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Dual Topologies for the Space of Multi–Functions

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Abstract. Dual topology for the function space topologies for multifunctions are introduced and investigated. It is found that a topology $\mathfrak{T}$ on $C_{\mathcal{M}}(Y,Z)$ is splitting (resp. admissible) if and only if its dual pair $(\mathbb{T}^+,\mathbb{T}^-)$ is splitting (resp. admissible). Similarly, the pair $(\mathbb{T}^+,\mathbb{T}^-)$ is splitting (resp. admissible) if and only if its dual $\mathfrak{T}(\mathbb{T}^+,\mathbb{T}^-)$ is splitting (resp. admissible).

1. Introduction

In the recent past, study of function space structures have gained attention from various quarters of researchers. In [16], different sets of conditions under which the Isbell topology, the compact-open or the natural topologies may coincide have been discussed. A unified theory for hyperspaces and function spaces has been investigated in [4]. Function space topologies and their dual spaces have been introduced and studied for generalized topological spaces in [7] and [9] respectively. Function space structures between X and Y, when one or both of X and Y are equipped with uniformities have been investigated in [11] and [10].

These studies have been further complimented by their applications in other fields. For example, the notion of admissibility of function space topology between topological vector spaces is found to play a key-role in obtaining solutions to various vector variational inequality problems [12, 17]. The continuous multifunctions in the study of function spaces have been investigated by several researchers [13–15, 18–21]. At the same time, topological properties of multivalued functions have also been applied in the recent past in various diverse fields such as in vector equilibrium problems, variational inequalities, optimization theory, etc. [1–3, 5].

These developments have motivated us to investigate the function space topologies for multivalued functions. In [8], we have introduced and studied the topologies on $C_{\mathcal{M}}(Y,Z)$, the family of continuous multivalued functions between topological spaces Y and Z. In the present paper, we investigate the dual topologies for these function spaces. It is found that the open sets of the domain space, which are pre-images of the continuous multifunctions, behave in a nice way. They can be used to define the dual topology of $C_{\mathcal{M}}(Y,Z)$. Unlike, in single-valued continuous mappings, here we get a pair of topologies: $\mathbb{T}^+$ on $\mathcal{O}_Z^+(Y)$ and $\mathbb{T}^-$ on $\mathcal{O}_Z^-(Y)$, respectively. It is found that a topology $\mathfrak{T}$ on $C_{\mathcal{M}}(Y,Z)$ is splitting (resp. admissible) if and only if its dual pair $(\mathbb{T}^+,\mathbb{T}^-)$ is splitting (resp. admissible). Similarly, the pair $(\mathbb{T}^+,\mathbb{T}^-)$ is a splitting (resp. admissible) pair if and only if its dual topology $\mathfrak{T}(\mathbb{T}^+,\mathbb{T}^-)$ on $C_{\mathcal{M}}(Y,Z)$ is splitting (resp. admissible).

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2. Preliminaries

In this section, we recall some definitions and results which will be used further to obtain the main results.

Definition 2.1. A *multifunction* $F : X \to Y$ is a point-to-set correspondence from X to Y.

We always assume that $F(x) \neq \emptyset$ for all $x \in X$. For each $B \subseteq Y$, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. For each $A \subseteq X$, $F(A) = \bigcup_{x \in A} F(x)$.

The collection of all the multifunctions from X to Y is denoted by $Y^X_{\mathcal{M}}$.

The following definitions and results are taken from available literature.

Definition 2.2. Let (X, τ) and (Y, μ) be two topological spaces. Then $F : X \to Y$ is called:

(i) *upper semi continuous* (or *u.s.c.*, in brief) at $x \in X$ if for each open set $V \subseteq Y$ with $F(x) \subseteq V$, there exists an open set U of X such that $x \in U$ and $F(U) \subseteq V$;
(ii) *lower semi continuous* (or *l.s.c*, in brief) at $x \in X$ if for each open set $V \subseteq Y$ with $F(x) \cap V \neq \emptyset$, there exists an open set U of X such that $x \in U$ and $F(u) \cap V \neq \emptyset$ for every $u \in U$;
(iii) *continuous* at $x \in X$, if it is both *u.s.c.* and *l.s.c.* at x;
(iv) *continuous* (resp. *u.s.c.*, *l.s.c.*) if it is continuous (resp. *u.s.c.*, *l.s.c.*) at each point of X.

Theorem 2.3. *Let (X, τ) and (Y, μ) be two topological spaces. Then the following conditions are equivalent for a multifunction $F : X \to Y$:*

(i) *F is l.s.c. (resp. u.s.c.);*
(ii) *$F^-(U)$ (resp. $F^+(U)$) is open in X for each open subset U of Y;*
(iii) *$F^+(A)$ (resp. $F^-(A)$) is closed in X for each closed subset A of Y.*

Definition 2.4. A multifunction $F : X \to Y$ is called a *closed map* if $F(A)$ is closed in Y whenever A is closed in X.

