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Schrödinger Equation with Asymptotically Linear Nonlinearities

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Abstract. In this paper, we investigate a quasilinear Schrödinger problem under Dirichlet boundary condition in a regular domain with asymptotically linear nonlinearities. We use Cerami version of the mountain pass theorem to prove the existence of solution without using the Ambrosetti-Rabionovitz condition or any of its refinements. Then, we prove that the same techniques work when the nonlinearity is superlinear and subcritical at infinity.

1. Introduction and main results

In this paper, we consider the following quasilinear elliptic problem

$$\begin{cases} -\Delta v + \sigma(x)v &= h(x,v) \quad \text{in} \quad \Omega \\ \qquad\qquad v &= 0 \quad\qquad\;\; \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N > 2$, is a regular bounded domain in $\mathbb{R}^N$ and $\sigma \in C(\overline{\Omega})$ is a positive function. The equation (1) arises in models of combustion [10, 11], describing the thermal explosions [10] and the nonlinear heat generation [14]. Also, We find them in the description of the gravitational equilibrium of polytropic stars [7, 12], in the study of electromagnetic radiation, seismology and acoustics.
The results of the study of the problem (1) differs according to the type of the nonlinearities $h(x,t)$, the diffusion source. In this paper, we are interested to study the case when the function $h(x,t)$ is not necessarily linear but asymptotically linear, that is

$$\lim_{t\to+\infty} \frac{h(x,t)}{t} = \alpha < \infty.$$

In the beginning Mironescu and Rădulescu, in [18], supposed that $\sigma(x) \equiv 0$ and $h(x,t)$ depends only on t and has the form $h(x,t) = \lambda f(t)$ where λ is a positive parameter and f is a positive convex C^1 function satisfying

$$\lim_{t\to+\infty} \frac{f(t)}{t} = a < \infty,$$

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and they proved that there exists a critical value $\lambda^* > 0$ such that the problem has a positive solution if $\lambda < \lambda^*$ and it does not have positive solutions if $\lambda > \lambda^*$. When $\lambda = \lambda^*$, the sign of the value $l := \lim_{t\to+\infty}(f(t) - at)$ determines whether or not a branch of unstable positive solutions exists.
After that, in [2] with the same type of nonlinearities, Abid et al. consider the problem (1) when $\sigma(x) = const.$ and generalized the results of [18]. Also, with these type of nonlinearities many problems have been treated and we can refer to [1–3, 6, 15–18, 21, 22] and the references therein.
For a more large class of asymptotically linear nonlinearities and for the case when $h(x,t)$ depends on x and t, we suppose that

(H1) $h(x,t)$ is a nonnegative continuous function on $\overline{\Omega} \times \mathbb{R}$ and $h(x,t) \equiv 0$ for $t \le 0$ and $x \in \overline{\Omega}$.

(H2) $\lim\limits_{t\to 0} \dfrac{h(x,t)}{t} = p(x)$, $\lim\limits_{t\to+\infty} \dfrac{h(x,t)}{t} = \alpha \in (0,\infty)$ uniformly in a.e.$x \in \Omega$, and $\|p(x)\|_\infty < \nu_1$, where $\nu_1 > 0$ is the first eigenvalue of $(-\Delta + \sigma(x), H_0^1(\Omega))$.

(H3) The function $\dfrac{h(x,t)}{t}$ is nondecreasing with respect to $t > 0$, for a.e. $x \in \Omega$.

This type of conditions was first introduced by Zhou in [27] in order to study equation (1) when $\sigma(x) \equiv 0$ and then they are used in [25, 26] for some elliptic problems.
In this paper, we consider the case when $\sigma(x)$ is a positive continuous function and we prove the following theorem.

