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Grüss Type Operator Inequalities on Time Scales

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Abstract. In this paper we formulate and prove some operator inequalities of Grüss type on time scales.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was initiated by Hilger [15] in his Ph.D. thesis in 1988 in order to contain both difference and differential calculus in a consistent way. Since then, many authors have expounded on various aspects of the theory of dynamic equations on time scales. For example, the monographes [4, 5, 12] and the references cited therein. The present paper is designed to provide the reader with an exposition of some operator inequalities of Grüss type on time scales.

The analysis on time scales is a relatively new area of mathematics that unifies and generalizes discrete and continuous theories. Moreover, it is a crucial tool in many computational and numerical applications. The subject is being applied to many different fields in which dynamic processes can be described with discrete or continuous models(see [1, 4, 8] and references therein). One of the important subjects being developed within the theory of time scales is the study of some inequalities on time scales [2].

The Grüss inequality is of great interest in differential and difference equations, as well as many other areas of mathematics [8, 9, 21, 22, 27]. The classical inequality was proved by G. Grüss in 1935 [13]. It states that if f and g are two continuous functions on $[a, b]$ satisfying $\varphi \le f(t) \le \Phi$ and $\gamma \le g(t) \le \Gamma$ for all $t \in [a, b]$, then

$$\left| \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2} \int_a^b f(t)dt \int_a^b g(t)dt \right| \le \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).$$

In recent years, the investigations of Grüss type inequalities and Ostrowski-Grüss type inequalities on time scales have been interesting topics in the literature and various their generalizations on time scales have been established(see [6], [7], [17], [18], [19], [20], [24], [25] and [26] and references therein).

The main aim of this paper is to be deducted some operator inequalities of Grüss type on time scales and some inequalities related to the forward jump operator. The paper is organized as follows. In Section 2 we recall some basic facts for bounded self-adjoint operators on Hilbert spaces. In Section 3 we formulate and prove some Grüss type inequalities on time scales. In Section 4 we give some inequalities related to the forward jump operator.

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2. Bounded self-adjoint operators

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space over the field of the complex numbers $\mathbb{C}$. A bounded linear operator A, defined on H, is self-adjoint, i.e., $A = A^*$ if and only if $\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in H$ and if A is self-adjoint, then

$$\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|.$$

With $\mathcal{B}(H)$ we will denote the Banach algebra of all bounded linear operators, defined on H.

Definition 2.1 ([14]). *Let A and B be self-adjoint operators on H. Then $A \le B$ or equivalently $B \ge A$ if*

$$\langle Ax, x\rangle \le \langle Bx, x\rangle$$

for all $x \in H$. In particular, A is called positive if $A \ge O$.

Note that for any operators $A \in \mathcal{B}(H)$ the opeartors AA^* and A^*A are positive and self-adjoint operators on H.

Definition 2.2 ([14]). *Let $A \in \mathcal{B}(H)$. Then the set*

$$Sp(A) = \{\lambda \in \mathbb{C} : A - \lambda I \quad \text{is not invertible}\},$$

is called the spectrum of the operator A.

Theorem 2.3 ([14], Property (P)). *Let A be a bounded self-adjoint operator on a Hilbert space H. The homeomorphism $\phi \to \phi(A)$ of $C(\mathbb{R})$ into $\mathcal{B}(H)$ is order preserving, meaning that if $\phi, \psi \in C(\mathbb{R})$ are real-valued functions on $Sp(A)$ and $\phi(\lambda) \ge \psi(\lambda)$ for any $\lambda \in Sp(A)$, then $\phi(A) \ge \psi(A)$ in the operator order of $\mathcal{B}(H)$.*

We conclude this section with the following Grüss type inequality, which we use in the next section.

Theorem 2.4 ([10]). *Let A be a self-adjoint operator on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars m and M. If h and g are continuous on $[m, M]$ and*

$$\gamma = \min_{t\in[m,M]} h(t), \quad \Gamma = \max_{t\in[m,M]} h(t),$$

then

$$\begin{aligned} &|\langle h(A)g(A)x, x\rangle - \langle h(A)x, x\rangle \cdot \langle g(A)x, x\rangle| \\ &\quad\le \frac{1}{2}(\Gamma - \gamma)\left(\|g(A)x\|^2 - \langle g(A)x, x\rangle^2\right)^{\frac{1}{2}} \\ &\quad\le \frac{1}{4}(\Gamma - \gamma)(L - l) \end{aligned}$$

for each $x \in H$ with $\|x\| = 1$, where

$$l = \min_{t\in[m,M]} g(t), \quad L = \max_{t\in[m,M]} g(t).$$

3. Grüss type inequalities for operators

Suppose that $\mathbb{T}$ is a time scale with forward jump operator and delta differentiation operator σ and Δ, respectively. We start with the following useful inequality. For its proof we refer the reader to the monograph [3].