In [8], Gupta and Sarma, introduced the notion of admissibility and splittingness on $C_{\mathcal{M}}(Y, Z)$, the space of all set-valued continuous functions as follow:

Definition 2.5. ([8]) Let (Y, τ) and (Z, μ) be two topological spaces. Let (X, λ) be another topological space. For a multifunction $G : X \times Y \to Z$, we define a map $G^* : X \to C_{\mathcal{M}}(Y, Z)$ by $G^*(x)(y) = G(x, y)$.

The mappings G and G^* related in this way are called *associated maps*.

Definition 2.6. ([8]) Let (Y, τ) and (Z, μ) be two topological spaces. A topology $\mathfrak{T}$ on $C_{\mathcal{M}}(Y, Z)$ is called

(i) *admissible* (resp. *upper admissible*, *lower admissible*) if the evaluation mapping $E : C_{\mathcal{M}}(Y, Z) \times Y \to Z$ defined by $E(F, y) = F(y)$ is continuous (resp. *u.s.c*, *l.s.c.*).
(ii) *splitting* (resp. *upper splitting*, *lower splitting*) if for each topological space X, continuity (resp. *u.s.c.*, *l.s.c.*) of $G : X \times Y \to Z$ implies the continuity of $G^* : X \to C_{\mathcal{M}}(Y, Z)$, where G^* is the associated map of G.

3. Dual topology for multifunctions

The open sets of the domain space which can be realized as pre-images of continuous multifunctions are found to behave in a nice way. They can be used to define the so-called "dual topology" of $C_{\mathcal{M}}(Y, Z)$. Interesting relationships can be observed between the space of multifunctions and its dual. In this section, we investigate such relationships with regard to splittingness, admissibility etc. of a space of multifunction and its dual.

First we define, for topological spaces (Y, τ) and (Z, μ),

$$O_Z^+(Y) = \{F^+(U) \mid F \in C_{\mathcal{M}}(Y,Z), U \in \mu\}$$
$$O_Z^-(Y) = \{F^-(U) \mid F \in C_{\mathcal{M}}(Y,Z), U \in \mu\}.$$

A topology on $O_Z^+(Y)$ (resp. $O_Z^-(Y)$) is called an *upper topology* (resp. a *lower topology*) with respect to $C_{\mathcal{M}}(Y,Z)$.

Definition 3.1. Let (Y,τ) and (Z,μ) be two topological spaces and $C_{\mathcal{M}}(Y,Z)$ be the set of all continuous multifunctions from Y to Z. Then for subsets $\mathbb{H}^+ \subseteq O_Z^+(Y), \mathbb{H}^- \subseteq O_Z^-(Y), \mathcal{H}^+, \mathcal{H}^- \subseteq C_{\mathcal{M}}(Y,Z)$ and $U \in \mu$ we define

$(\mathbb{H}^+, U) = \{F \in C_{\mathcal{M}}(Y,Z) \mid F^+(U) \in \mathbb{H}^+\}$,
$(\mathbb{H}^-, U) = \{F \in C_{\mathcal{M}}(Y,Z) \mid F^-(U) \in \mathbb{H}^-\}$
and
$(\mathcal{H}^+, U) = \{F^+(U) \mid U \in \mu, F \in \mathcal{H}^+\}$,
$(\mathcal{H}^-, U) = \{F^-(U) \mid U \in \mu, F \in \mathcal{H}^-\}$.

Definition 3.2. Let (Y,τ) and (Z,μ) be two topological spaces. Let $\mathbb{T}^+$ and $\mathbb{T}^-$ be an upper and a lower topology with respect to $C_{\mathcal{M}}(Y,Z)$. Then we define

$$\mathcal{S}(\mathbb{T}^+) = \{(\mathbb{H}^+, U) \mid \mathbb{H}^+ \in \mathbb{T}^+, U \in \mu\}$$
$$\text{and } \mathcal{S}(\mathbb{T}^-) = \{(\mathbb{H}^-, U) \mid \mathbb{H}^- \in \mathbb{T}^-, U \in \mu\}$$

Lemma 3.3. *$\mathcal{S}(\mathbb{T}^+)$ (resp. $\mathcal{S}(\mathbb{T}^-)$) is a subbasis for a topology on $C_{\mathcal{M}}(Y,Z)$.*

Proof. Let $F \in C_{\mathcal{M}}(Y,Z)$. Then, we have $F^+(Z) = Y$ (resp. $F^-(Z) = Y$)) which belongs to $\mathbb{H}^+$ (resp. $\mathbb{H}^-$). This holds for all $F \in C_{\mathcal{M}}(Y,Z)$. Therefore $C_{\mathcal{M}}(Y,Z) = \bigcup(\mathbb{H}^+, U)$ (resp. $\bigcup(\mathbb{H}^-, U)$). Hence $\mathcal{S}(\mathbb{T}^+)$ (resp. $\mathcal{S}(\mathbb{T}^-)$) is a subbasis for a topology on $C_{\mathcal{M}}(Y,Z)$. □

The topologies on $C_{\mathcal{M}}(Y,Z)$ obtained from $\mathcal{S}(\mathbb{T}^+)$ and $\mathcal{S}(\mathbb{T}^-)$ are denoted by $\mathfrak{T}(\mathbb{T}^+)$ and $\mathfrak{T}(\mathbb{T}^-)$ respectively.