Theorem 1.1. *Assume* (*H*1) *and* (*H*2) *hold, then we have.*

(i) If $\alpha < \nu_1$ *and* (*H*3) *holds, then the problem* (1) *does not have a positive solution.*

(ii) If $\alpha > \nu_1$*, then the problem* (1) *has a positive solution.*

(iii) If $\alpha = \nu_1$ *and* (*H*3) *holds, then* (1) *has a positive solution* $v \in H_0^1(\Omega)$ *if and only if there exists a constant* $c_0 > 0$ *such that* $v = c_0\phi_1$ *and* $h(x,v) = \nu_1 v$ *a.e. in* Ω*, where* ϕ_1 *is a positive eigenfunction associated to* ν_1*.*

For the proof of the existence of nontrivial solution, we use the variational method. We use a mountain pass theorem and we prove the compactness condition without using the Ambrosetti-Rabionovitz condition (AR) [4, 19]: There exists a constant $\mu > N$ and a constant $T > 0$ such that for all $|t| \ge T$ and $x \in \Omega$,

$$0 < \mu \int_0^t h(x,s)ds \le h(x,t)t.$$

Since the condition (AR) gives

$$\lim_{t\to+\infty} \frac{H(x,t)}{t^2} = +\infty,$$

for

$$H(x,t) = \int_0^t h(x,s)ds \tag{2}$$

and so $\lim\limits_{t\to+\infty} \dfrac{h(x,t)}{t} = +\infty$. Therefore, we can not suppose and use such condition for the asymptotically linear nonlinearities.
For other conditions imposed to solve the compactness problem, we can refer to [8, 9, 13, 20, 23, 24, 27] and the references therein.

The techniques that we use in the proof of the Theorem 1.1 are still valid when the nonlinearities are superlinear and subcritical. More precisely, we prove the following second result.

Theorem 1.2. *Suppose that* (*H*1), (*H*2) *and* (*H*3) *hold and* $\alpha = +\infty$.
If $\lim\limits_{t\to+\infty} \dfrac{h(x,t)}{t^{r-1}} = 0$ *uniformly in* $x \in \Omega$*, for some real* r *with* $r \in (2, 2^*)$*. Then the problem* (1) *has a positive solution, where*

$$2^* = \begin{cases} \frac{2N}{N-2} & if \quad N > 2 \\ +\infty & if \quad N \le 2. \end{cases}$$

Through this paper, the constants $C, C_1, C_2, \cdots$, may change from line to another.

2. Preliminaries and variational setting

Let Ω be a bounded open domain in $\mathbb{R}^N$, $N \geq 2$. For $v \in L^p(\Omega)$, $1 \leq p < \infty$, we denote

$$\|v\|_p = \Big(\int_\Omega |v|^p\, dx\Big)^{\frac{1}{p}}$$

the well known Lebesgue norm in the space $L^p(\Omega)$. When $v \in H_0^1(\Omega)$, we set

$$\|v\| = \Big(\int_\Omega |\nabla v|^2\, dx\Big)^{\frac{1}{2}}$$

the standard norm in $H_0^1(\Omega)$ and we define

$$\|v\|_\sigma = \Big(\int_\Omega [|\nabla v|^2 + \sigma(x)v^2]\, dx\Big)^{\frac{1}{2}} \tag{3}$$

the norm coming from the inner product

$$\langle u, v\rangle_\sigma = \int_\Omega [\nabla u.\nabla v + \sigma(x)uv]dx. \tag{4}$$

In this paper, we consider the following definition of solutions (weak solutions) for the problem (1).

Definition 2.1. *A function $v \in H_0^1(\Omega)$ is called a solution of the problem* (1) *if*

$$\int_\Omega \nabla v \nabla \phi dx + \int_\Omega \sigma(x) v\phi dx = \int_\Omega h(x, v)\phi dx, \tag{5}$$

for all $\phi \in H_0^1(\Omega)$.

Since the equation (1) has variational form, let $\mathcal{I}$ be the functional defined on $H_0^1(\Omega)$ by

$$\mathcal{I}(v) = \frac{1}{2}\int_\Omega [|\nabla v|^2 + \sigma(x)v^2]dx - \int_\Omega H(x, v)dx \tag{6}$$

where the function $H(x, s)$ is given by (2).
To prove the existence of nonzero critical point of $\mathcal{I}$, we use the following different version of the mountain pass theorem introduced in [9].

Theorem 2.2. *[9] Let X be a real Banach space and $\mathcal{I} \in C^1(X, \mathbb{R})$ a functional satisfying*

(i) There exist δ, $\tau > 0$ such that $\forall v \in \partial B(0, \delta), \mathcal{I}(v) \geq \tau$.
(ii) There exists $x_1 \in X$ such that $\|x_1\| > \delta$ and $\mathcal{I}(x_1) < 0$.
(iii) $\max\{\mathcal{I}(0), \mathcal{I}(x_1)\} < \tau$.

