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Estimation of $\mathfrak{f}$-Divergence and Shannon Entropy by Bullen Type Inequalities via Fink's Identity

Muhammad Adeel[a], Khuram Ali Khan[b], Đilda Pečarić[c], Josip Pečarić[d]

[a]*Department of Mathematics, University of Sargodha, Sargodha-40100, Pakistan and University of central Punjab, Lahore, Pakistan*
[b]*Department of Mathematics, University of Sargodha, Sargodha-40100, Pakistan*
[c]*University North, Koprivnica, Croatia*
[d]*Croatian Academy of Sciences and Arts, Zagreb, Croatia*

Abstract. By using Fink's identity some new generalizations of Levinson type inequalities for n-convex functions are obtained. In seek of applications of our results to information theory, new generalizations based on $\mathfrak{f}$-divergence estimates are also proven. Moreover, some inequalities for Shannon entropies are deduced as well.

1. Introduction and Preliminaries

The theory of convex functions has encountered a fast advancement. This can be attributed to a few causes: firstly, applications of convex functions are directly involve in modern analysis, secondly, many important inequalities are results applications of convex functions and convex functions are closely related to inequalities (see [1]).
Ky Fan's inequality is generalized by Levinson [21], (see also [23, p. 32, Theorem 1]) for 3-convex functions as follow:

Theorem 1.1. *Let $f : I = (0, 2\alpha) \to \mathbb{R}$ with $f^{(3)}(t) \geq 0$. Let $x_\rho \in (0, \alpha)$ and $p_\rho > 0$. Then*

$$\frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho f(x_\rho) - f\Big(\frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho x_\rho\Big) \leq \frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho f(2\alpha - x_\rho) - f\Big(\frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho (2\alpha - x_\rho)\Big). \tag{1}$$

Functional form of (1) is defined as follows:

$$\begin{aligned} J_1(f(\cdot)) \quad = \quad & \frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho f(2a - x_\rho) - f\Big(\frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho (2a - x_\rho)\Big) - \frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho f(x_\rho) \\ & + f\Big(\frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho x_\rho\Big). \end{aligned} \tag{2}$$

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Email addresses: adeel.uosmaths@gmail.com (Muhammad Adeel), khuramsms@gmail.com (Khuram Ali Khan), gildapeca@gmail.com (Đilda Pečarić), pecaric@element.hr (Josip Pečarić)

Working with the divided differences, assumptions of differentiability on f can be weakened.
In [24], Popoviciu noted that (1) is valid on $(0, 2a)$ for 3-convex functions, while in [6], (see also [23, p. 32, Theorem 2]) Bullen gave different proof of Popoviciu's result and also the converse of (1).

Theorem 1.2. *(a) Let $f : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a 3-convex function and $x_\rho, y_\rho \in [\zeta_1, \zeta_2]$ for $\rho = 1, 2, \cdots, \mu$ be such that*

$$\max\{x_1 \ldots x_\mu\} \leq \min\{y_1 \ldots y_\mu\}, \quad x_1 + y_1 = \ldots = x_\mu + y_\mu \tag{3}$$

and $p_\rho > 0$ then

$$\frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho f(x_\rho) - f\Big(\frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho x_\rho\Big) \leq \frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho f(y_\rho) - f\Big(\frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho y_\rho\Big). \tag{4}$$

(b) If f is continuous and $p_\rho > 0$, (4) holds for all x_ρ, y_ρ satisfying (3), then f is 3-convex.

Functional form of (4) is defined as follows:

$$J_2(f(\cdot)) = \frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho f(y_\rho) - f\Big(\frac{1}{P_\mu}\sum_{\rho=1}^{\mu} q_\rho y_\rho\Big) - \frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho f(x_\rho) + f\Big(\frac{1}{P_\mu}\sum_{\rho=1}^{\mu} p_\rho x_\rho\Big). \tag{5}$$

Remark 1.3. *It is essential to take note of that under the suppositions of Theorem 1.1 and Theorem 1.2, if the function f is 3-convex then $J_i(f(\cdot)) \geq 0$ for $i = 1, 2$, and $J_i(f(\cdot)) = 0$ for $f(x) = x$ or $f(x) = x^2$ or f is a constant function.*

In [25], (see also [23, p. 32, Theorem 4]) Pečarić weakened the assumption (3) and prove that inequality (4) still holds, i. e. the following result holds:

Theorem 1.4. *Let $f : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a 3-convex function, $p_\rho > 0$. Also, let $x_\rho, y_\rho \in [\zeta_1, \zeta_2]$ be such that $x_\rho + y_\rho = 2\check{c}$, for $\rho = 1, \cdots, \mu$ $x_\rho + x_{\mu-\rho+1} \leq 2\check{c}$ and $\frac{p_\rho x_\rho + p_{\mu-\rho+1} x_{\mu-\rho+1}}{p_\rho + p_{\mu-\rho+1}} \leq \check{c}$. Then (4) holds.*

In [22], Mercer made a notable work by replacing the condition of symmetric distribution of points x_ρ and y_ρ with symmetric variances of points x_ρ and y_ρ. Second condition is weaker condition.

Theorem 1.5. *Let f be a 3-convex function on $[\zeta_1, \zeta_2]$, p_ρ are positive such that $\sum_{\rho=1}^{\mu} p_\rho = 1$. Also, let x_ρ, y_ρ satisfy $\max\{x_1 \ldots x_\rho\} \leq \min\{y_1 \ldots y_\rho\}$ and*

$$\sum_{\rho=1}^{\mu} p_\rho \left(x_\rho - \sum_{\rho=1}^{\mu} p_\rho x_\rho \right)^2 = \sum_{\rho=1}^{\mu} p_\rho \left(y_\rho - \sum_{\rho=1}^{\mu} p_\rho y_\rho \right)^2, \tag{6}$$

then (4) holds.

In the present paper, we use Fink's identity and prove many interesting results. The following theorem is proved by Fink in [2].