Definition 3.4. Let (Y,τ) and (Z,μ) be two topological spaces. Then for each pair $(\mathbb{T}^+, \mathbb{T}^-)$ of upper and lower topology, the topology generated by

$$\{(\mathbb{H}^+, U) \cap (\mathbb{H}^-, U) \mid U \in \mu, \mathbb{H}^+ \in \mathbb{T}^+, \mathbb{H}^- \in \mathbb{T}^-\}$$

on $C_{\mathcal{M}}(Y,Z)$ is called the *dual* of the pair $(\mathbb{T}^+, \mathbb{T}^-)$ and is denoted by $\mathfrak{T}(\mathbb{T}^+, \mathbb{T}^-)$.

From the construction itself, it is clear that every pair of upper and lower topology generates a unique topology on $C_{\mathcal{M}}(Y,Z)$. Similarly, we show that every topology on $C_{\mathcal{M}}(Y,Z)$ generates a pair of upper and lower topology.

Definition 3.5. Let (Y,τ) and (Z,μ) be two topological spaces and let $\mathfrak{T}$ be a topology on $C_{\mathcal{M}}(Y,Z)$. Then we define

$$\mathcal{S}^+(\mathfrak{T}) = \{(\mathcal{H}^+, U) \mid \mathcal{H}^+ \in \mathfrak{T}, U \in \mu\}$$
$$\mathcal{S}^-(\mathfrak{T}) = \{(\mathcal{H}^-, U) \mid \mathcal{H}^- \in \mathfrak{T}, U \in \mu\}$$

Lemma 3.6. *$\mathcal{S}^+(\mathfrak{T})$ (resp. $\mathcal{S}^-(\mathfrak{T})$) is a subbasis for $O_Z^+(Y)$ (resp. $O_Z^-(Y)$).*

Proof. Let $V \in O_Z^+(Y)$ (resp. $V \in O_Z^-(Y)$). Then there exists a $F \in C_{\mathcal{M}}(Y,Z)$ and $U \in \mu$ such that $V = F^+(U)$ (resp. $V = F^-(U)$). Now for $\mathcal{H}^+ = C_{\mathcal{M}}(Y,Z) \in \mathfrak{T}$ (resp. $\mathcal{H}^- = C_{\mathcal{M}}(Y,Z) \in \mathfrak{T}$), we have $V \in (\mathcal{H}^+, U)$ (resp. $V \in (\mathcal{H}^-, U)$). Hence $O_Z^+(Y) = \bigcup(\mathcal{H}^+, U)$ (resp. $\bigcup(\mathcal{H}^-, U)$). Hence $\mathcal{S}^+(\mathfrak{T})$ (resp. $\mathcal{S}^-(\mathfrak{T})$) is a subbasis for $O_Z^+(Y)$ (resp. $O_Z^-(Y)$). □

We elaborate the above lemma with the help of the following examples.

Example 3.7. Let $Y = \mathbb{R}$ be the set of all real numbers with usual topology τ and let $Z = \mathbb{Z}$ be the set of all integers and p be a fixed prime. Then a topology μ known as *p-adic topology* [22] on Z is generated by taking as basis, the sets of the form

$$U_\alpha(n) = \{n + \lambda p^\alpha \mid \lambda \in \mathbb{Z}\}$$

Let $C_{\mathcal{M}}(\mathbb{R}, \mathbb{Z})$ be the collection of all the continuous multifunctions from $\mathbb{R}$ to $\mathbb{Z}$. Consider the compact-open topology for $C_{\mathcal{M}}(\mathbb{R}, \mathbb{Z})$, defined in [8], having a sub-base defined as

$$\mathcal{S}_{co}^{\mathcal{M}} = \{(C, U_\alpha(n)) \mid C \text{ is compact in } Y \text{ and } U_\alpha(n) \in \mu\}$$

for some α and $n \in \mathbb{Z}$.

Let us define

$$\mathcal{S}^+(\mathfrak{T}) = \{\big((C, U_\alpha(n))^+, U_\beta(m)\big) \mid \text{for some } \alpha, \beta \in \mathbb{Z} \text{ and } C \text{ is a compact subset in } \mathbb{R}\}$$
$$\text{where, } \big((C, U_\alpha(n))^+, U_\beta(m)\big) = \{F^+(U_\beta(m)) \mid F \in C_{\mathcal{M}}(\mathbb{R}, \mathbb{Z}) \text{ and } F(C) \subseteq U_\alpha(n)\}$$

Similarly, we define

$$\mathcal{S}^-(\mathfrak{T}) = \{\big((C, U_\alpha(n))^-, U_\beta(m)\big) \mid \text{for some } \alpha, \beta \in \mathbb{Z} \text{ and } C \text{ is compact in } \mathbb{R}\}$$
$$\text{where, } \big((C, U_\alpha(n))^-, U_\beta(m)\big) = \{F^-(U_\beta(m)) \mid F \in C_{\mathcal{M}}(\mathbb{R}, \mathbb{Z}) \text{ and } F(C) \subseteq U_\alpha(n)\}$$

It can be easily verified that $\mathcal{S}^+(\mathfrak{T})$ and $\mathcal{S}^-(\mathfrak{T})$ form subbasis for topologies on $O^+_{\mathbb{Z}}(\mathbb{R})$ and $O^-_{\mathbb{Z}}(\mathbb{R})$ respectively, justifying Lemma 3.6.