Let c be the number characterized by

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0, 1], X);\ \gamma(0) = 0 \ \text{ and } \ \gamma(1) = x_1\}$ the set of continuous paths joining 0 *and x_1 in X. Then, $c \geq \tau$ and there exists a sequence (v_n) in X satisfying the Cerami conditions:*

$$\mathcal{I}(v_n) \to c \quad as \ n \to +\infty \tag{7}$$

and

$$(1 + \|v_n\|)\|\mathcal{I}'(v_n)\|_* \to 0 \quad as \ n \to +\infty. \tag{8}$$

In the Theorem 1.1, we consider ϕ_1 a normalised positive eigenfunction associated to ν_1, the first eigenvalue of the operator $-\Delta + \sigma(x)$ with Dirichlet boundary condition on the open domain Ω, that is

$$\begin{cases} -\Delta\phi_1 + \sigma(x)\phi_1 & = & \nu_1\phi_1 & \text{in} & \Omega \\ \phi_1 & = & 0 & \text{on} & \partial\Omega \\ \int_\Omega \phi_1^2 dx & = & 1. & & \end{cases} \tag{9}$$

3. Proof of the Theorem 1.1

The proof of the Theorem 1.1 will be given in many steps. After introducing the Energy $\mathcal{I}$ by the formula (6), we have interest to use the norm $||u||_\sigma$ given by (3). So, the first elementary result is the following.

Lemma 3.1. *The space $H_0^1(\Omega)$ endowed with the norm $||v||_\sigma$ is a Hilbert space.*

Proof. Recall that $(H_0^1(\Omega), ||.||)$ is a Hilbert space. For $v \in H_0^1(\Omega)$, we have

$$||v|| \leq ||v||_\sigma.$$

Since $\sigma^+ = \sup_{x\in\overline{\Omega}} \sigma(x) > 0$, by using the Poincaré inequality, the two norms are equivalent and so $(H_0^1(\Omega), ||.||_\sigma)$ is a Banach space. From (3) and (4), the space $(H_0^1(\Omega), ||.||_\sigma)$ is a Hilbert space. □

Next, we prove the first geometric property of the functional $\mathcal{I}$.

Lemma 3.2. *Suppose that (H1) and (H2) hold, then we have.*

(i) There exist $\delta > 0$ and $\tau > 0$ such that $\forall v \in \partial B(0,\delta), \mathcal{I}(v) \geq \tau$.
(ii) When $\nu_1 < \alpha$, $\mathcal{I}(t\phi_1) \to -\infty$ as $t \to +\infty$.

Proof. (*i*) Let $\epsilon > 0$, there exists $C = C(\epsilon) \geq 0$ such that for all $t \geq 0$ and for all $q \geq 1$, we have

$$h(x,t) \leq (||p(x)||_\infty + \epsilon)t + C|t|^q \tag{10}$$

and then

$$H(x,t) \leq \frac{1}{2}(||p(x)||_\infty + \epsilon)t^2 + C|t|^{q+1}. \tag{11}$$

Let $1 < q < 2^* - 1$, by Sobolev embedding theorem $||v||_{q+1}^{q+1} \leq C_1||v||_\sigma^{q+1}$, for some positive constant C_1. From (3) and (6), we get

$$\mathcal{I}(v) = \frac{1}{2}||v||_\sigma^2 - \int_\Omega H(x,v)\,dx. \tag{12}$$

By (11), we get

$$\mathcal{I}(v) \geq \frac{1}{2}||v||_\sigma^2 - \frac{1}{2}(||p(x)||_\infty + \epsilon)||v||_2^2 - C||v||_{q+1}^{q+1}.$$

So,

$$\mathcal{I}(v) \geq \frac{1}{2}||v||_\sigma^2 - \frac{1}{2}(||p(x)||_\infty + \epsilon)||v||_2^2 - C_2||v||_\sigma^{q+1}$$

and then,

$$\mathcal{I}(v) \geq \frac{1}{2}(1 - \frac{||p(x)||_\infty + \epsilon}{\nu_1})||v||_\sigma^2 - C_2||v||_\sigma^{q+1}. \tag{13}$$

If we consider $\epsilon > 0$ such that $\|p(x)\|_\infty + \epsilon < \nu_1$, then we can choose $\|v\|_\sigma = \delta$ small enough in order to have $\mathcal{I}(v) \geq \tau$ for some $\tau > 0$ sufficiently small.