Theorem 1.6. *Let $\mathfrak{f} : [\zeta_1, \zeta_2] \to \mathbb{R}$, $n \geq 1$ and $\mathfrak{f}^{(n-1)}$ is absolutely continuous on $[\zeta_1, \zeta_2]$, where $\zeta_1, \zeta_2 \in \mathbb{R}$. Then*

$$\begin{aligned} \mathfrak{f}(x) &= \frac{n}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2} \mathfrak{f}(t)dt \\ &\quad - \sum_{w=1}^{n-1}\Big(\frac{n-w}{w!}\Big)\Big(\frac{\mathfrak{f}^{(w-1)}(\zeta_1)(x-\zeta_1)^w - \mathfrak{f}^{(w-1)}(\zeta_2)(x-\zeta_2)^w}{\zeta_2 - \zeta_1}\Big) \\ &\quad + \frac{1}{(n-1)!(\zeta_2 - \zeta_1)}\int_{\zeta_1}^{\zeta_2} (x-t)^{n-1} W^{[\zeta_1,\zeta_2]}(t,x)\mathfrak{f}^{(n)}(t)dt, \end{aligned} \tag{7}$$

where

$$W^{[\zeta_1,\zeta_2]}(t,x) = \begin{cases} (t-\zeta_1), & \zeta_1 \leq t \leq x \leq \zeta_2, \\ (t-\zeta_2), & \zeta_1 \leq x < t \leq \zeta_2. \end{cases} \tag{8}$$

2. Main results

Motivated by identity (5), we construct the following identity with help of (7) coming from Fink's identity.

2.1. Bullen type inequalities for higher order convex functions

First we define the following functional. $\mathcal{F}$: Let $\mathfrak{f} : I_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a function, $x_1, \ldots, x_\mu$ and $y_1, \ldots, y_\nu \in I_1$ such that

$$\max\{x_1 \ldots x_\mu\} \leq \min\{y_1 \ldots y_\nu\}, \quad x_1 + y_1 = \ldots = x_\mu + y_\nu. \tag{9}$$

Also, let $(p_1, \ldots, p_\mu) \in \mathbb{R}^\mu$ and $(q_1, \ldots, q_\nu) \in \mathbb{R}^\nu$ be such that $\sum_{\rho=1}^{\mu} p_\rho = 1$, $\sum_{\varrho=1}^{\nu} q_\varrho = 1$ and $x_\rho, y_\varrho, \sum_{\rho=1}^{\mu} p_\rho x_\rho$, $\sum_{\varrho=1}^{\nu} q_\varrho y_\varrho \in I_1$. Then

$$\mathbb{J}(\mathfrak{f}(\cdot)) = \sum_{\varrho=1}^{\nu} q_\varrho \mathfrak{f}(y_\varrho) - \mathfrak{f}\Big(\sum_{\varrho=1}^{\nu} q_\varrho y_\varrho\Big) - \sum_{\rho=1}^{\mu} p_\rho \mathfrak{f}(x_\rho) + \mathfrak{f}\Big(\sum_{\rho=1}^{\mu} p_\rho x_\rho\Big). \tag{10}$$

Theorem 2.1. *Assume $\mathcal{F}$ and $\mathfrak{f} : I_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ are such that $\mathfrak{f}^{(n-1)}$ is absolutely continuous. Then*

$$\begin{aligned}\mathbb{J}(\mathfrak{f}(\cdot)) \;=\; & \sum_{w=3}^{n-1} \Big(\frac{n-w}{w!(\zeta_2 - \zeta_1)}\Big)\Big(\mathfrak{f}^{(w-1)}(\zeta_2)\mathbb{J}(\cdot - \zeta_2)^w - \mathfrak{f}^{(w-1)}(\zeta_1)\mathbb{J}(\cdot - \zeta_1)^w\Big) \\ & + \frac{1}{(n-1)!(\zeta_2 - \zeta_1)} \int_{\zeta_1}^{\zeta_2} \mathbb{J}(\cdot - t)^{n-1} W^{[\zeta_1,\zeta_2]}(t, \cdot)\mathfrak{f}^{(n)}(t)dt.\end{aligned} \tag{11}$$

Proof. Using Fink's identity (7) in the functional (10), we have

$$\begin{aligned}\mathbb{J}(\mathfrak{f}(\cdot)) \;=\; & \sum_{\varrho=1}^{\nu} q_\varrho \Big[\frac{n}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \mathfrak{f}(t)dt \\ & - \sum_{w=1}^{n-1} \Big(\frac{n-w}{w!}\Big)\Big(\frac{\mathfrak{f}^{(w-1)}(\zeta_1)(y_\varrho - \zeta_1)^w - \; mathfrakf^{(w-1)}(\zeta_2)(y_\varrho - \zeta_2)^w}{\zeta_2 - \zeta_1}\Big) \\ & + \frac{1}{(n-1)!(\zeta_2 - \zeta_1)} \int_{\zeta_1}^{\zeta_2} (y_\varrho - t)^{n-1} W^{[\zeta_1,\zeta_2]}(t, y_\varrho)\mathfrak{f}^{(n)}(t)dt\Big] \\ & - \Big[\frac{n}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \mathfrak{f}(t)dt \\ & - \sum_{w=1}^{n-1} \Big(\frac{n-w}{w!}\Big)\Big(\frac{\mathfrak{f}^{(w-1)}(\zeta_1)(\sum_{\varrho=1}^{\nu} q_\varrho y_\varrho - \zeta_1)^w - \mathfrak{f}^{(w-1)}(\zeta_2)(\sum_{\varrho=1}^{\nu} q_\varrho y_\varrho - \zeta_2)^w}{\zeta_2 - \zeta_1}\Big) \\ & + \frac{1}{(n-1)!(\zeta_2 - \zeta_1)} \int_{\zeta_1}^{\zeta_2} \Big(\sum_{\varrho=1}^{\nu} q_\varrho y_\varrho - t\Big)^{n-1} W^{[\zeta_1,\zeta_2]}\Big(t, \sum_{\varrho=1}^{\nu} q_\varrho y_\varrho\Big)\mathfrak{f}^{(n)}(t)dt\Big] \\ & - \Big[\sum_{\rho=1}^{\mu} p_\rho \Big(\frac{n}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \mathfrak{f}(t)dt \\ & - \sum_{w=1}^{n-1} \Big(\frac{n-w}{w!}\Big)\Big(\frac{\mathfrak{f}^{(w-1)}(\zeta_1)(x_\rho - \zeta_1)^w - \mathfrak{f}^{(w-1)}(\zeta_2)(x_\rho - \zeta_2)^w}{\zeta_2 - \zeta_1}\Big)\end{aligned}$$

$$+\frac{1}{(n-1)!(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}(x_\rho-t)^{n-1}W^{[\zeta_1,\zeta_2]}(t,x_\rho)\mathfrak{f}^{(n)}(t)dt\Big)\Big]$$
$$+\Big[\frac{n}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\mathfrak{f}(t)dt$$
$$-\sum_{w=1}^{n-1}\Big(\frac{n-w}{w!}\Big)\Big(\frac{\mathfrak{f}^{(w-1)}(\zeta_1)(\sum_{\rho=1}^{\mu}p_\rho x_\rho-\zeta_1)^w-\mathfrak{f}^{(w-1)}(\zeta_2)(\sum_{\rho=1}^{\mu}p_\rho x_\rho-\zeta_2)^w}{\zeta_2-\zeta_1}\Big)$$
$$+\frac{1}{(n-1)!(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}\Big(\sum_{\rho=1}^{\mu}p_\rho x_\rho-t\Big)^{n-1}W^{[\zeta_1,\zeta_2]}\Big(t,\sum_{\rho=1}^{\mu}p_\rho x_\rho\Big)\mathfrak{f}^{(n)}(t)dt\Big].$$