Similarly, we have the following:

Example 3.8. Let $Y = \mathbb{R}$ be the set of all real numbers with usual topology τ and let $Z = \mathbb{Z}$ be the set of all integers equipped with *p-adic topology* on Z as in Example 3.7. Let $C_{\mathcal{M}}(\mathbb{R}, \mathbb{Z})$ be the collection of all the continuous multifunctions from $\mathbb{R}$ to $\mathbb{Z}$. Consider the open-open topology for $C_{\mathcal{M}}(\mathbb{R}, \mathbb{Z})$, defined in [8], having a sub-base defined as

$$\mathcal{S}_{\tau,\mu}^{\mathcal{M}} = \{(U, V_\alpha(n)) \mid U \in \tau \text{ and } V_\alpha(n) \in \mu\}$$

for some α and $n \in \mathbb{Z}$.

As in Example 3.7, let us define

$$\mathcal{S}^+(\mathfrak{T}) = \{\big((U, V_\alpha(n))^+, V_\beta(m)\big) \mid \text{for some } \alpha, \beta \in \mathbb{Z} \text{ and } U \in \tau$$
$$\text{where, } \big((U, V_\alpha(n))^+, V_\beta(m)\big) = \{F^+(V_\beta(m)) \mid F \in C_{\mathcal{M}}(\mathbb{R}, \mathbb{Z}) \text{ and } F(U) \subseteq V_\alpha(n)\}$$

Similarly, we define

$$\mathcal{S}^-(\mathfrak{T}) = \{\big((U, V_\alpha(n))^-, V_\beta(m)\big) \mid \text{for some } \alpha, \beta \in \mathbb{Z} \text{ and } U \in \tau$$
$$\text{where, } \big((U, V_\alpha(n))^-, V_\beta(m)\big) = \{F^-(V_\beta(m)) \mid F \in C_{\mathcal{M}}(\mathbb{R}, \mathbb{Z}) \text{ and } F(U) \subseteq V_\alpha(n)\}$$

Like in Example 3.7, it can be verified that $\mathcal{S}^+(\mathfrak{T})$ and $\mathcal{S}^-(\mathfrak{T})$ form subbasis for topologies on $O^+_{\mathbb{Z}}(\mathbb{R})$ and $O^-_{\mathbb{Z}}(\mathbb{R})$ respectively, justifying Lemma 3.6.

Definition 3.9. The pair $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ is called the *dual* of $\mathfrak{T}$, where the topologies $\mathbb{T}^+(\mathfrak{T})$ and $\mathbb{T}^-(\mathfrak{T})$ are obtained from $\mathcal{S}^+(\mathfrak{T})$ and $\mathcal{S}^-(\mathfrak{T})$ on $O^+_Z(Y)$ and $O^-_Z(Y)$, respectively.

Now we define splittingness and admissibility on $O^+_Z(Y)$ and $O^-_Z(Y)$. Then we investigate the possible relationships between a topology on $C_{\mathcal{M}}(Y, Z)$ and its dual pair and vice-versa with respect to these properties.

Definition 3.10. Let $\mathbb{T}^+$ and $\mathbb{T}^-$ be topologies on $O_Z^+(Y)$ and $O_Z^-(Y)$ respectively. Then the topology generated by $\{\mathbb{H}^+ \cap \mathbb{K}^- \mid \mathbb{H}^+ \in \mathbb{T}^+, \mathbb{K}^- \in \mathbb{T}^-\}$ on $O_Z^+(Y) \cap O_Z^-(Y)$ is called the *topology generated* by the pair $(\mathbb{T}^+, \mathbb{T}^-)$ and is denoted by $\mathbb{T}^*$.

In the following discussion, the topology on $O_Z^+(Y) \cap O_Z^-(Y)$ is taken to be $\mathbb{T}^*$, whereas topology μ is immaterial for our discussion.

Definition 3.11. Let (Y, τ) and (Z, μ) be two topological spaces and (X, λ) be another topological space. Then for a multifunction $G : X \times Y \to Z$ and its associated map G^*, the map $\overline{G} : X \times \mu \to O_Z^+(Y) \cap O_Z^-(Y)$ is defined by $\overline{G}(x, U) = [G^*(x)]^+ (U) = [G^*(x)]^- (U)$ for every $U \in \mu$ and $x \in X$.

Definition 3.12. Let (Y, τ) and (Z, μ) be two topological spaces and (X, λ) be another topological space. A map $M : X \times \mu \to O_Z^+(Y) \cap O_Z^-(Y)$ is called *continuous with respect to the first variable* if the map $M_U : X \to O_Z^+(Y) \cap O_Z^-(Y)$ defined by $M_U(x) = M(x, U)$ is continuous for every $x \in X$ and for each fixed $U \in \mu$.