(*ii*) Suppose that $\nu_1 < \alpha < +\infty$. Let $t > 0$ and consider

$$\mathcal{I}(t\phi_1) = \frac{t^2}{2}\int_\Omega [|\nabla\phi_1|^2 + \sigma(x)|\phi_1|^2]dx - \int_\Omega H(x, t\phi_1)\, dx. \tag{14}$$

From (9), we obtain

$$\mathcal{I}(t\phi_1) = \frac{t^2}{2}\nu_1 - \int_\Omega H(x, t\phi_1)\, dx. \tag{15}$$

Then, by the use of the Fatou's Lemma,

$$\lim_{t\to\infty} \frac{\mathcal{I}(t\phi_1)}{t^2} \leq \frac{1}{2}\nu_1 - \int_\Omega \lim_{t\to\infty} \frac{H(x, t\phi_1)}{(t\phi_1)^2}\phi_1^2\, dx.$$

Since $h(x, t)$ is asymptotically linear, we get

$$\lim_{t\to\infty} \frac{H(x, t)}{t^2} = \frac{\alpha}{2}\cdot \tag{16}$$

Therefore,

$$\lim_{t\to\infty} \frac{\mathcal{I}(t\phi_1)}{t^2} \leq \frac{1}{2}(\nu_1 - \alpha). \tag{17}$$

Then, $\lim_{t\to\infty} \mathcal{I}(t\phi_1) = -\infty$.

□

Proof of the Theorem 1.1

(*i*) Assume that $0 < \alpha < \nu_1$ and $(H1) - (H3)$ hold. Suppose that $v \in H_0^1(\Omega)$ is a positive solution of the problem (1). In this case, from the conditions $(H1) - (H3)$, we get

$$\int_\Omega [|\nabla v|^2 + \sigma(x)v^2]dx = \int_\Omega h(x, v)v dx \leq \int_\Omega \alpha v^2 dx. \tag{18}$$

So, $\nu_1 \leq \alpha$ and this contradicts the hypothesis of this first case. Then, Theorem 1.1 (*i*) is proved.

(*ii*) Assume that $\nu_1 < \alpha$ and $(H1) - (H2)$ hold.
The functional $\mathcal{I}$ introduced by (6) is C^1 and satisfies $\mathcal{I}(0) = 0$.
By Lemma 3.2, there exist $\delta > 0$, $\tau > 0$ and there exists $x_1 \in H_0^1(\Omega)$ such that $\|x_1\| > \delta$ and $\mathcal{I}(x_1) < 0$. Since

$$\max\{\mathcal{I}(0), \mathcal{I}(x_1)\} < \tau,$$

by the Theorem 2.2, there exists a sequence $(v_n) \subset H_0^1(\Omega)$ verifying (7) and (8). The idea is to prove that the sequence (v_n) has a convergent subsequence in $H_0^1(\Omega)$ to a nonzero function v and then v will be a critical point of $\mathcal{I}$ and so a nontrivial solution of the problem (1). After that, by the maximum principle, the solution v will be positive.

From (7) and (8), we get

$$\mathcal{I}(v_n) = \frac{1}{2}\|v_n\|_\sigma^2 - \int_\Omega H(x, v_n)\, dx \to c \quad \text{as} \quad n \to +\infty \tag{19}$$

and

$$\|\mathcal{I}'(v_n)\|_* \to 0 \quad \text{as} \quad n \to +\infty. \tag{20}$$

If the sequence (v_n) is bounded in $H_0^1(\Omega)$, then there exists $v \in H_0^1(\Omega)$ and a subsequence still denoted (v_n) satisfying

$$\begin{aligned} v_n &\rightharpoonup v \ \text{ weakly in } H_0^1(\Omega) \ \text{ as } n \to +\infty \\ v_n &\to v \ \text{ strongly in } L^2(\Omega) \ \text{ as } n \to +\infty \\ v_n(x) &\to v(x) \ \text{ a.e in } \Omega \ \text{ as } n \to +\infty. \end{aligned}$$