After some simple calculation we have

$$\begin{aligned}\mathbb{J}(\mathfrak{f}(\cdot)) = & -\sum_{\varrho=1}^{\nu}q_\varrho\Big[\sum_{w=3}^{n-1}\Big(\frac{n-w}{w!}\Big)\Big(\frac{\mathfrak{f}^{(w-1)}(\zeta_1)(y_\varrho-\zeta_1)^w-\mathfrak{f}^{(w-1)}(\zeta_2)(y_\varrho-\zeta_2)^w}{\zeta_2-\zeta_1}\Big)\Big]\\
&+\sum_{w=3}^{n-1}\Big(\frac{n-w}{w!}\Big)\Big(\frac{\mathfrak{f}^{(w-1)}(\zeta_1)(\sum_{\varrho=1}^{\nu}q_\varrho y_\varrho-\zeta_1)^w-\mathfrak{f}^{(w-1)}(\zeta_2)(\sum_{\varrho=1}^{\nu}q_\varrho y_\varrho-\zeta_2)^w}{\zeta_2-\zeta_1}\Big)\\
&+\sum_{\rho=1}^{\mu}p_\rho\Big[\sum_{w=3}^{n-1}\Big(\frac{n-w}{w!}\Big)\Big(\frac{\mathfrak{f}^{(w-1)}(\zeta_1)(x_\rho-\zeta_1)^w-\mathfrak{f}^{(w-1)}(\zeta_2)(x_\rho-\zeta_2)^w}{\zeta_2-\zeta_1}\Big)\Big]\\
&-\sum_{w=3}^{n-1}\Big(\frac{n-w}{w!}\Big)\Big(\frac{\mathfrak{f}^{(w-1)}(\zeta_1)(\sum_{\rho=1}^{\mu}p_\rho x_\rho-\zeta_1)^w-\mathfrak{f}^{(w-1)}(\zeta_2)(\sum_{\rho=1}^{\mu}p_\rho x_\rho-\zeta_2)^w}{\zeta_2-\zeta_1}\Big)\\
&+\sum_{\varrho=1}^{\nu}q_\varrho\Big[\frac{1}{(n-1)!(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}(y_\varrho-t)^{n-1}W^{[\zeta_1,\zeta_2]}(t,y_\varrho)\mathfrak{f}^{(n)}(t)dt\Big]\\
&-\Big[\frac{1}{(n-1)!(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}(\sum_{\varrho=1}^{\nu}q_\varrho y_\varrho-t)^{n-1}W^{[\zeta_1,\zeta_2]}(t,\sum_{\varrho=1}^{\nu}q_\varrho y_\varrho)\mathfrak{f}^{(n)}(t)dt\Big]\\
&-\sum_{\rho=1}^{\mu}p_\rho\Big[\frac{1}{(n-1)!(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}(x_\rho-t)^{n-1}W^{[\zeta_1,\zeta_2]}(t,x_\rho)\mathfrak{f}^{(n)}(t)dt\Big]\\
&+\Big[\frac{1}{(n-1)!(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}(\sum_{\rho=1}^{\mu}p_\rho x_\rho-t)^{n-1}W^{[\zeta_1,\zeta_2]}(t,\sum_{\rho=1}^{\mu}p_\rho x_\rho)\mathfrak{f}^{(n)}(t)dt\Big].\end{aligned}$$

By using definition of $\mathbb{J}(\cdot)$, we have

$$\begin{aligned}\mathbb{J}(\mathfrak{f}(\cdot)) = & \sum_{w=3}^{n-1}\Big(\frac{n-w}{w!(\zeta_2-\zeta_1)}\Big)\Big(\mathfrak{f}^{(w-1)}(\zeta_2)\mathbb{J}(\cdot-\zeta_2)^w-\mathfrak{f}^{(w-1)}(\zeta_1)\mathbb{J}(\cdot-\zeta_1)^w\Big)\\
&+\frac{1}{(n-1)!(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}\mathbb{J}(\cdot-t)^{n-1}W^{[\zeta_1,\zeta_2]}(t,\cdot)\mathfrak{f}^{(n)}(t)dt,\end{aligned}$$

where

$$\begin{aligned}\mathbb{J}(\cdot-\zeta_1)^w = & \sum_{\varrho=1}^{\nu}q_\varrho(y_\varrho-\zeta_1)^w-\Big(\sum_{\varrho=1}^{\nu}q_\varrho y_\varrho-\zeta_1\Big)^w-\sum_{\rho=1}^{\mu}p_\rho(x_\rho-\zeta_1)^w\\
&+\Big(\sum_{\rho=1}^{\mu}p_\rho x_\rho-\zeta_1\Big)^w \qquad (12)\end{aligned}$$

and

$$\mathbb{J}(\cdot-\zeta_2)^w = \sum_{\varrho=1}^{\nu} q_\varrho (y_\varrho-\zeta_2)^w - \Big(\sum_{\varrho=1}^{\nu} q_\varrho y_\varrho - \zeta_2\Big)^w - \sum_{\rho=1}^{\mu} p_\rho (x_\rho-\zeta_2)^w + \Big(\sum_{\rho=1}^{\mu} p_\rho x_\rho - \zeta_2\Big)^w. \tag{13}$$

□

As an application of the above obtained identity, the next theorem gives generalization of Bullen type inequalities for n-convex functions involving Fink's identity.