Now we are in a position to define splittingness and admissibility of a pair of upper and lower topologies.

Definition 3.13. Let (Y, τ) and (Z, μ) be two topological spaces and (X, λ) be another topological space. Then a pair $(\mathbb{T}^+, \mathbb{T}^-)$ is called

- (*i*) a *splitting pair* (resp. *upper splitting pair*, *lower splitting pair*) if the continuity (resp. upper semi continuity, lower semi continuity) of a map $G : X \times Y \to Z$ implies the continuity with respect to the first variable of the map $\overline{G} : X \times \mu \to O_Z^+(Y) \cap O_Z^-(Y)$.
- (*ii*) an *admissible pair* (resp. *upper admissible pair*, *lower admissible pair*) if for every map $G^* : X \to C_{\mathcal{M}}(Y, Z)$, the continuity with respect to the first variable of the map $\overline{G} : X \times \mu \to O_Z^+(Y) \cap O_Z^-(Y)$ implies the continuity (resp. upper semi continuity, lower semi continuity) of the associated map $G : X \times Y \to Z$.

In the remaining part of this section, we investigate how duality links splittingness and admissibility of a topology on $C_{\mathcal{M}}(Y, Z)$ and its dual and vice-versa. The first set of theorems is about the pair $(\mathbb{T}^+, \mathbb{T}^-)$ and its dual $\mathfrak{T}(\mathbb{T}^+, \mathbb{T}^-)$.

Theorem 3.14. *The pair* $(\mathbb{T}^+, \mathbb{T}^-)$ *forms a splitting pair* (*resp. upper splitting pair, lower splitting pair*) *if and only if its dual topology* $\mathfrak{T}(\mathbb{T}^+, \mathbb{T}^-)$ *on* $C_{\mathcal{M}}(Y, Z)$ *is splitting* (*resp. upper splitting, lower splitting*).

Proof. Suppose, $(\mathbb{T}^+, \mathbb{T}^-)$ forms a splitting pair (resp. upper splitting pair, lower splitting pair), that is, for every space X, the continuity (u.s.c, l.s.c) of the map $F : X \times Y \to Z$ implies the continuity with respect to the first variable of the map $\overline{F} : X \times \mu \to O_Z^+(Y) \cap O_Z^-(Y)$. We have to show that the topology $\mathfrak{T}(\mathbb{T}^+, \mathbb{T}^-)$ on $C_{\mathcal{M}}(Y, Z)$ is splitting (resp. upper splitting, lower splitting), that is for every space X, the continuity (resp, u.s.c, l.s.c) of the map $F : X \times Y \to Z$ implies the continuity of the associated map $F^* : X \to C_{\mathcal{M}}(Y, Z)$. Thus, it is sufficient to show that the continuity with respect to the first variable of the map $\overline{F} : X \times \mu \to O_Z^+(Y) \cap O_Z^-(Y)$ implies the continuity of the associated map $F^* : X \to C_{\mathcal{M}}(Y, Z)$.

Let $x \in X$, and $(\mathbb{H}^+, U) \cap (\mathbb{K}^-, U) \in \mathfrak{T}(\mathbb{T}^+, \mathbb{T}^-)$ be a subbasic open neighbourhood of $F^*(x)$. Then $F^*(x) \in (\mathbb{H}^+, U)$ and $F^*(x) \in (\mathbb{K}^-, U)$, which implies $[F^*(x)]^+(U) \in \mathbb{H}^+$ and $[F^*(x)]^-(U) \in \mathbb{K}^-$. Therefore, we have $\overline{F}_U(x) \in \mathbb{H}^+$ and $\overline{F}_U(x) \in \mathbb{K}^-$. Since $\overline{F}_U(x) \in \mathbb{H}^+ \cap \mathbb{K}^-$. Also $\overline{F} : X \times \mu \to O_Z^+(Y) \cap O_Z^-(Y)$ is continuous with respect the first variable and $\mathbb{H}^+ \cap \mathbb{K}^-$ is an open neighbourhood of $\overline{F}_U(x)$. Thus there exists an open neighbourhood V of x such that $\overline{F}_U(V) \subseteq \mathbb{H}^+ \cap \mathbb{K}^-$. Now, for $y \in V$, we have $\overline{F}_U(y) \in \mathbb{H}^+ \cap \mathbb{K}^-$. Thus, $\overline{F}_U(y) \in \mathbb{H}^+$ and $\overline{F}_U(y) \in \mathbb{K}^-$, which implies $F^*(y) \in (\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$ for all $y \in V$. Hence $F^*(V) \subseteq (\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$. Therefore F^* is continuous.

