = \frac{1}{10i}\binom{u}{i},\ i = \left[\frac{u}{2}\right]+1,\dots,u,\ \mu_j = \frac{1}{10j}\binom{v}{j},\ j = \left[\frac{v}{2}\right]+1,\dots,v.$$

In Figures 2-3, we give approximations of $\mathcal{B}_{u,v}^{\nu,\mu}$ *operators with* $u = v = 10$ *and related error of approximation, respectively. In Figure 4, we extensively examine errors with different values of* u *and* v*.*

Example 5.3. *Consider the following function*

$$h(y,z) = \frac{\left(z^3 - 2z^3y^2\right)\sin(2\pi z)}{2(y^3+1)}$$

on $(y,z) \in [0,1] \times [0,1]$*. We choose the following shape parameters for this example:*

$$\nu_i = -0.000005\binom{u}{i},\ i = 1,2,\dots,\left[\frac{u}{2}\right],\ \mu_j = -0.000005\binom{v}{j},\ j = 1,2,\dots,\left[\frac{v}{2}\right],$$
$$\nu_i = -0.000001\binom{u}{i-1},\ i = \left[\frac{u}{2}\right]+1,\dots,u,\ \mu_j = -0.000001\binom{v}{j-1},\ j = \left[\frac{v}{2}\right]+1,\dots,v.$$

In Figures 5-6, we provide convergence of $\mathcal{B}_{u,v}^{\nu,\mu}$ *and related errors of approximation for the given function. In Figure 7, we extensively examine errors with different values of* u *and* v*. We also compare the maximum error of* $\mathcal{B}_{u,v}^{\nu,\mu}$ *with the classical Bernstein operators in Table 2.*

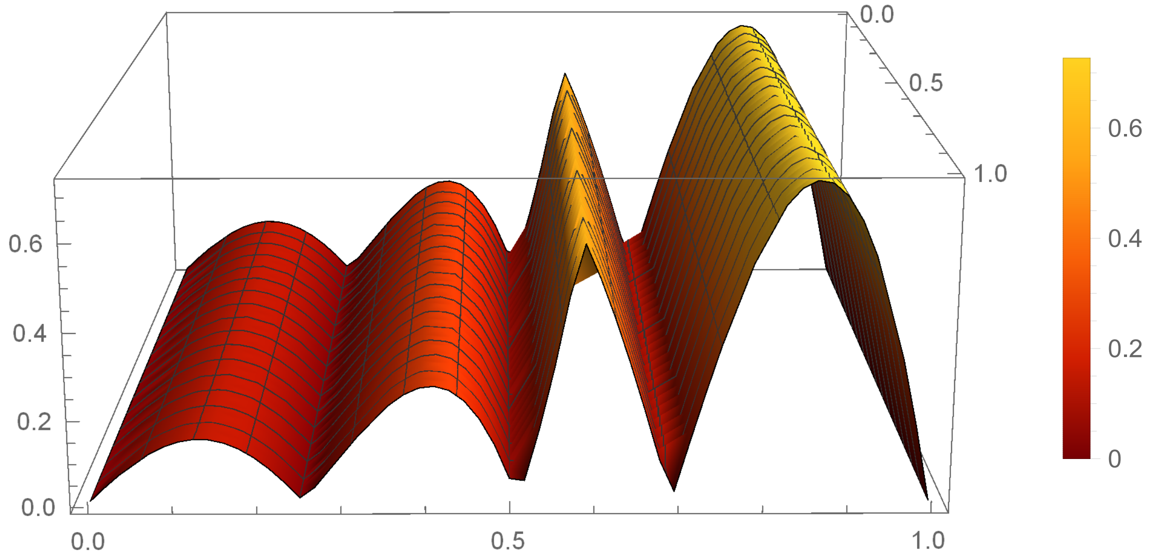
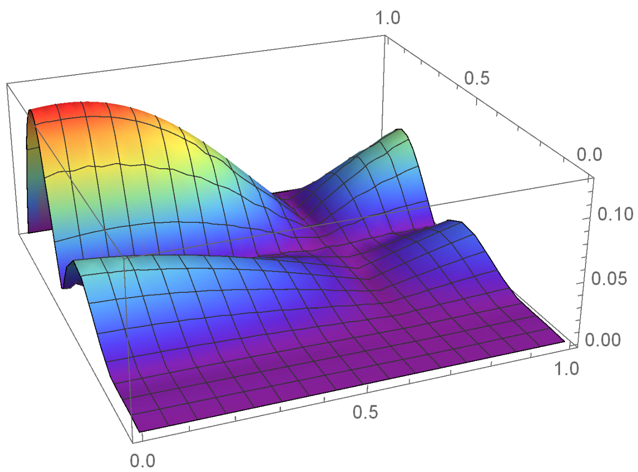
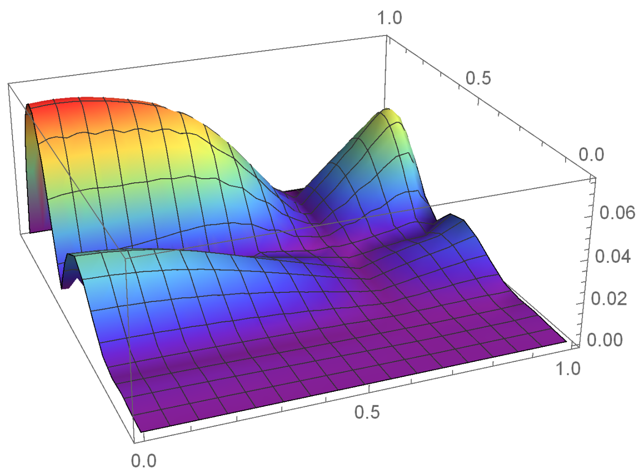