Lemma 3.1. *Let $f : \mathbb{T} \to \mathbb{R}$ be convex and Δ-differentiable on an interval $I \subset \mathbb{T}$ such that f^Δ is increasing on I. Then*

$$f(y) - f(x) \geq f^\Delta(x)(y - x)$$

for any $x, y \in I$.

By Lemma 3.1, we have the following result.

Corollary 3.2. *Let $f, g : \mathbb{T} \to \mathbb{R}$ be convex and Δ-differentiable on an interval $I \subset \mathbb{T}$ such that f^Δ and g^Δ are increasing on I. Then*

$$f(y)g(y) - f(x)g(x) \geq \left(f^\Delta(x)g(x) + f(\sigma(x))g^\Delta(x)\right)(y - x)$$

and

$$f(y)g(y) - f(x)g(x) \geq \left(f^\Delta(x)g(\sigma(x)) + f(x)g^\Delta(x)\right)(y - x)$$

for any $x, y \in I$.

The following theorem is an analogue of the well-known inequality for convex functions (see [23]).

Theorem 3.3. *Let $I \subset \mathbb{T}$ be an interval and $f : \mathbb{T} \to \mathbb{R}$ be convex and Δ-differentiable on I whose derivative f^Δ is continuous and increasing on I. If A is a self-adjoint operator on a Hilbert space H with $\langle Ax, x\rangle \in \mathbb{T}$, $x \in H$, and $Sp(A) \subseteq [m, M] \subset I$, then*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle$$

for any $x \in H$ with $\|x\| = 1$.

Proof. By Lemma 3.1, we have

$$f(t) - f(s) \geq f^\Delta(s)(t - s)$$

for any $t, s \in I$. Let

$$s = \langle Ax, x\rangle \in [m, M]$$

for any $x \in H$ with $\|x\| = 1$. Since $Sp(A) \subseteq [m, M]$, we obtain

$$f(t) - f(\langle Ax, x\rangle) \geq f^\Delta(\langle Ax, x\rangle)(t - \langle Ax, x\rangle)$$

for any $x \in H$ with $\|x\| = 1$ and for any $t \in I$. If we fix $x \in H$ with $\|x\| = 1$ in the last inequality and we apply the property (P), then we get

$$\begin{aligned} &\langle (f(A) - f(\langle Ax, x\rangle) I_H) x, x\rangle \\ &\quad \geq \left\langle f^\Delta(\langle Ax, x\rangle)(A - \langle Ax, x\rangle I_H) x, x\right\rangle \end{aligned}$$

or

$$\langle f(A)x, x\rangle - \langle f(\langle Ax, x\rangle) I_H x, x\rangle \geq \left\langle f^\Delta(\langle Ax, x\rangle) Ax, x\right\rangle - \left\langle f^\Delta(\langle Ax, x\rangle)\langle Ax, x\rangle x, x\right\rangle,$$

or

$$\begin{aligned} \langle f(A)x, x\rangle - \langle f(\langle Ax, x\rangle) I_H x, x\rangle &\geq \left\langle f^\Delta(\langle Ax, x\rangle) Ax, x\right\rangle - \left\langle f^\Delta(\langle Ax, x\rangle) Ax, x\right\rangle \\ &= 0. \end{aligned}$$

It follows that

$$\langle f(\langle Ax, x\rangle) I_H x, x\rangle \leq \langle f(A)x, x\rangle$$

for any $x \in H$ with $\|x\| = 1$. This completes the proof. □

Corollary 3.4. *Assume that f is as in Theorem 3.3. If A_j are self-adjoint operators with $\langle A_j x, x\rangle \in \mathbb{T}$, $x \in H$, and $Sp(A_j) \subseteq [m, M] \subset I$, $p_j \geq 0$ with $\sum_{j=1}^{n} p_j = 1$, $j \in \{1, \ldots, n\}$, then*

$$f\left(\sum_{j=1}^{n} p_j \langle A_j x, x\rangle\right) \leq \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle$$

for any $x \in H$ with $\|x\| = 1$.