Theorem 2.2. *Let all the assumptions of Theorem 2.1 be satisfied and let for $n \geq 1$*

$$\mathbb{J}(\cdot-t)^{n-1} W^{[\zeta_1,\zeta_2]}(t,\cdot) \geq 0 \tag{14}$$

If $\mathfrak{f}$ is n-convex function, then we have

$$\mathbb{J}(\mathfrak{f}(\cdot)) \geq \sum_{w=3}^{n-1}\Big(\frac{n-w}{w!(\zeta_2-\zeta_1)}\Big) \times\Big(\mathfrak{f}^{(w-1)}(\zeta_2)\mathbb{J}(\cdot-\zeta_2)^w - \mathfrak{f}^{(w-1)}(\zeta_1)\mathbb{J}(\cdot-\zeta_1)^w\Big). \tag{15}$$

Proof. Since $\mathfrak{f}^{(n-1)}$ is absolutely continuous on $[\zeta_1,\zeta_2]$, therefore $\mathfrak{f}^{(n)}$ exists almost everywhere. As $\mathfrak{f}$ is n-convex we have $\mathfrak{f}^{(n)}(x) \geq 0$ for all $x \in [\zeta_1,\zeta_2]$. Applying Theorem 2.1, we obtain (15). □

Remark 2.3. *In Theorem 2.2, reverse inequality in (14) leads to reverse inequality in (15).*

If we put $\nu=\mu$, $p_\rho = q_\varrho$ and by using positive weights in (10), then $\mathbb{J}(\cdot)$ converted to the functional $J_2(\cdot)$ defined in (5), also in this case, (11), (12), (13), (14) and (15) become

$$J_2(\mathfrak{f}(\cdot)) = \sum_{\rho=1}^{\mu} p_\rho \mathfrak{f}(y_\varrho) - \mathfrak{f}\Big(\sum_{\rho=1}^{\mu} p_\rho y_\varrho\Big) - \sum_{\rho=1}^{\mu} p_\rho \mathfrak{f}(x_\rho) + \mathfrak{f}\Big(\sum_{\rho=1}^{\mu} p_\rho x_\rho\Big), \tag{16}$$

$$J(\mathfrak{f}(\cdot)) = \sum_{w=3}^{n-1}\Big(\frac{n-w}{w!(\zeta_2-\zeta_1)}\Big)\Big(\mathfrak{f}^{(w-1)}(\zeta_2)J_2(\cdot-\zeta_2)^w - \mathfrak{f}^{(w-1)}(\zeta_1)J_2(\cdot-\zeta_1)^w\Big) + \frac{1}{(n-1)!(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2} J_2(\cdot-t)^{n-1}W^{[\zeta_1,\zeta_2]}(t,\cdot)\mathfrak{f}^{(n)}(t)dt, \tag{17}$$

$$J_2(\cdot-\zeta_1)^w = \sum_{\rho=1}^{\mu} p_\rho (y_\varrho-\zeta_1)^w - \Big(\sum_{\rho=1}^{\mu} p_\rho y_\varrho - \zeta_1\Big)^w - \sum_{\rho=1}^{\mu} p_\rho (x_\rho-\zeta_1)^w + \Big(\sum_{\rho=1}^{\mu} p_\rho x_\rho - \zeta_1\Big)^w, \tag{18}$$

$$J_2(\cdot-\zeta_2)^w = \sum_{\rho=1}^{\mu} p_\rho (y_\varrho-\zeta_2)^w - \Big(\sum_{\rho=1}^{\mu} p_\rho y_\varrho - \zeta_2\Big)^w - \sum_{\rho=1}^{\mu} p_\rho (x_\rho-\zeta_2)^w + \Big(\sum_{\rho=1}^{\mu} p_\rho x_\rho - \zeta_2\Big)^w, \tag{19}$$

$$J_2(\cdot - t)^{n-1}W^{[\zeta_1,\zeta_2]}(t,\cdot) \geq 0 \tag{20}$$

and

$$\begin{aligned} J_2(\mathfrak{f}(\cdot)) \quad &\geq \quad \sum_{w=3}^{n-1}\left(\frac{n-w}{w!(\zeta_2-\zeta_1)}\right) \\ &\quad \times\left(\mathfrak{f}^{(w-1)}(\zeta_2)J_2(\cdot-\zeta_2)^w - \mathfrak{f}^{(w-1)}(\zeta_1)J_2(\cdot-\zeta_1)^w\right). \end{aligned} \tag{21}$$

Now, we will give generalization of Bullen type inequality for μ-tuples.

Theorem 2.4. *Let all the assumptions of Theorem 2.1 be satisfied in addition with the condition that* $\mathbf{p} = (p_1, ..., p_\mu)$ *be positive* μ*-tuples such that that* $\sum_{\rho=1}^{n} p_\rho = 1$ *and consider* $\mathfrak{f} : [\zeta_1, \zeta_2] \to \mathbb{R}$ *be n-convex function. Then*
(i) *If n is odd and* $n > 4$, *then for every n-convex function* $\mathfrak{f} : [\zeta_1, \zeta_2] \to \mathbb{R}$, *(21) holds.*
(ii) *Let the inequality (21) be satisfied. If the function*

$$\begin{aligned} \Theta(\cdot) \quad &= \quad \sum_{w=3}^{n-1}\left(\frac{n-w}{w!(\zeta_2-\zeta_1)}\right) \\ &\quad \times\left(\mathfrak{f}^{(w-1)}(\zeta_2)J_2(\cdot-\zeta_2)^w - \mathfrak{f}^{(w-1)}(\zeta_1)J_2(\cdot-\zeta_1)^w\right) \end{aligned} \tag{22}$$

is 3*-convex, the right hand side of (21) is non negative and we have inequality*

$$J_2(\mathfrak{f}(\cdot)) \geq 0. \tag{23}$$

Proof. (i) Let

$$\Psi(\cdot) := (\cdot - t)^{n-1}W^{[\zeta_1,\zeta_2]}(t,\cdot) = \begin{cases} (\cdot - t)^{n-1}(t-\zeta_1), & \zeta_1 \leq t \leq \cdot \leq \zeta_2, \\ (\cdot - t)^{n-1}(t-\zeta_2), & \zeta_1 \leq \cdot \leq t \leq \zeta_2, \end{cases}$$

we have

$$\Psi'''(\cdot) := \begin{cases} (n-1)(n-2)(n-3)(\cdot - t)^{n-4}(t-\zeta_1), & \zeta_1 \leq t \leq \cdot \leq \zeta_2, \\ (n-1)(n-2)(n-3)(\cdot - t)^{n-4}(t-\zeta_2), & \zeta_1 \leq \cdot \leq t \leq \zeta_2. \end{cases}$$

So, Ψ is 3-convex for odd n, where $n > 4$. Hence, by using Remark 1.3, (20) holds for odd values of n. Therefore using Theorem 2.2, we have (21).
(ii) $J(.)$ is linear functional, so we can rewrite the right hand side. of (21) in the form $J(\Theta(\cdot))$, where Θ is defined in (22) and will be obtained after reorganization of this side. Since Θ is assumed to be 3-convex, therefore using the given conditions and by following Remark 1.3, the non negativity of the right hand side. of (21) is immediate and we have (23) for μ-tuples. □

Next we have generalized form (for real weights) of Levinson's type inequality for $2n$ points given in [25](see also [23]).