From (20), for all $\phi \in H_0^1(\Omega)$ we have

$$\int_\Omega [\nabla v_n . \nabla \phi + \sigma(x) v_n \phi] dx - \int_\Omega h(x, v_n)\phi \, dx \to 0 \ \text{ as } \ n \to +\infty, \tag{21}$$

that is

$$-\Delta v_n + \sigma(x) v_n - h(x, v_n) \to 0 \quad \text{in} \quad H_0^{-1}(\Omega). \tag{22}$$

where $H_0^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$.
Note that by $(H2)$, $h(x, v_n) \to h(x, v)$ in $L^2(\Omega)$ and since the dual space of $L^2(\Omega)$ is the space $L^2(\Omega)$ and $L^2(\Omega) \hookrightarrow H_0^{-1}(\Omega)$, we have

$$-\Delta v_n + \sigma(x) v_n \to h(x, v) \quad \text{in} \quad H_0^{-1}(\Omega). \tag{23}$$

Therefore, by using the fact that the operator $L = -\Delta + \sigma(x)$ is an isomorphism from $H_0^1(\Omega)$ to $H_0^{-1}(\Omega)$, we get

$$v_n \to L^{-1}(h(x, v)) \quad \text{in} \quad H_0^1(\Omega). \tag{24}$$

From (24) and the uniqueness of the limit, we deduce that the sequence (v_n) converges to the function v in $H_0^1(\Omega)$. The sequence (v_n) induced by the Theorem 2.2 is relatively compact and the limit of its convergent subsequence is a critical point of the functional $\mathcal{I}$.

To finish the proof of the part (*ii*), we have to prove that (v_n) is bounded in $H_0^1(\Omega)$. For this, we argue by contradiction and we suppose that (v_n) is not bounded in $H_0^1(\Omega)$. So, up to a subsequence, $\|v_n\|_\sigma \to +\infty$. Let

$$z_n = \frac{v_n}{\|v_n\|_\sigma}, \quad t_n = \|v_n\|_\sigma. \tag{25}$$

Since z_n is bounded in $H_0^1(\Omega)$, there exists $z \in H_0^1(\Omega)$ such that

$$z_n \rightharpoonup z \qquad \text{weakly in } H_0^1(\Omega),$$

$$z_n \to z \qquad \text{strongly in } L^2(\Omega),$$

and

$$z_n(x) \to z(x) \qquad \text{a.e in } \Omega.$$

We claim that

$$-\Delta z + \sigma(x) z = \alpha \, z^+ \quad \text{in } \Omega. \tag{26}$$

For the proof of the claim (26), we divide (21) by $t_n = \|v_n\|_\sigma$ we get

$$\int_\Omega [\nabla z_n.\nabla\phi + \sigma(x)z_n\phi]dx - \int_\Omega \frac{h(x,v_n)}{\|v_n\|_\sigma}\phi\, dx \to 0 \text{ for all } \phi \in H^1_0(\Omega). \tag{27}$$

That is,

$$-\Delta z_n + \sigma(x)z_n - \frac{h(x,v_n)}{\|v_n\|_\sigma} \to 0 \text{ in } H_0^{-1}(\Omega). \tag{28}$$

Since

$$\frac{h(x,v_n)}{\|v_n\|_\sigma} = \frac{h(x,v_n)}{v_n}z_n$$

and $v_n = \|v_n\|_\sigma z_n$. Then, $\lim_{n\to+\infty} v_n = +\infty$ if $z(x) > 0$ and so by using conditions $(H1)$ and $(H2)$, we get

$$\lim_{n\to+\infty}\frac{h(x,v_n)}{v_n}z_n = \alpha z^+$$

in $\{x \in \Omega / z_n(x) \to z(x) \text{ and } z(x) \neq 0\}$.
If $z_n(x) \to z(x)$ and $z(x) = 0$, from $(H1)$ we deduce that $\frac{h(x,v_n)}{v_n}z_n$ converges to zero. Thus

$$\frac{h(x,v_n)}{v_n}z_n \text{ converges to } \alpha z^+ \text{ a.e. in } \Omega.$$

Now, the sequence $z_n \to z$ in $L^2(\Omega)$. By Theorem IV.9 in [5], the sequence (z_n) is dominated in $L^2(\Omega)$, up to a subsequence.
Therefore, $\frac{h(x,v_n)}{v_n}z_n$ is dominated and then converges to αz^+ in $L^2(\Omega)$.
Since $L^2(\Omega) \hookrightarrow H_0^{-1}(\Omega)$, from (28), we get the equation (26) and so the claim is proved.