Proof. By Theorem 3.3, we get

$$f\left(\langle A_j x, x\rangle\right) \leq \left\langle f(A_j)x, x\right\rangle, \quad j \in \{1, \ldots, n\},$$

whereupon

$$p_j f\left(\langle A_j x, x\rangle\right) \leq p_j \left\langle f(A_j)x, x\right\rangle, \quad j \in \{1, \ldots, n\},$$

and since f is convex, we have

$$\begin{aligned}\left\langle f\left(\sum_{j=1}^{n} p_j \langle A_j x, x\rangle\right) I_H x, x\right\rangle &\leq \left\langle \sum_{j=1}^{n} p_j f\left(\langle A_j x, x\rangle\right) I_H x, x\right\rangle \\ &\leq \sum_{j=1}^{n} p_j \left\langle f(A_j)x, x\right\rangle \\ &\leq \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$. This completes the proof. □

We need the following theorem for presenting our main result in this section.

Theorem 3.5. *Let $I \subset \mathbb{T}$ and $f, g : \mathbb{T} \to \mathbb{R}$ be convex and Δ-differentiable on I whose derivatives f^{Δ} and g^{Δ} are continuous and increasing on I. If A is a self-adjoint operator on a Hilbert space H with $\langle Ax, x\rangle \in \mathbb{T}$, $x \in H$, and $Sp(A) \subseteq [m, M] \subset I$, then*

$$\begin{aligned}&\langle f(A)g(A)x, x\rangle - f(\langle Ax, x\rangle)g(\langle Ax, x\rangle) \\ &\quad \leq \quad \langle (fg)^{\Delta}(A)Ax, x\rangle - \left\langle (fg)^{\Delta}(A)x, x\right\rangle \langle Ax, x\rangle\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. By Corollary 3.2, we have

$$f(t)g(t) - f(s)g(s) \geq \left(f^{\Delta}(s)g(s) + f(\sigma(s))g^{\Delta}(s)\right)(t - s)$$

for any $t, s \in I$. Let $t = \langle Ax, x\rangle \in [m, M]$ for any $x \in H$ with $\|x\| = 1$. Since $Sp(A) \subseteq [m, M]$, we obtain

$$f(\langle Ax, x\rangle)g(\langle Ax, x\rangle) - f(s)g(s) \geq \left(f^{\Delta}(s)g(s) + f(\sigma(s))g^{\Delta}(s)\right)(\langle Ax, x\rangle - s)$$

for any $s \in I$. If we fix $x \in H$ with $\|x\| = 1$ and we apply the property (P), by the last inequality, we get

$$\begin{aligned}&\langle (f(\langle Ax, x\rangle)g(\langle Ax, x\rangle)I_H - f(A)g(A))\, x, x\rangle \\ &\quad \geq \quad \left\langle \left(f^{\Delta}(A)g(A) + f(\sigma(A))g^{\Delta}(A)\right)(\langle Ax, x\rangle I_H - A)x, x\right\rangle\end{aligned}$$

or

$$\begin{aligned} &f(\langle Ax,x\rangle)g(\langle Ax,x\rangle)-\langle f(A)g(A)x,x\rangle \\ &\geq \left\langle\left(f^{\Delta}(A)g(A)+f(\sigma(A))g^{\Delta}(A)\right)x,x\right\rangle\langle Ax,x\rangle \\ &- \langle\left(f^{\Delta}(A)g(A)+f(\sigma(A))g^{\Delta}(A)\right)Ax,x\rangle \end{aligned}$$

This completes the proof. □

By Theorem 3.5, we give the following Grüss type inequalities for operators on time scales.

Theorem 3.6. *Let $I\subset\mathbb{T}$ and $f,g:\mathbb{T}\to\mathbb{R}$ be two convex functions, Δ-differentiable and increasing on I. If A is a self-adjoint operator on a Hilbert space H with $\langle Ax,x\rangle\in\mathbb{T}$, $x\in H$, and $Sp(A)\subseteq[m,M]\subset I$, then*

$$\begin{aligned} &\left|\langle f(A)g(A)x,x\rangle-\langle f(A)x,x\rangle\langle g(A)x,x\rangle\right| \\ &\leq \left|\langle f(A)g(A)x,x\rangle-f(\langle Ax,x\rangle)g(\langle Ax,x\rangle)\right| \\ &\leq \begin{cases} \frac{1}{2}(M-m)(\|(fg)^{\Delta}(A)x\|^2-\langle(fg)^{\Delta}(A)x,x\rangle^2)^{\frac{1}{2}} \\ \\ \frac{1}{2}\left((fg)^{\Delta}(M)-(fg)^{\Delta}(m)\right)\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)^{\frac{1}{2}} \end{cases} \\ &\leq \frac{1}{4}(M-m)\left((fg)^{\Delta}(M)-(fg)^{\Delta}(m)\right), \end{aligned}$$

where $x\in H$ and $\|x\|=1$.