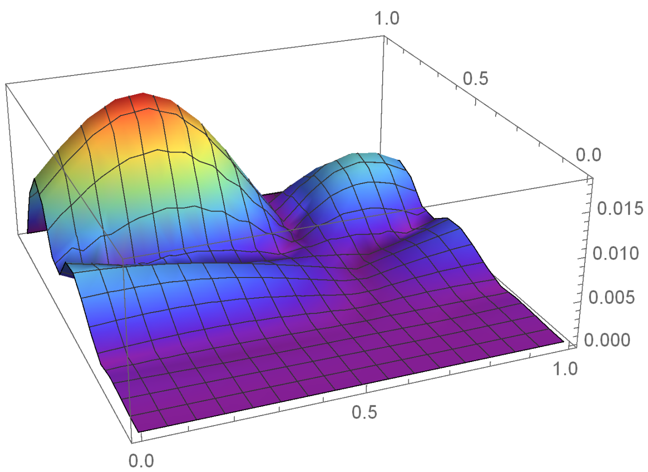
Figure 3: Errors of approximation for Example 5.2



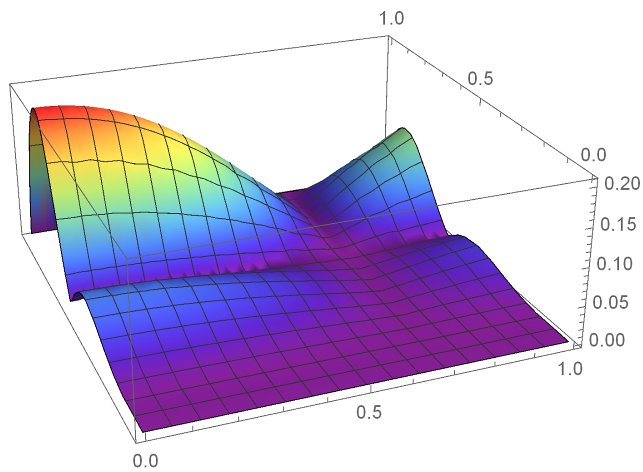
(a) $u, v = 6$



(b) $u, v = 15$

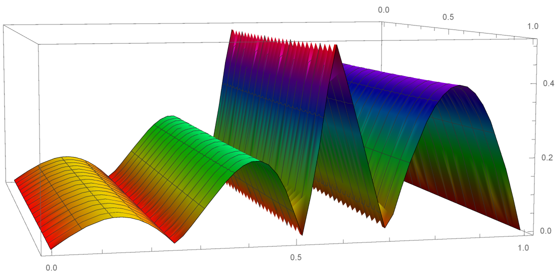


(c) $u = 10$ and $v = 100$

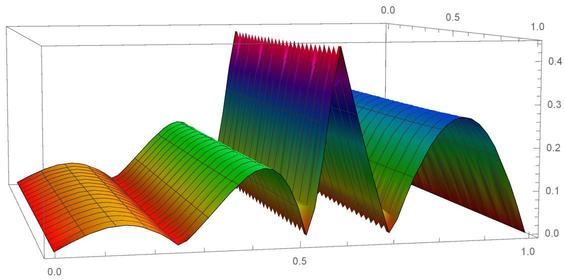


(d) $u = 100$ and $v = 5$

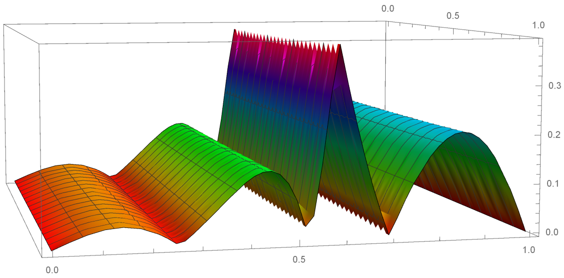
Figure 7: Approximation errors of $\mathcal{B}_{u,v}^{\nu,\mu}$ to the function $h(y,z)$ for different v and v values