Therefore,

$$\begin{cases} -\Delta z + \sigma(x)z &= \alpha z^+ \quad \text{in} \quad \Omega \\ \qquad\qquad z &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{29}$$

By the maximum principle, $z > 0$ and then $z = z^+$ satisfies the problem (29).
It follows that $z = c\phi_1$, for some constant $c >$ and $\alpha = \nu_1$, which contradicts the fact that $\nu_1 < \alpha < \infty$.

(iii) Let $\alpha = \nu_1$. Suppose that v is a positive solution for the problem (1). If we take $\phi = \phi_1$ in (5), we obtain

$$\int_\Omega [\nabla v.\nabla\phi_1 + \sigma(x)v.\phi_1]\, dx = \int_\Omega h(x,v)\phi_1 dx. \tag{30}$$

Conversely, consider the equation (9) and take v as a test function, we obtain

$$\int_\Omega [\nabla v.\nabla\phi_1 + \sigma(x)v.\phi_1]\, dx = \alpha\int_\Omega v\phi_1 dx. \tag{31}$$

So,

$$\int_\Omega (h(x,v) - \alpha v)\phi_1 dx = 0.$$

Now, from $(H1) - (H3)$ and the fact that $\phi_1 > 0$, we get $h(x,v) = \alpha v$ a.e. in Ω. Hence, $h(x,v) = \nu_1 v$ a.e. in Ω and the result follows from the fact that the eigenvalue ν_1 is simple.
Conversely, If $\alpha = \nu_1$, $h(x,v) = \nu_1 v$ and $v = c_0\phi_1$ for some constant $c_0 > 0$. Then, v is an eigenfunction satisfying (9) and so a solution of the problem (1).

□

4. Proof of the Theorem 1.2

We start by proving the geometric properties for the functional $\mathcal{I}$ introduced by (6).

Lemma 4.1. *Suppose that* $(H1)-(H3)$ *hold,* $\alpha = +\infty$ *and the function* $h(x,t)$ *is subcritical at* $t = +\infty$ *uniformly on* x *a.e. in* Ω*. We have:*

(i) There exist $\delta, \beta > 0$ *positive constants such that* $\mathcal{I}(v) \geq \beta$ *for all* $v \in H_0^1(\Omega)$ *with* $\|v\| = \delta$*.*
(ii) $\lim\limits_{t\to+\infty} \mathcal{I}(t\phi_1) = -\infty$.

Proof. (i) In this subcritical case, the condition

$$\lim_{t\to+\infty} \frac{h(x,t)}{t^{r-1}} = 0 \quad \text{for some } \ r \in (2, 2^*) \tag{32}$$

and the first part of the condition $(H2)$ gives that for any $\epsilon > 0$, there exists $C = C(\epsilon) \geq 0$ such that for all $t \in \mathbb{R}$ and $x \in \Omega$

$$H(x,t) \leq \frac{1}{2}(\|p(x)\|_\infty + \epsilon)t^2 + C|t|^r, \tag{33}$$

and so

$$\mathcal{I}(v) \geq \frac{1}{2}\|v\|_\sigma^2 - \frac{1}{2}(\|p(x)\|_\infty + \epsilon)\|v\|_2^2 - C\|v\|_r^r.$$

Since $2 < r < 2^*$, by Sobolev embedding theorem we have $\|v\|_r^r \leq C_1\|v\|_\sigma^r$, for some constant $C_1 > 0$ and then

$$\mathcal{I}(v) \geq \frac{1}{2}\|v\|_\sigma^2 - \frac{1}{2}(\|p(x)\|_\infty + \epsilon)\|v\|_2^2 - C_2\|v\|_\sigma^r. \tag{34}$$

We know that one characterization of ν_1 is that $\nu_1\|v\|_2^2 \leq \|v\|_\sigma^2$, for all $v \in H_0^1(\Omega)$ and so

$$\mathcal{I}(v) \geq \frac{1}{2}(1 - \frac{\|p(x)\|_\infty + \epsilon}{\nu_1})\|v\|_\sigma^2 - C_2\|v\|_\sigma^r. \tag{35}$$

Now, we can choose $\epsilon > 0$ in (35) such that $\|p(x)\|_\infty + \epsilon < \nu_1$ and $\|v\| = \delta$ small enough in order to have $\mathcal{I}(v) \geq \beta$ for $\beta > 0$ sufficiently small.