Theorem 2.5. *Let* $\mathfrak{f} : I_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ *be such that* $\mathfrak{f}^{(n-1)}$ *is absolutely continuous,* $(p_1, \ldots, p_\mu) \in \mathbb{R}^\mu$, $(q_1, \ldots, q_\nu) \in \mathbb{R}^\nu$ *be such that* $\sum_{\rho=1}^{\mu} p_\rho = 1$, $\sum_{\varrho=1}^{\nu} q_\varrho = 1$, $\sum_{\varrho=1}^{\nu} q_\varrho y_\varrho$ *and* $\sum_{\rho=1}^{\mu} p_\rho x_\rho \in I_1$. *Also, let* $x_1, \ldots, x_\mu$ *and* $y_1, \ldots, y_\nu \in I_1$ *such that* $x_\rho + y_\varrho = 2\check{c}$ *and* $x_\rho + x_{\mu-\rho+1} \leq 2\check{c}$, $\frac{p_\rho x_\rho + p_{\mu-\rho+1}x_{\mu-\rho+1}}{p_\rho + p_{\mu-\rho+1}} \leq \check{c}$. *Then (11) holds.*

Proof. Using Theorem 2.1 with the conditions given in statement we get the required result. □

In the following theorem we obtain generalizations of Levinson's type functional for n-convex functions.

Theorem 2.6. *Let* $\mathfrak{f} : I_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ *be such that* $\mathfrak{f}^{(n-1)}$ *is absolutely continuous,* $(p_1, \ldots, p_\mu) \in \mathbb{R}^\mu$, $(q_1, \ldots, q_\nu) \in \mathbb{R}^\nu$ *are such that* $\sum_{\rho=1}^{\mu} p_\rho = 1$, $\sum_{\varrho=1}^{\nu} q_\varrho = 1$, $\sum_{\varrho=1}^{\nu} q_\varrho y_\varrho$ *and* $\sum_{\rho=1}^{\mu} p_\rho x_\rho \in I_1$. *Also, let* $x_1, \ldots, x_\mu$ *and* $y_1, \ldots, y_\nu \in I_1$ *be such that* $x_\rho + y_\varrho = 2\check{c}$ *and* $x_\rho + x_{\mu-\rho+1} \leq 2\check{c}$, $\frac{p_\rho x_\rho + p_{\mu-\rho+1}x_{\mu-\rho+1}}{p_\rho + p_{\mu-\rho+1}} \leq \check{c}$. *If (14) is valid then inequality (15) is also valid.*

Proof. Proof is similar to Theorem 2.2. □

Theorem 2.7. *Let $\mathfrak{f} : I_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ be such that $\mathfrak{f}^{(n-1)}$ is absolutely continuous, $(p_1, \dots, p_\mu) \in \mathbb{R}^\mu$ be such that $\sum_{\rho=1}^{\mu} p_\rho = 1$. Also, let $x_1, \dots, x_\mu$ and $y_1, \dots, y_n \in I_1$ be such that $x_\rho + y_\varrho = 2\check{c}$ and $x_\rho + x_{\mu-\rho+1} \leq 2\check{c}$, $\frac{p_\rho x_\rho + p_{\mu-\rho+1} x_{\mu-\rho+1}}{p_\rho + p_{\mu-\rho+1}} \leq \check{c}$. Then*
(i) *If n is odd and $n > 4$, then for every n-convex function $\mathfrak{f} : [\zeta_1, \zeta_2] \to \mathbb{R}$, (21) holds.*
(ii) *Let the inequality (21) be satisfied. If the function Θ is same as defined in (22), is 3-convex the right hand side of (21) is non negative, then inequality (23) holds.*

Proof. By using Theorem 2.6 and Remark 1.3, we get the inequality (23). □

In the next result we have Levinson's type inequality (for real weights) under Mercer's condition.

Corollary 2.8. *Let $\mathfrak{f} : I_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ be such that $\mathfrak{f}^{(n-1)}$ is absolutely continuous and x_ρ, y_ϱ satisfy $\max\{x_1 \dots x_\mu\} \leq \min\{y_1 \dots y_\nu\}$ and*

$$\sum_{\rho=1}^{\mu} p_\rho \left(x_\rho - \sum_{\rho=1}^{\mu} p_\rho x_\rho \right)^2 = \sum_{\rho=1}^{\mu} q_\varrho \left(y_\varrho - \sum_{\rho=1}^{\mu} q_\varrho y_\varrho \right)^2. \tag{24}$$

Also, let $(p_1, \dots, p_\mu) \in \mathbb{R}^\mu$ be such that $\sum_{\rho=1}^{\mu} p_\rho = 1$. Then (17) holds, where $J_2(\mathfrak{f}(\cdot))$ is defined in (16).

Proof. Choose x_ρ and y_ϱ such that the conditions $\max\{x_1 \dots x_\mu\} \leq min\{y_1 \dots y_n\}$ and (24) holds for positive weights in Theorem 2.1, we get required the result. □

2.2. Levinson's inequality for higher order convex functions

Motivated by identity (2), we construct the following identities for two type of data points with help of (7) coming from Fink's identity.
$\mathcal{H}$: Let $\mathfrak{f} : I_2 = [0, 2a] \to \mathbb{R}$ be a function, $x_1, \dots, x_\mu \in (0, a)$, $(p_1, \dots, p_\mu) \in \mathbb{R}^\mu$, $(q_1, \dots, q_\nu) \in \mathbb{R}^\nu$ are real numbers such that $\sum_{\rho=1}^{\mu} p_\rho = 1$ and $\sum_{\varrho=1}^{\nu} q_\varrho = 1$. Also, let $x_\rho, \sum_{\varrho=1}^{\nu} q_\varrho(2a - x_\rho)$ and $\sum_{\rho=1}^{\mu} p_\rho \in I_2$. Then

$$\begin{aligned} \mathbf{J}(\mathfrak{f}(\cdot)) &= \sum_{\varrho=1}^{\nu} q_\varrho f(2a - x_\rho) - \mathfrak{f}\Big(\sum_{\varrho=1}^{\nu} q_\varrho (2a - x_\rho) \Big) - \sum_{\rho=1}^{\mu} p_\rho \mathfrak{f}(x_\rho) \\ &\quad + \mathfrak{f}\Big(\sum_{\rho=1}^{\mu} p_\rho x_\rho \Big). \end{aligned} \tag{25}$$