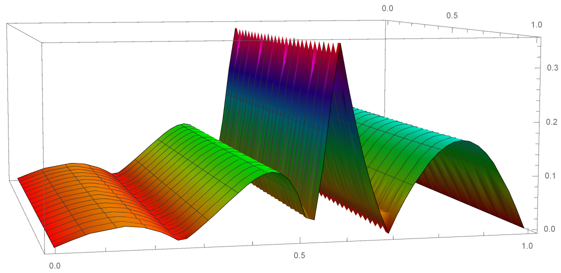
(a) $u, v = 20$



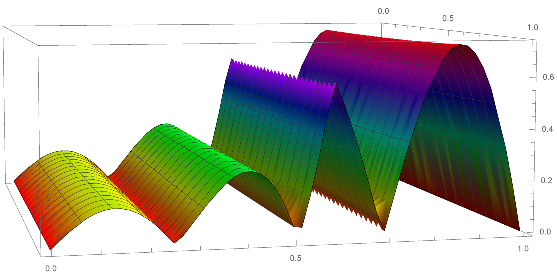
(b) $u, v = 30$



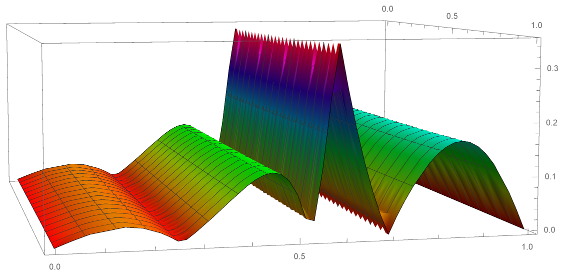
(c) $u, v = 40$



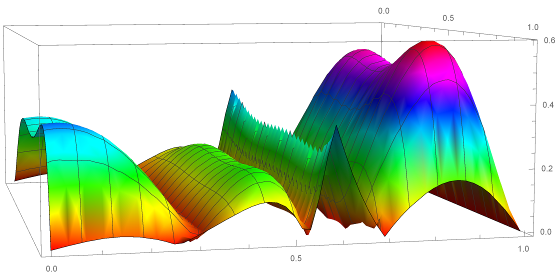
(d) $u, v = 50$



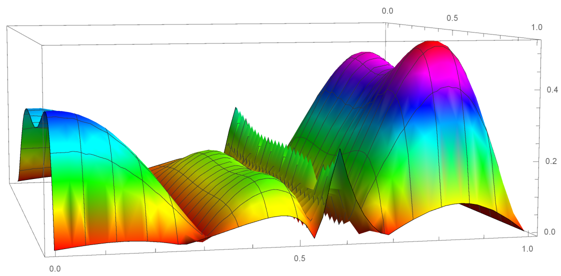
(e) $u = 50, v = 10$



(f) $u = 20, v = 50$



(g) $u = 50, v = 10$



(h) $u = 20, v = 50$

Figure 4: Approximation errors of $\mathcal{B}_{u,v}^{\nu,\mu}$ to the function $h(y,z)$ for different v and v values