Conversely, let $\mathfrak{T}(\mathbb{T}^+, \mathbb{T}^-)$ be splitting (resp. upper splitting, lower splitting), we have to show that the pair $(\mathbb{T}^+, \mathbb{T}^-)$ is splitting pair (resp. upper splitting pair, lower splitting pair). For this, it is sufficient to show that $\overline{F} : X \times \mu \to O_Z^+(Y) \cap O_Z^-(Y)$ is continuous with respect to the first variable provided that the map $F^* : X \to C_{\mathcal{M}}(Y, Z)$ is continuous. Let, for a fixed $U \in \mu$ and $x \in X$, $\mathbb{H} \in O_Z^+(Y) \cap O_Z^-(Y)$ be an open neighbourhood of $\overline{F}(x, U)$. Then $\mathbb{H} = \mathbb{H}^+ \cap \mathbb{K}^-$, where $\mathbb{H}^+ \in \mathbb{T}^+$ and $\mathbb{K}^- \in \mathbb{T}^-$. That is $\overline{F}(x, U) \in \mathbb{H} = \mathbb{H}^+ \cap \mathbb{K}^-$

which implies $\overline{F}(x, U) \in \mathbb{H}^+$ as well as $\overline{F}(x, U) \in \mathbb{K}^-$. Thus $[F^*(x)]^+(U) \in \mathbb{H}^+$ and $[F^*(x)]^-(U) \in \mathbb{K}^-$. Hence $F^*(x) \in (\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$. Now the map F^* is given to be continuous and $(\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$ is an open neighbourhood of $F^*(x)$. Thus there exists an open neighbourhood V of x such that $F^*(V) \subseteq (\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$. Now consider, $y \in V$, we have $F^*(y) \in (\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$. Therefore, $F^*(y) \in (\mathbb{H}^+, U)$ and $F^*(y) \in (\mathbb{K}^-, U)$. Hence, we have $\overline{F}_U(y) \in \mathbb{H}^+$ and $\overline{F}_U(y) \in \mathbb{K}^-$, that is, $\overline{F}_U(y) \in \mathbb{H}^+ \cap \mathbb{K}^- = \mathbb{H}$, for all $y \in V$. Hence $\overline{F}_U(V) \subseteq \mathbb{H}$. Hence the map $\overline{F}$ is continuous with respect to the first variable. Thus, the pair $(\mathbb{T}^+, \mathbb{T}^-)$ is a splitting pair (resp. upper splitting pair, lower splitting pair). □

Theorem 3.15. *The pair* $(\mathbb{T}^+, \mathbb{T}^-)$ *is an admissible pair* (*resp. upper admissible pair, lower admissible pair*) *if and only if its dual topology* $\mathfrak{T}(\mathbb{T}^+, \mathbb{T}^-)$ *on* $C_{\mathcal{M}}(Y, Z)$ *is admissible* (*resp. upper admissible, lower admissible*).

Proof. Let the pair $(\mathbb{T}^+, \mathbb{T}^-)$ be an admissible pair (resp. upper admissible pair, lower admissible pair), that is for every space X and for every map $G^* : X \to C_{\mathcal{M}}(Y, Z)$, the continuity of the map $\overline{G} : X \times \mu \to O^+_Z(Y) \cap O^-_Z(Y)$ with respect the first variable implies the continuity (resp. u.s.c, l.s.c) of the map $G : X \times Y \to Z$. We have to prove that the topology $\mathfrak{T}(\mathbb{T}^+, \mathbb{T}^-)$ is admissible (resp. upper admissible, lower admissible), that is the continuity of $G^* : X \to C_{\mathcal{M}}(Y, Z)$ implies the continuity (resp. u.s.c, l.s.c) of the associated map $G : X \times Y \to Z$. Thus it is sufficient to prove that $\overline{G} : X \times \mu \to O^+_Z(Y) \cap O^-_Z(Y)$ is continuous with respect to the first variable provided the map $G^* : X \to C_{\mathcal{M}}(Y, Z)$ is continuous.

Let we have, for fixed $U \in \mu$ and $x \in X$, a subbasic open neighbourhood $\mathbb{H} = \mathbb{H}^+ \cap \mathbb{K}^-$ of $\overline{G}(x, U)$, where $\mathbb{H}^+ \in \mathbb{T}^+$ and $\mathbb{K}^- \in \mathbb{T}^-$. Therefore $\overline{G}(x, U) \in \mathbb{H}$. That is, $\overline{G}_U(x) \in \mathbb{H} = \mathbb{H}^+ \cap \mathbb{K}^-$ which implies $\overline{G}_U(x) \in \mathbb{H}^+$ and $\overline{G}_U(x) \in \mathbb{K}^-$. Therefore $[G^*(x)]^+(U) = \overline{G}_U(x) \in \mathbb{H}^+$ and $[G^*(x)]^-(U) \in \mathbb{K}^-$ also. Thus $G^*(x) \in (\mathbb{H}^+, U)$ as well as $G^*(x) \in (\mathbb{K}^-, U)$, which implies $G^*(x) \in (\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$. Since the map G^* is given to be continuous and $(\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$ is a subbasic open neighbourhood of $G^*(x)$, therefore there exists an open neighbourhood V of x such that $G^*(V) \subseteq (\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$. Now, for $y \in V$, we have $G^*(y) \in (\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$, that is $[G^*(y)]^+(U) \in \mathbb{H}^+$ and $[G^*(y)]^-(U) \in \mathbb{K}^-$. Thus $\overline{G}_U(y) \in \mathbb{H}^+ \cap \mathbb{K}^-$ for all $y \in V$. Hence, $\overline{G}_U(V) \subseteq \mathbb{H}^+ \cap \mathbb{K}^-$. Therefore the map $\overline{G}$ is continuous with respect to the first variable. Hence the topology $\mathfrak{T}(\mathbb{T}^+, \mathbb{T}^-)$ is admissible (resp. upper admissible, lower admissible).