Theorem 2.9. *Assume $\mathcal{H}$ and $\mathfrak{f} : I_2 = [\zeta_1, \zeta_2] \to \mathbb{R}$ be such that $\mathfrak{f}^{(n-1)}$ is absolutely continuous. Then for $0 \leq \zeta_1 < \zeta_2 \leq 2a$, we have*

$$\begin{aligned} \mathbf{J}(\mathfrak{f}(\cdot)) &= \sum_{w=3}^{n-1} \Big(\frac{n-w}{w!(\zeta_2 - \zeta_1)} \Big) \Big(\mathfrak{f}^{(w-1)}(\zeta_2) \mathbf{J}(\cdot - \zeta_2)^w - \mathfrak{f}^{(w-1)}(\zeta_1) \mathbf{J}(\cdot - \zeta_1)^w \Big) \\ &\quad + \frac{1}{(n-1)!(\zeta_2 - \zeta_1)} \int_{\zeta_1}^{\zeta_2} \mathbf{J}(\cdot - t)^{n-1} W^{[\zeta_1, \zeta_2]}(t, \cdot) \mathfrak{f}^{(n)}(t) dt, \end{aligned} \tag{26}$$

where

$$\begin{aligned} \mathbf{J}(\cdot - \zeta_1)^w &= \sum_{\varrho=1}^{\nu} q_\varrho ((2a - x_\rho) - \zeta_1)^w - \Big(\sum_{\varrho=1}^{\nu} q_\varrho (2a - x_\rho) - \zeta_1 \Big)^w \\ &\quad - \sum_{\rho=1}^{\mu} p_\rho (x_\rho - \zeta_1)^w + \Big(\sum_{\rho=1}^{\mu} p_\rho x_\rho - \zeta_1 \Big)^w \end{aligned} \tag{27}$$

and

$$\begin{aligned}\mathbf{J}(\cdot-\zeta_2)^w &= \sum_{\varrho=1}^{\nu} q_\varrho((2a-x_\rho)-\zeta_2)^w - \Big(\sum_{\varrho=1}^{\nu} q_\varrho(2a-x_\rho)-\zeta_2\Big)^w \\ &\quad -\sum_{\rho=1}^{\mu} p_\rho(x_\rho-\zeta_2)^w + \Big(\sum_{\rho=1}^{\mu} p_\rho x_\rho-\zeta_2\Big)^w. \end{aligned} \tag{28}$$

Proof. By replacing I_1 with I_2 and y_ϱ with $(2a-x_\rho)$ in Theorem 2.1 we get (26). □

In the following theorem we obtain generalizations of Levinson's inequality (for real weights) for n-convex functions.

Theorem 2.10. *Let all the assumptions of Theorem 2.9 be satisfied and let for $n \geq 1$*

$$\mathbf{J}(\cdot-t)^{n-1}W^{[\zeta_1,\zeta_2]}(t,\cdot) \geq 0. \tag{29}$$

If $\mathfrak{f}$ is n-convex function, then we have

$$\begin{aligned}\mathbf{J}(\mathfrak{f}(\cdot)) &\geq \sum_{w=3}^{n-1}\Big(\frac{n-w}{w!(\zeta_2-\zeta_1)}\Big) \\ &\quad \times\Big(\mathfrak{f}^{(w-1)}(\zeta_2)\mathbf{J}(\cdot-\zeta_2)^w - \mathfrak{f}^{(w-1)}(\zeta_1)\mathbf{J}(\cdot-\zeta_1)^w\Big). \end{aligned} \tag{30}$$

Proof. Similar to Theorem 2.2 by using given conditions in the statement. □

Remark 2.11. *Reverse inequality in (29) leads to reverse inequality in (30).*

If we put $\nu=\mu$, $p_\rho=q_\varrho$ and by using positive weights in (25), then $\mathbf{J}(\cdot)$ converted to the functional $J_1(\cdot)$ defined in (2), also in this case, (26), (27), (28), (29) and (30) become

$$\begin{aligned}J_1(\mathfrak{f}(\cdot)) &= \sum_{\rho=1}^{\mu} p_\rho f(2a-x_\rho) - \mathfrak{f}(\sum_{\rho=1}^{\mu} p_\rho(2a-x_\rho)) - \sum_{\rho=1}^{\mu} p_\rho \mathfrak{f}(x_\rho) \\ &\quad +\mathfrak{f}(\sum_{\rho=1}^{\mu} p_\rho x_\rho), \end{aligned} \tag{31}$$

$$\begin{aligned}J_1(\mathfrak{f}(\cdot)) &= \sum_{w=3}^{n-1}\Big(\frac{n-w}{w!(\zeta_2-\zeta_1)}\Big)\Big(\mathfrak{f}^{(w-1)}(\zeta_2)J_1(\cdot-\zeta_2)^w - \mathfrak{f}^{(w-1)}(\zeta_1)J_1(\cdot-\zeta_1)^w\Big) \\ &\quad +\frac{1}{(n-1)!(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2} J_1(\cdot-t)^{n-1}W^{[\zeta_1,\zeta_2]}(t,\cdot)\mathfrak{f}^{(n)}(t)dt, \end{aligned} \tag{32}$$

$$\begin{aligned}J_1(\cdot-\zeta_1)^w &= \sum_{\rho=1}^{\mu} p_\rho((2a-x_\rho)-\zeta_1)^w - \Big(\sum_{\rho=1}^{\mu} p_\rho(2a-x_\rho)-\zeta_1\Big)^w \\ &\quad -\sum_{\rho=1}^{\mu} p_\rho(x_\rho-\zeta_1)^w + \Big(\sum_{\rho=1}^{\mu} p_\rho x_\rho-\zeta_1\Big)^w, \end{aligned} \tag{33}$$

$$J_1(\cdot - \zeta_2)^w = \sum_{\rho=1}^{\mu} p_\rho((2a - x_\rho) - \zeta_2)^w - \Big(\sum_{\rho=1}^{\mu} p_\rho(2a - x_\rho) - \zeta_2\Big)^w - \sum_{\rho=1}^{\mu} p_\rho(x_\rho - \zeta_2)^w + \Big(\sum_{\rho=1}^{\mu} p_\rho x_\rho - \zeta_2\Big)^w, \tag{34}$$

$$J_1(\cdot - t)^{n-1} W^{[\zeta_1,\zeta_2]}(t,\cdot) \geq 0 \tag{35}$$

and

$$J_1(\mathfrak{f}(\cdot)) \geq \sum_{w=3}^{n-1} \Big(\frac{n-w}{w!(\zeta_2-\zeta_1)}\Big) \times\Big(\mathfrak{f}^{(w-1)}(\zeta_2) J_1(\cdot - \zeta_2)^w - \mathfrak{f}^{(w-1)}(\zeta_1) J_1(\cdot - \zeta_1)^w\Big). \tag{36}$$

Now we will give generalization of Levinson's type inequality for positive μ-tuples.