Proof. By the Grüss type inequality, we get

$$\begin{aligned} &\langle(fg)^{\Delta}(A)Ax,x\rangle-\left\langle(fg)^{\Delta}(A)x,x\right\rangle\langle Ax,x\rangle \\ &\leq \frac{1}{2}(M-m)(\|(fg)^{\Delta}(A)x\|^2-\langle(fg)^{\Delta}(A)x,x\rangle^2)^{\frac{1}{2}} \\ &\leq \frac{1}{4}(M-m)\left((fg)^{\Delta}(M)-(fg)^{\Delta}(m)\right) \end{aligned}$$

and

$$\begin{aligned} &\langle(fg)^{\Delta}(A)Ax,x\rangle-\left\langle(fg)^{\Delta}(A)x,x\right\rangle\langle Ax,x\rangle \\ &\leq \frac{1}{2}\left((fg)^{\Delta}(M)-(fg)^{\Delta}(m)\right)\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{4}(M-m)\left((fg)^{\Delta}(M)-(fg)^{\Delta}(m)\right), \end{aligned}$$

where $x\in H$ and $\|x\|=1$. Hence, by Theorem 3.5, we obtain

$$\begin{aligned} &\langle f(A)g(A)x,x\rangle-f(\langle Ax,x\rangle)g(\langle Ax,x\rangle) \\ &\leq \begin{cases} \frac{1}{2}(M-m)(\|(fg)^{\Delta}(A)x\|^2-\langle(fg)^{\Delta}(A)x,x\rangle^2)^{\frac{1}{2}} \\ \\ \frac{1}{2}\left((fg)^{\Delta}(M)-(fg)^{\Delta}(m)\right)\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)^{\frac{1}{2}} \end{cases} \\ &\leq \frac{1}{4}(M-m)\left((fg)^{\Delta}(M)-(fg)^{\Delta}(m)\right), \end{aligned}$$

where $x\in H$ and $\|x\|=1$. Also, note that by Theorem 3.3 for the self-adjoint operator A and the convex function f, we have

$$f(\langle Ax,x\rangle)\leq\langle f(A)x,x\rangle$$

for every $x\in H$ such that $\|x\|=1$. This completes the proof. □

4. Some inequalities related to the forward jump operator

When we say that H is a Hilbert space on a time scale $\mathbb{T}$, we have in mind a set of functions $x : \mathbb{T} \to \mathbb{R}$ in which is defined an inner product $\langle \cdot, \cdot \rangle$. x^{σ^k} is an element of this set for any $k \in \mathbb{N}$, the inner product of x^{σ^p} and y^{σ^k}, $x, y \in H$, $p, k \in \mathbb{N}$, is well defined, and this set is a vector space and it is complete with respect to the norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. If A is an operator on H, then we define

$$(\sigma(A))^p x := Ax^{\sigma^p}, \quad p \in \mathbb{N}, \quad x \in H.$$

Theorem 4.1. *Let $I \subset \mathbb{T}$, A and B be self-adjoint operators on a Hilbert space H with $\langle A^l x, x\rangle \in \mathbb{T}$, $\langle B^l x, x\rangle \in \mathbb{T}$, $x \in H$, and $Sp(A^l), Sp(B^l) \subseteq [m, M] \subset I$, $l \in \{1, \dots, p\}$. If $\langle A^\sigma x, x\rangle \geq \langle Bx, x\rangle$ for $x \in H$, then*

$$\begin{aligned}
&(\langle Ax, x\rangle - \langle Bx, x\rangle)^p + \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx, x\rangle I_H)^j \, (A - \langle Bx, x\rangle I_H)^{p-j-1} \, Ax, x \right\rangle \\
\geq \;& \langle (A - \langle Bx, x\rangle I_H)^p \, x, x\rangle + \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx, x\rangle I_H)^j \, (A - \langle Bx, x\rangle I_H)^{p-j-1} \, \langle Ax, x\rangle x, x \right\rangle
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, $p \in \mathbb{N}$ such that $p > 2$, and $l \in \{1, \dots, p\}$.