Conversely, let $\mathfrak{T}(\mathbb{T}^+, \mathbb{T}^-)$ be given to be admissible (resp. upper admissible, lower admissible), we have to show that the pair $(\mathbb{T}^+, \mathbb{T}^-)$ forms an admissible pair (resp. upper admissible pair, lower admissible pair). For this, it is sufficient to show that the continuity with respect to the first variable of the map $\overline{G} : X \times \mu \to O^+_Z(Y) \cap O^-_Z(Y)$ implies continuity of the map $G^* : X \to C_{\mathcal{M}}(Y, Z)$.

Let $x \in X$ and $(\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$ be a subbasic open neighbourhood of $G^*(x)$, that is $G^*(x) \in (\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$. Thus $[G^*(x)]^+(U) \in \mathbb{H}^+$ and $[G^*(x)]^-(U) \in \mathbb{K}^-$. Hence $\overline{G}_U(x) \in \mathbb{H}^+ \cap \mathbb{K}^-$. Since the map $\overline{G}$ is given to be continuous with respect to the first variable and $\mathbb{H}^+ \cap \mathbb{K}^-$ is a subbaisc open neighbourhood of $\overline{G}_U(x)$, thus there exists an open neighbourhood V of x such that $\overline{G}_U(V) \subseteq \mathbb{H}^+ \cap \mathbb{K}^-$. Hence for $y \in V$, we have $\overline{G}_U(y) \in \mathbb{H}^+$ and $\overline{G}_U(y) \in \mathbb{K}^-$ which implies $[G^*(y)]^+(U) \in \mathbb{H}^+$ and $[G^*(y)]^-(U) \in \mathbb{K}^-$. Therefore $G^*(y) \in (\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$ for all $y \in V$ and hence $G^*(V) \subseteq (\mathbb{H}^+, U) \cap (\mathbb{K}^-, U)$. Thus the pair $(\mathbb{T}^+, \mathbb{T}^-)$ forms an admissible pair (resp. upper admissible pair, lower admissible pair). □

Now, we provide the relationship between a topology on $C_{\mathcal{M}}(Y, Z)$ and its dual.

Theorem 3.16. *Topology* $\mathfrak{T}$ *on* $C_{\mathcal{M}}(Y, Z)$ *is splitting* (*resp. upper splitting, lower splitting*) *if and only if its dual pair* $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ *forms a splitting pair* (*resp. upper splitting pair, lower splitting pair*).

Proof. Let $\mathfrak{T}$ be a splitting (resp. upper splitting, lower splitting) topology on $C_{\mathcal{M}}(Y, Z)$. We have to show that the pair $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ forms a splitting pair (resp. upper splitting pair, lower splitting pair). It is sufficient to show that the continuity of the map $G^* : X \to C_{\mathcal{M}}(Y, Z)$ implies the continuity of the map $\overline{G} : X \times \mu \to O^+_Z(Y) \cap O^-_Z(Y)$ with respect to the first variable.

Let $x \in X$ and $\mathcal{H} \in \mathfrak{T}$ be an open neighbourhood of $G^*(x)$. Then for any fixed $U \in \mu$, $(\mathcal{H}, U) \in \mathbb{T}^*$ is an open neighbourhood of $\overline{G}(x, U)$. That is, $\overline{G}(x, U) \in (\mathcal{H}, U)$. Now $\overline{G}(x, U) = [G^*(x)]^+(U) = [G^*(x)]^-(U) \in (\mathcal{H}, U)$, by definition. This implies $G^*(x) \in \mathcal{H}$. Since the map G^* is given to be continuous and $\mathcal{H}$ be an open neighbourhood of $G^*(x)$, therefore there exists an open neighbourhood V of x such that $G^*(V) \subseteq \mathcal{H}$. Now, consider for $y \in V$, we have $G^*(y) \in \mathcal{H}$, that is $[G(y)]^+(U) \in (\mathcal{H}, U)$ and $[G(y)]^-(U) \in (\mathcal{H}, U)$. Hence $\overline{G}(y, U) \in (\mathcal{H}, U)$, for all $y \in V$. Therefore $\overline{G}_U(V) \subseteq (\mathcal{H}, U)$ and the map $\overline{G}$ is continuous with respect to the first variable. Hence the result.

Conversely, let the pair $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ form a splitting pair (resp. upper splitting pair, lower splitting pair). We have to show that the topology $\mathfrak{T}$ on $C_{\mathcal{M}}(Y, Z)$ is splitting (resp. upper splitting, lower splitting). It is equivalent to show that the map $G^* : X \to C_{\mathcal{M}}(Y, Z)$ is continuous provided the map $\overline{G} : X \times \mu \to O_Z^+(Y) \cap O_Z^-(Y)$ is continuous with respect to the first variable.