(ii) Since the positive function ϕ_1 is in $C(\Omega)$. Let $\Omega_0 \subset \mathbb{R}^N$ be an open domain such that $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$ and let $\gamma > 0$ be a number satisfying $\phi_1(x) \geq \gamma > 0$ for all $x \in \Omega_0$. From $(H3)$, we obtain

$$0 \leq 2H(x,t) \leq th(x,t), \tag{36}$$

and then the function $\dfrac{H(x,t)}{t^2}$ is nondecreasing with respect to $t > 0$ for a.e. $x \in \Omega_0$.
$\alpha = +\infty$ implies that

$$\lim_{t\to+\infty} \frac{H(x,t)}{t^2} = +\infty.$$

So, for all $x \in \Omega_0$ and $t > 0$,

$$\frac{H(x, t\phi_1(x))}{t^2\phi_1^2(x)} \geq \frac{H(x, t\gamma)}{t^2\gamma^2}. \tag{37}$$

For all $B > 0$, there exists t_1 verifying for all $t \geq t_1$ and for all $x \in \Omega_0$

$$\frac{H(x, t\phi_1(x))}{t^2\phi_1^2(x)} \geq B. \tag{38}$$

$$\frac{\mathcal{I}(t\phi_1)}{t^2} = \frac{1}{2}\int_\Omega [|\nabla\phi_1|^2 + \sigma(x)\phi_1^2]dx - \int_\Omega \frac{H(x,t\phi_1)}{(t\phi_1)^2}\phi_1^2\,dx. \tag{39}$$

So,

$$\frac{\mathcal{I}(t\phi_1)}{t^2} \le \frac{1}{2}\int_\Omega \phi_1^2 dx - \int_{\Omega_0} \frac{H(x,t\phi_1)}{(t\phi_1)^2}\phi_1^2\,dx. \tag{40}$$

From (38) and (40) we get

$$\frac{\mathcal{I}(t\phi_1)}{t^2} \le \frac{1}{2}\nu_1 \int_\Omega \phi_1^2 dx - B\int_{\Omega_0} \phi_1^2\,dx,$$

and so

$$\frac{\mathcal{I}(t\phi_1)}{t^2} \le \frac{1}{2}\nu_1 - B\gamma^2|\Omega_0|.$$

We can choose $B > 0$ large enough such that

$$\frac{\mathcal{I}(t\phi_1)}{t^2} \le -C < 0,$$

where $C > 0$ is a positive constant. Therefore

$$\lim_{t\to+\infty} \mathcal{I}(t\phi_1) = -\infty.$$

□

Before starting the proof of the second existence result Theorem 1.2, we recall the following result which the proof is similar to [27, Lemma 2.3].

Lemma 4.2. *Let $\mathcal{I}$ the functional defined by (6). Suppose that (H3) holds and*

$$\langle \mathcal{I}'(v_n), v_n\rangle \to 0 \ \text{ as } \ n \to +\infty.$$

Then, (v_n) has a subsequence, still denoted (v_n), satisfying for all $t > 0$ and for all $n > 0$

$$\mathcal{I}(tv_n) \le \frac{1+t^2}{2n} + \mathcal{I}(v_n).$$

Proof of the Theorem 1.2

Suppose that $\alpha = +\infty$, the conditions $(H1) - (H3)$ hold and $h(x,t)$ is subcritical at $+\infty$ uniformly a.e. on $x \in \Omega$. From Lemma 4.1 and Theorem 2.2, there exists a sequence (v_n) satisfying the Cerami conditions (7) and (8) and so (19) and (20). We have only to prove that the sequence (v_n) is bounded in $H_0^1(\Omega)$ and the rest will be the same as in the proof of the Theorem 1.1 (*ii*).
Suppose that (v_n) is not bounded in $H_0^1(\Omega)$, then up to a subsequence $\|v_n\|_\sigma \to +\infty$, when $n \to +\infty$. Let $d > 0$ be a positive number and set

$$z_n = \frac{v_n}{d\|v_n\|_\sigma}, \;\; t_n = \frac{1}{d\|v_n\|_\sigma}. \tag{41}$$