Theorem 2.12. *Let all the assumptions of Theorem 2.9 be satisfied in addition with the condition that* $\mathbf{p} = (p_1, ..., p_\mu)$ *be positive n -tuples such that that* $\sum_{\rho=1}^{n} p_\rho 1$*, and consider* $\mathfrak{f} : [0, 2a] \to \mathbb{R}$ *is n-convex function. Then*
(i) *If n is odd and* $n > 4$*, then for every n-convex function* $\mathfrak{f} : [0, 2a] \to \mathbb{R}$*, (36) holds.*
(ii) *Let the inequality (36) be satisfied. If the function*

$$\Theta_1(\cdot) = \sum_{w=3}^{n-1} \Big(\frac{n-w}{w!(\zeta_2-\zeta_1)}\Big) \times\Big(\mathfrak{f}^{(w-1)}(\zeta_2) J_1(\cdot - \zeta_2)^w - \mathfrak{f}^{(w-1)}(\zeta_1) J_1(\cdot - \zeta_1)^w\Big) \tag{37}$$

is 3*-convex, the right hand side of (36) is non negative then*

$$J_1(\mathfrak{f}(\cdot)) \geq 0. \tag{38}$$

Proof. By using Theorem 2.10 and Remark 1.3, the result holds. □

Remark 2.13. *Cebyšev, Grüss and Ostrowski-type new bounds related to obtained generalizations can also be obtained. Moreover we can also give related mean value theorems by using non-negative functionals (11) and (26) to construct the new families of n-exponentially convex functions and Cauchy means related to these functionals as given in Section 4 of [3].*

3. Application to information theory

The idea of Shannon entropy is the central job of information speculation, now and again implied as measure of uncertainty. The entropy of a random variable is described with respect to a probability distribution and it can be shown that it is a decent measure of random. The assignment Shannon entropy is to assess the typical least number of bits expected to encode a progression of pictures subject to the letters, including the size and the repetition of the symbols.

Divergences between probability distributions can be interpreted as measure of the difference between them. An assortment of sorts of divergences exist, for example the $\mathfrak{f}$-divergences (especially, Kullback-Leibler divergences, Hellinger distance and total variation distance), Rényi divergences, Jensen-Shannon divergences, etc (see [15, 16]). There are a lot of papers dealing with the subject of inequalities and entropies, see, e.g., [5, 7, 9, 12, 13] and the references therein. The Jensen's inequality which deals one kind of data points, Levinson's inequality deals with two type of data points.

3.1. Csiszár divergence

In [8, 10] Csiszár gave the following definition:

Definition 3.1. *Let $\mathfrak{f}$ be a convex function from $\mathbb{R}^+$ to $\mathbb{R}^+$. Let $\tilde{\mathbf{r}}, \tilde{\mathbf{k}} \in \mathbb{R}_+^\mu$ be such that $\sum_{\rho=1}^{\mu} r_\rho = 1$ and $\sum_{\rho=1}^{\mu} k_\rho = 1$. Then f-divergence functional is defined by*

$$I_{\mathfrak{f}}(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) := \sum_{\rho=1}^{\mu} k_\rho \mathfrak{f}\left(\frac{r_\rho}{k_\rho}\right).$$

By defining the following:

$$\mathfrak{f}(0) := \lim_{x\to 0^+} \mathfrak{f}(x); \quad 0\mathfrak{f}\left(\frac{0}{0}\right) := 0; \quad 0\mathfrak{f}\left(\frac{a}{0}\right) := \lim_{x\to 0^+} x\mathfrak{f}\left(\frac{a}{0}\right), \quad a > 0,$$

he stated that nonnegative probability distributions can also be used.
Using the definition of $\mathfrak{f}$-divergence functional, Horv́ath *et al.* [14], gave the following functional:

Definition 3.2. *Let I be an interval contained in $\mathbb{R}$ and $\mathfrak{f} : I \to \mathbb{R}$ be a function. Also, let $\tilde{\mathbf{r}} = (r_1, \ldots, r_\mu) \in \mathbb{R}^\mu$ and $\tilde{\mathbf{k}} = (k_1, \ldots, k_\mu) \in (0, \infty)^\mu$ be such that*

$$\frac{r_\rho}{k_\rho} \in I, \quad \rho = 1, \ldots, \mu.$$

Then

$$\hat{I}_{\mathfrak{f}}(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) := \sum_{\rho=1}^{\mu} k_\rho \mathfrak{f}\left(\frac{r_\rho}{k_\rho}\right). \tag{39}$$

We apply Theorem 2.2 for n-convex functions to $\hat{I}_{\mathfrak{f}}(\tilde{\mathbf{r}}, \tilde{\mathbf{k}})$.

Theorem 3.3. *Let $\tilde{\mathbf{r}} = \left(r_1, \ldots, r_\mu\right) \in \mathbb{R}^\mu$, $\tilde{\mathbf{w}} = (w_1, \ldots, w_\nu) \in \mathbb{R}^\nu$, $\tilde{\mathbf{k}} = \left(k_1, \ldots, k_\mu\right) \in (0, \infty)^\mu$ and $\tilde{\mathbf{t}} = (t_1, \ldots, t_\nu) \in (0, \infty)^\nu$ be such that*

$$\frac{r_\rho}{k_\rho} \in I, \quad \rho = 1, \ldots, \mu,$$

and

$$\frac{w_\varrho}{t_\varrho} \in I, \quad \varrho = 1, \ldots, \nu.$$

Also, let $\mathfrak{f} : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ be such that $\mathfrak{f}^{(n-1)}$ is absolutely continuous, then for $n \geq 4$ (n is odd), we have

$$\begin{aligned} J_{cis}(\mathfrak{f}(\cdot)) \quad &\geq \quad \sum_{w=3}^{n-1} \left(\frac{n-w}{w!(\zeta_2 - \zeta_1)}\right) \\ &\quad \times\Big(\mathfrak{f}^{(w-1)}(\zeta_2)\mathfrak{I}(\cdot - \zeta_2)^w - \mathfrak{f}^{(w-1)}(\zeta_1)\mathfrak{I}(\cdot - \zeta_1)^w\Big), \end{aligned} \tag{40}$$