Proof. Note that the function $f(x) = (x - a)^p$, $x \in H$, $x \geq a$ and $p > 2$, is a convex and increasing function, and

$$f^\Delta(x) = \sum_{j=0}^{p-1} (\sigma(x) - a)^j (x - a)^{p-j-1}.$$

Then, by Lemma 3.1, we get

$$(y - a)^p - (t - a)^p \geq \sum_{j=0}^{p-1} (\sigma(t) - a)^j (t - a)^{p-j-1} (y - t) \tag{1}$$

for any $y, t, a \in I$, $y \geq a$, $t \geq a$. Let

$$y = \langle Ax, x\rangle, \quad a = \langle Bx, x\rangle, \quad x \in H, \quad \|x\| = 1.$$

Since $Sp(A^l), Sp(B^l) \subseteq [m, M]$, $x \in H$, $\|x\| = 1$, for any $l \in \{1, \dots, p\}$, we get

$$\begin{aligned}
&(\langle Ax, x\rangle - \langle Bx, x\rangle)^p - (t - \langle Bx, x\rangle)^p \\
\geq \;& \sum_{j=0}^{p-1} (\sigma(t) - \langle Bx, x\rangle)^j \, (t - \langle Bx, x\rangle)^{p-j-1} \, (\langle Ax, x\rangle - t) \, .
\end{aligned}$$

If we fix $x \in H$ with $\|x\| = 1$ in the last inequality and applying the property (P), we have

$$\begin{aligned}
&\langle ((\langle Ax, x\rangle - \langle Bx, x\rangle)^p \, I_H - (A - \langle Bx, x\rangle I_H)^p \, x, x\rangle \\
\geq \;& \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx, x\rangle I_H)^j \, (A - \langle Bx, x\rangle I_H)^{p-j-1} \, (\langle Ax, x\rangle I_H - A) \, x, x \right\rangle
\end{aligned}$$

or

$$\begin{aligned}
&(\langle Ax, x\rangle - \langle Bx, x\rangle)^p - \langle (A - \langle Bx, x\rangle I_H)^p \, x, x\rangle \\
\geq \;& \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx, x\rangle I_H)^j \, (A - \langle Bx, x\rangle I_H)^{p-j-1} \, \langle Ax, x\rangle x, x \right\rangle \\
&- \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx, x\rangle I_H)^j \, (A - \langle Bx, x\rangle I_H)^{p-j-1} \, Ax, x \right\rangle.
\end{aligned}$$

Equivalently,

$$
\begin{aligned}
&(\langle Ax,x\rangle - \langle Bx,x\rangle)^p + \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx,x\rangle I_H)^j \, (A - \langle Bx,x\rangle I_H)^{p-j-1} \, Ax, x \right\rangle \\
&\geq \quad \langle (A - \langle Bx,x\rangle I_H)^p \, x, x\rangle + \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx,x\rangle I_H)^j \, (A - \langle Bx,x\rangle I_H)^{p-j-1} \, \langle Ax,x\rangle x, x \right\rangle .
\end{aligned}
$$

This completes the proof. □

When $B = O$ in Theorem 4.1, we get the following corollary.

Corollary 4.2. *Let $I \subset \mathbb{T}$ and A be a self-adjoint operator on a Hilbert space H with $\langle A^l x, x\rangle \in \mathbb{T}$, $x \in H$, and $Sp(A^l) \subseteq [m, M] \subset I$, $l \in \{1, \dots, p\}$. If $f(t) = t^l \geq 0$, $t \in [m, M]$, $l \in \{1, \dots, p\}$, then*

$$
\begin{aligned}
&\langle Ax,x\rangle^p + \sum_{j=0}^{p} \left\langle (\sigma(A))^j A^{p-j} x, x \right\rangle \\
&\geq \quad \langle A^p x, x\rangle + \sum_{j=0}^{p} \left\langle (\sigma(A))^j A^{p-j-1} \langle Ax,x\rangle x, x \right\rangle
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$, $p \in \mathbb{N}$ such that $p > 2$, and $l \in \{1, \dots, p\}$.