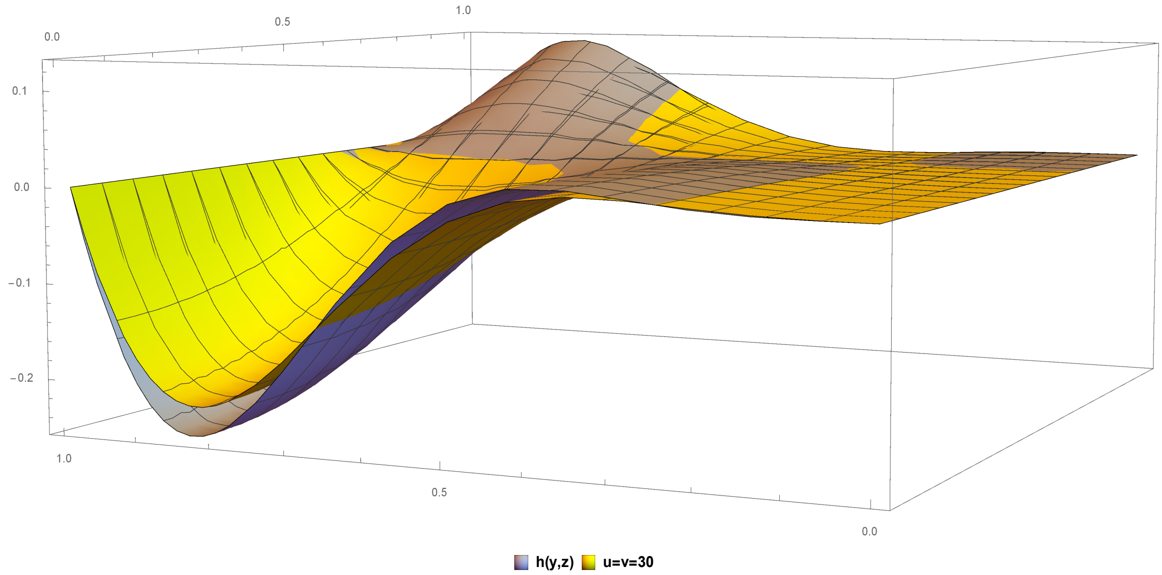


Figure 5: Approximation of $\mathcal{B}_{u,v}^{\nu,\mu}$ operators for $u = v = 30$

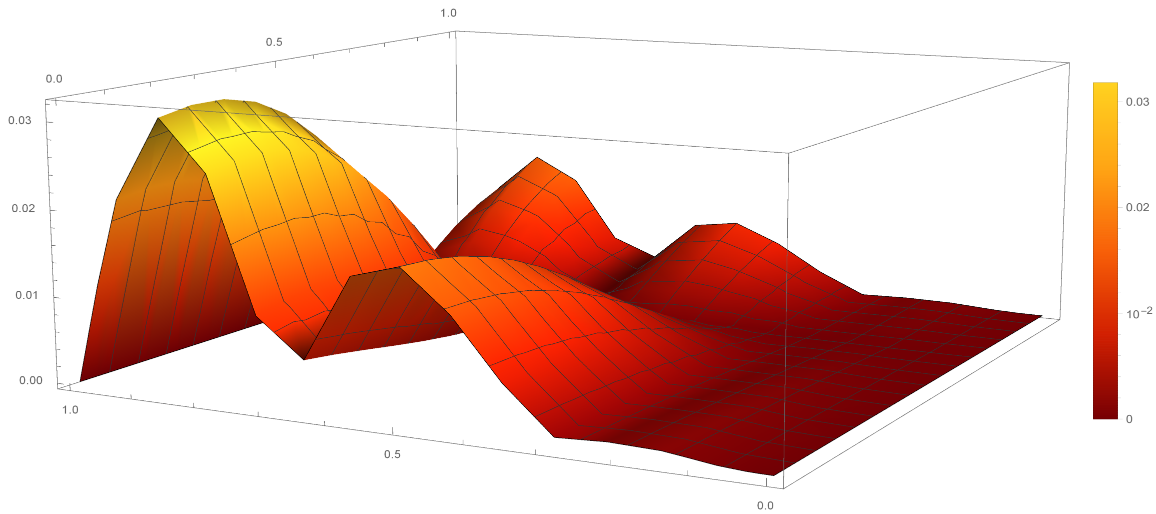


Figure 6: Error of approximation

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