So, there exists $z \in H_0^1(\Omega)$ such that, up to a subsequence $z_n \rightharpoonup z$ weakly in $H_0^1(\Omega)$, $z_n \to z$ strongly in $L^2(\Omega)$ and $z_n(x) \to z(x)$ a.e in Ω.
As consequence,

$$z_n^+ \to z^+ \quad \text{in} \quad L^2(\Omega),$$

where $z_n^+ = \dfrac{z_n + |z_n|}{2}$ and

$$z_n^+ \to z^+ \quad \text{a.e. in} \quad \Omega.$$

From the formula (6)

$$\mathcal{I}(z_n) = \frac{1}{2}\|z_n\|_\sigma^2 - \int_\Omega H(x, z_n)dx.$$

From the condition $(H1)$, we get

$$\mathcal{I}(z_n) = \frac{1}{2}\|z_n\|_\sigma^2 - \int_\Omega H(x, z_n^+)dx. \tag{42}$$

Let $\Omega_+ = \{x \in \Omega; z^+(x) > 0\}$. For $x \in \Omega_+$,

$$v_n^+(x) = dz_n^+(x)\|v_n\|_\sigma \to +\infty$$

and so, for any $B > 0$, there exists $n_1 = n_1(x) > 0$ such that for all $n \geq n_1$, we have

$$\frac{h(x, v_n^+(x))}{v_n^+(x)} \geq B. \tag{43}$$

Also, $z_n^+(x) \to z^+(x)$ then there exists $n_2 = n_2(x) > 0$ such that for all $n \geq n_2$, we have

$$z_n^+(x) \geq \frac{z^+(x)}{2}\cdot \tag{44}$$

From (42) and (43), we get

$$\frac{h(x, v_n^+(x))}{v_n^+(x)}(z_n^+(x))^2 \geq B\frac{(z^+(x))^2}{4}\cdot$$

So, for n large enough and for all $x \in \Omega_+$,

$$\lim_{n\to+\infty} \frac{h(x, v_n^+(x))}{v_n^+(x)}(z_n^+(x))^2 \geq B\frac{(z^+(x))^2}{4}\cdot \tag{45}$$

From (21), by taking the test function $\phi = v_n$, we get

$$\|v_n\|_\sigma^2 - \int_\Omega h(x, v_n)v_n \, dx \to 0,$$

hence

$$\frac{1}{d^2} - \int_\Omega \frac{h(x, v_n)}{v_n}(z_n)^2 \, dx \to 0. \tag{46}$$

From (46) and the condition $(H1)$, we obtain

$$\lim_{n\to+\infty} \int_\Omega \frac{h(x, v_n^+)}{v_n^+}(z_n^+)^2 \, dx = \frac{1}{d^2}\cdot \tag{47}$$

Therefore,

$$\begin{aligned} \frac{1}{d^2} &\geq \lim_{n\to+\infty} \int_{\Omega_+} \frac{h(x, v_n^+)}{v_n^+}(z_n^+)^2 \, dx \\ &\geq \int_{\Omega_+} \lim_{n\to+\infty} \frac{h(x, v_n^+(x))}{v_n^+(x)}(z_n^+(x))^2 \end{aligned}$$

By (45), we have then

$$\frac{1}{d^2} \geq \frac{B}{4} \int_{\Omega_+} (z^+(x))^2 \, dx \tag{48}$$

and this is for all $B > 0$. So, $|\Omega_+| = 0$ and then $z^+ \equiv 0$.
From (42), we obtain

$$\lim_{n\to+\infty} \mathcal{I}(z_n) = \frac{1}{2d^2}. \tag{49}$$

On the other hand, by the Lemma 4.2 and up to a subsequence, we get

$$\mathcal{I}(z_n) = \mathcal{I}(t_n v_n) \leq \frac{1}{2n}(1 + t_n^2) + \mathcal{I}(v_n). \tag{50}$$

From (19), (41),(49) and (50)

$$\frac{1}{2d^2} \leq c$$

for all $d > 0$. This is impossible and so the sequence (v_n) is bounded in $H_0^1(\Omega)$ and Theorem 1.2 follows. □

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