Theorem 4.3. *Let $I \subset \mathbb{T}$, A and B be self-adjoint operators on a Hilbert space H with $\langle A^l x, x\rangle \in \mathbb{T}$, $\langle B^l x, x\rangle \in \mathbb{T}$, $x \in H$, and $Sp(A^l), Sp(B^l) \subseteq [m, M] \subset I$, $l \in \{1, \dots, p\}$. Let also, $\langle A^\sigma x, x\rangle \geq \langle Bx, x\rangle$, $x \in H$. Then*

$$
\begin{aligned}
&\langle (A - \langle Bx,x\rangle I_H)^p \, x, x\rangle + \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx,x\rangle I_H)^j \, (\langle Ax,x\rangle - \langle Bx,x\rangle)^{p-j-1} \, \langle Ax,x\rangle x, x \right\rangle \\
&\qquad \geq (\langle Ax,x\rangle - \langle Bx,x\rangle)^p + \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx,x\rangle I_H)^j \, (\langle Ax,x\rangle - \langle Bx,x\rangle)^{p-j-1} \, Ax, x \right\rangle
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$, $p \in \mathbb{N}$, $p > 2$, and $l \in \{1, \dots, p\}$.

Proof. We take

$$
t = \langle Ax,x\rangle, \quad a = \langle Bx,x\rangle, \quad x \in H, \quad \|x\| = 1,
$$

in the inequality (1). Then we obtain

$$
\begin{aligned}
&(y - \langle Bx,x\rangle)^p - (\langle Ax,x\rangle - \langle Bx,x\rangle)^p \\
&\geq \quad \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx,x\rangle)^j \, (\langle Ax,x\rangle - \langle Bx,x\rangle)^{p-j-1} \, (y - \langle Ax,x\rangle) .
\end{aligned}
$$

We fix $x \in H$, $\|x\| = 1$, in the last inequality and applying the property (P), we get

$$
\begin{aligned}
&\langle ((A - \langle Bx,x\rangle I_H)^p - (\langle Ax,x\rangle - \langle Bx,x\rangle)^p \, I_H) \, x, x\rangle \\
&\geq \quad \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx,x\rangle I_H)^j \, (\langle Ax,x\rangle - \langle Bx,x\rangle)^{p-j-1} \, (A - \langle Ax,x\rangle I_H) \, x, x \right\rangle
\end{aligned}
$$

or

$$\begin{aligned}
&\langle (A - \langle Bx, x\rangle I_H)^p x, x\rangle - (\langle Ax, x\rangle - \langle Bx, x\rangle)^p \\
&\geq \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx, x\rangle I_H)^j (\langle Ax, x\rangle - \langle Bx, x\rangle)^{p-j-1} Ax, x \right\rangle \\
&\quad - \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx, x\rangle I_H)^j (\langle Ax, x\rangle - \langle Bx, x\rangle)^{p-j-1} \langle Ax, x\rangle x, x \right\rangle,
\end{aligned}$$

or

$$\begin{aligned}
&\langle (A - \langle Bx, x\rangle I_H)^p x, x\rangle + \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx, x\rangle I_H)^j (\langle Ax, x\rangle - \langle Bx, x\rangle)^{p-j-1} \langle Ax, x\rangle x, x \right\rangle \\
&\quad \geq (\langle Ax, x\rangle - \langle Bx, x\rangle)^p + \left\langle \sum_{j=0}^{p-1} (\sigma(A) - \langle Bx, x\rangle I_H)^j (\langle Ax, x\rangle - \langle Bx, x\rangle)^{p-j-1} Ax, x \right\rangle.
\end{aligned}$$

This completes the proof. □

If $B = O$ in Theorem 4.3, we get the following corollary.

Corollary 4.4. *Let $I \subset \mathbb{T}$ and A be a self-adjoint operator on a Hilbert space H with $\langle A^l x, x\rangle \in \mathbb{T}$, $x \in H$, and $Sp(A^l) \subseteq [m, M] \subset I$, $l \in \{1, \ldots, p\}$. Then*

$$\begin{aligned}
&\langle A^p x, x\rangle + \left\langle \sum_{j=0}^{p-1} (\sigma(A))^j \langle Ax, x\rangle^{p-j} x, x \right\rangle \\
&\geq (\langle Ax, x\rangle)^p + \left\langle \sum_{j=0}^{p-1} (\sigma(A))^j \langle Ax, x\rangle^{p-j-1} Ax, x \right\rangle
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, $p \in \mathbb{N}$ such that $p > 2$, and $l \in \{1, \ldots, p\}$.