where

$$\begin{aligned} \mathfrak{I}(\cdot - \zeta_1)^w \quad = \quad & \sum_{\varrho=1}^{\nu} \frac{t_\varrho}{\sum_{\varrho=1}^{\nu} t_\varrho}\Big(\frac{w_\varrho}{t_\varrho} - \zeta_1\Big)^w - \Big(\sum_{\varrho=1}^{\nu} \frac{w_\varrho}{\sum_{\varrho=1}^{\nu} t_\varrho} - \zeta_1\Big)^w \\ & - \sum_{\rho=1}^{\mu} \frac{k_\rho}{\sum_{\rho=1}^{\mu} k_\rho}\Big(\frac{r_\rho}{k_\rho} - \zeta_1\Big)^w + \Big(\sum_{\rho=1}^{\mu} \frac{r_\rho}{\sum_{\rho=1}^{\mu} k_\rho} - \zeta_1\Big)^w \end{aligned} \tag{41}$$

and

$$\begin{aligned}\mathfrak{I}(\cdot-\zeta_2)^w &= \sum_{\varrho=1}^{\nu}\frac{t_\varrho}{\sum_{\varrho=1}^{\nu}t_\varrho}\Big(\frac{w_\varrho}{t_\varrho}-\zeta_2\Big)^w-\Big(\sum_{\varrho=1}^{\nu}\frac{w_\varrho}{\sum_{\varrho=1}^{\nu}t_\varrho}-\zeta_2\Big)^w\\ &- \sum_{\rho=1}^{\mu}\frac{k_\rho}{\sum_{\rho=1}^{\mu}k_\rho}\Big(\frac{r_\rho}{k_\rho}-\zeta_2\Big)^w+\Big(\sum_{\rho=1}^{\mu}\frac{r_\rho}{\sum_{\rho=1}^{\mu}k_\rho}-\zeta_2\Big)^w. \end{aligned}\tag{42}$$

Proof. Let

$$\Psi(\cdot):=(\cdot-t)^{n-1}W^{[\zeta_1,\zeta_2]}(t,\cdot)=\begin{cases}(\cdot-t)^{n-1}(t-\zeta_1), & \zeta_1\le t\le\cdot\le\zeta_2,\\ (\cdot-t)^{n-1}(t-\zeta_2), & \zeta_1\le\cdot\le t\le\zeta_2,\end{cases}$$

we have

$$\Psi'''(\cdot):=\begin{cases}(n-1)(n-2)(n-3)(\cdot-t)^{n-4}(t-\zeta_1), & \zeta_1\le t\le\cdot\le\zeta_2,\\ (n-1)(n-2)(n-3)(\cdot-t)^{n-4}(t-\zeta_2), & \zeta_1\le\cdot\le t\le\zeta_2.\end{cases}$$

So, Ψ is 3-convex for odd n, where $n>4$, (14) holds for odd values of n. Hence using $p_\rho=\frac{k_\rho}{\sum_{\rho=1}^{\mu}k_\rho}$, $x_\rho=\frac{r_\rho}{k_\rho}$, $q_\varrho=\frac{t_\varrho}{\sum_{\varrho=1}^{\nu}t_\varrho}$, $y_\varrho=\frac{w_\varrho}{t_\varrho}$ in Theorem 2.2, (15) becomes (40), where $\hat{I}_{\mathfrak{f}}(\tilde{\mathbf{r}},\tilde{\mathbf{k}})$ is defined in (39) and

$$\hat{I}_{\mathfrak{f}}(\tilde{\mathbf{w}},\tilde{\mathbf{t}}):=\sum_{\varrho=1}^{\nu}t_\varrho\mathfrak{f}\Big(\frac{w_\varrho}{t_\varrho}\Big).\tag{43}$$

□

3.2. Shannon Entropy

Definition 3.4. *(see [14]) The $\mathcal{S}$hannon entropy of positive probability distribution $\tilde{\mathbf{k}}=(k_1,\dots,k_\mu)$ is defined by*

$$\mathcal{S}:=-\sum_{\rho=1}^{\mu}k_\rho\log(k_\rho).\tag{44}$$

Corollary 3.5. *Let $\tilde{\mathbf{k}}=(k_1,\dots,k_\mu)$ and $\tilde{\mathbf{t}}=(t_1,\dots,t_\nu)$ be positive probability distributions. Also, let $\tilde{\mathbf{r}}=(r_1,\dots,r_\mu)\in(0,\infty)^\mu$ and $\tilde{\mathbf{w}}=(w_1,\dots,w_\nu)\in(0,\infty)^\nu$.*
If base of log is greater than 1 and n is odd, then

$$\begin{aligned}J_s(\cdot) &\ge \sum_{w=3}^{n-1}\Big(\frac{n-w}{w!(\zeta_2-\zeta_1)}\Big)\\ &\times\Big(\frac{(-1)^{w-2}(w-2)!}{(\zeta_2)^{w-1}}\mathfrak{I}(\cdot-\zeta_2)^w-\frac{(-1)^{w-2}(w-2)!}{(\zeta_1)^{w-1}}\mathfrak{I}(\cdot-\zeta_1)^w\Big),\end{aligned}\tag{45}$$

where

$$\begin{aligned}J_s(\cdot) &= \sum_{\varrho=1}^{\nu}t_\varrho\log(w_\varrho)+\tilde{\mathcal{S}}-\log\left(\sum_{\varrho=1}^{\nu}w_\varrho\right)-\sum_{\rho=1}^{\mu}k_\rho\log(r_\rho)+\mathcal{S}\\ &+\log\left(\sum_{\rho=1}^{\mu}r_\rho\right)\end{aligned}\tag{46}$$

and $\mathfrak{I}(\cdot-\zeta_1)^w$, $\mathfrak{I}(\cdot-\zeta_2)^w$ be the same as defined in (41) and (42) respectively.

Proof. The function $f : x \to \log(x)$ is n-convex for odd n ($n > 4$) and base of log is greater than 1. Therefore using $f(x) = \log(x)$ in (40), we have (45), where $\mathcal{S}$ is defined in (44) and

$$\tilde{\mathcal{S}} = -\sum_{\varrho=1}^{\nu} t_\varrho \log(t_\varrho).$$

□

Remark 3.6. *Inequality in* (45) *holds in reverse direction if n is even, because* $\mathfrak{f}(x) = \log(x)$ *is n-concave for* $n = 6, 8, \ldots$.

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