Theorem 4.5. *Let $I \subset \mathbb{T}$ and A be a self-adjoint operator on a Hilbert space H with $\langle A^l x, x\rangle \in \mathbb{T}$, $x \in H$, and $Sp(A^l) \subseteq [m, M] \subset I$, $l \in \{1, \ldots, p\}$. If $f(t) = t^l \geq 0$, $t \in [m, M]$, $l \in \{1, \ldots, p\}$, then*

$$\begin{aligned}
&\langle A^p x, x\rangle - (\langle Ax, x\rangle)^p \\
&\leq \begin{cases}
(M - m)\Big(\sum_{j=0}^{p-1} \left\| (\sigma(A))^j A^{p-j-1} x \right\|^2 \\
- \left\langle \sum_{j=0}^{p-1} (\sigma(A))^j A^{p-j-1} x, x \right\rangle^2 \Big)^{\frac{1}{2}} \\
\frac{p}{2} \left((\sigma(M))^p - m^p \right) \left(\|Ax\|^2 - \langle Ax, x\rangle^2 \right)^{\frac{1}{2}}
\end{cases} \\
&\leq \frac{p}{4}(M - m) \left((\sigma(M))^p - m^p \right)
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, $p \in \mathbb{N}$ such that $p > 2$, and $l \in \{1, \ldots, p\}$.

Proof. By the Grüss type inequality, we get

$$
\begin{aligned}
&\left\langle \sum_{j=0}^{p-1} (\sigma(A))^j A^{p-j} x, x \right\rangle - \left\langle \sum_{j=0}^{p-1} (\sigma(A))^j A^{p-j-1} x, x \right\rangle \cdot \langle Ax, x \rangle \\
\leq\ & \frac{1}{2}(M-m)\Bigg(\left\| \sum_{j=0}^{p-1} (\sigma(A))^j A^{p-j-1} x \right\|^2 - \left\langle \sum_{j=0}^{p-1} (\sigma(A))^j A^{p-j-1} x, x \right\rangle^2 \Bigg)^{\frac{1}{2}} \\
\leq\ & \frac{1}{4}(M-m)\left(\sum_{j=0}^{p-1} (\sigma(M))^j M^{p-j-1} - \sum_{j=0}^{p-1} (\sigma(m))^j m^{p-j-1} \right) \\
\leq\ & \frac{p}{4}(M-m)\left((\sigma(M))^p - m^p\right)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&\left\langle \sum_{j=0}^{p-1} (\sigma(A))^j A^{p-j} x, x \right\rangle - \left\langle \sum_{j=0}^{p-1} (\sigma(A))^j A^{p-j-1} x, x \right\rangle \cdot \langle Ax, x \rangle \\
\leq\ & \frac{1}{2}\left(\sum_{j=0}^{p-1} (\sigma(M))^j M^{p-j-1} - \sum_{j=0}^{p-1} (\sigma(m))^j m^{p-j-1} \right)\left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right)^{\frac{1}{2}} \\
\leq\ & \frac{p}{2}\left((\sigma(M))^p - m^p\right)\left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right)^{\frac{1}{2}} \\
\leq\ & \frac{p}{4}(M-m)((\sigma(M))^p - m^p).
\end{aligned}
$$

Hence, by Corollary 4.2, it follows that

$$
\begin{aligned}
&\langle A^p x, x \rangle - (\langle Ax, x \rangle)^p \\
\leq\ & \left\langle \sum_{j=0}^{p-1} (\sigma(A))^j A^{p-j} x, x \right\rangle - \left\langle \sum_{j=0}^{p-1} (\sigma(A))^j A^{p-j-1} x, x \right\rangle \langle Ax, x \rangle \\
\leq\ & \begin{cases} \frac{1}{2}(M-m)\Big(\sum_{j=0}^{p-1} \left\| (\sigma(A))^j A^{p-j-1} x \right\|^2 \\ - \left\langle \sum_{j=0}^{p-1} (\sigma(A))^j A^{p-j-1} x, x \right\rangle^2 \Big)^{\frac{1}{2}} \\ \frac{p}{2}\left((\sigma(M))^p - m^p\right)\left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right)^{\frac{1}{2}} \end{cases} \\
\leq\ & \frac{p}{4}(M-m)\left((\sigma(M))^p - m^p\right).
\end{aligned}
$$

This completes the proof. □

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