Let $x \in X$ and $\mathcal{H}$ be an open neighbourhood of $G^*(x)$, that is $G^*(x) \in \mathcal{H}$. For any fixed $U \in \mu$, we have $[G^*(x)]^+(U) \in (\mathcal{H}, U) \in \mathbb{T}^+(\mathfrak{T})$ and $[G^*(x)]^-(U) \in (\mathcal{H}, U) \in \mathbb{T}^-(\mathfrak{T})$. Therefore $\overline{G}(x, U) \in O_Z^+(Y) \cap O_Z^-(Y)$ for a fixed $U \in \mu$. Since the map $\overline{G}$ is given to be continuous with respect to the first variable, thus there exists an open neighbourhood V of x such that $\overline{G}_U(V) \subseteq (\mathcal{H}, U)$. Now consider, for $y \in V$, we have $\overline{G}_U(y) \in (\mathcal{H}, U)$ which implies $[G^*(y)]^+(U) \in (\mathcal{H}, U)$ and $[G^*(y)]^-(U) \in (\mathcal{H}, U)$. Therefore $G^*(y) \in \mathcal{H}$ for every $y \in V$. Hence the map G^* is continuous. □

In the following discussion, we explain the above result.

Let $\mathbb{T}^+(\mathfrak{T})$ and $\mathbb{T}^-(\mathfrak{T})$ be the topologies generated by $\mathcal{S}^+(\mathfrak{T})$ and $\mathcal{S}^-(\mathfrak{T})$ as discussed in Example 3.7 on $O_{\mathbb{Z}}^+(\mathbb{R})$ and $O_{\mathbb{Z}}^-(\mathbb{R})$ respectively. Then the pair $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ is dual of the compact-open topology generated by $\mathcal{S}_{co}^{\mathcal{M}}$ on $C_{\mathcal{M}}(\mathbb{R}, \mathbb{Z})$. We show that $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ is an upper splitting pair in this case.

Proposition 3.17. *Let $Y = \mathbb{R}$ and $Z = \mathbb{Z}$ be the set of real numbers and integers equipped with topologies τ and μ respectively. Let the pair $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ be the topologies generated by $\mathcal{S}^+(\mathfrak{T})$ and $\mathcal{S}^-(\mathfrak{T})$, as discussed in Example 3.7, on $O_{\mathbb{Z}}^+(\mathbb{R})$ and $O_{\mathbb{Z}}^-(\mathbb{R})$ respectively. Then $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ forms an upper splitting pair.*

Proof. Let (X, μ_1) be any topological space. The compact-open topology defined over $C_{\mathcal{M}}(\mathbb{R}, \mathbb{Z})$ in Example 3.7 is upper splitting [8]. Here, we have to show that the pair $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ forms an upper splitting pair. For this, it is sufficient to show that the continuity of the map $G^* : X \to C_{\mathcal{M}}(\mathbb{R}, \mathbb{Z})$ implies the continuity of the map $\overline{G} : X \times \mu \to O_{\mathbb{Z}}^+(\mathbb{R}) \cap O_{\mathbb{Z}}^-(\mathbb{R})$ with respect to the first variable.

Let $x \in X$ and $(C, U_\alpha(n)) \in \mathfrak{T}$ be an open neighbourhood of $G^*(x)$. Then for any fixed $U_\beta(m) \in \mu$, $((C, U_\alpha(n)), U_\beta(m)) \in \mathbb{T}^*$ is an open neighbourhood of $\overline{G}(x, U_\beta(m))$. That is, $\overline{G}(x, U_\beta(m)) \in ((C, U_\alpha(n)), U_\beta(m))$. Now $\overline{G}(x, U_\beta(m)) = [G^*(x)]^+(U_\beta(m)) = [G^*(x)]^-(U_\beta(m)) \in ((C, U_\alpha(n)), U_\beta(m))$, by definition. This implies $G^*(x) \in (C, U_\alpha(n))$. Since the map G^* is given to be continuous and $(C, U_\alpha(n))$ is an open neighbourhood of $G^*(x)$, therefore there exists an open neighbourhood V of x such that $G^*(V) \subseteq (C, U_\alpha(n))$. Now, consider for $y \in V$, we have $G^*(y) \in (C, U_\alpha(n))$, that is $[G(y)]^+(U_\beta(m)) \in ((C, U_\alpha(n)), U_\beta(m))$ and $[G(y)]^-(U_\beta(m)) \in ((C, U_\alpha(n)), U_\beta(m))$. Hence $\overline{G}(y, U_\beta(m)) \in ((C, U_\alpha(n)), U_\beta(m))$, for all $y \in V$. Therefore $\overline{G_{U_\beta(m)}}(V) \subseteq ((C, U_\alpha(n)), U_\beta(m))$ and the map $\overline{G}$ is continuous with respect to the first variable. Hence the result. □

Similarly, we can prove that

Theorem 3.18. *Topology $\mathfrak{T}$ on $C_{\mathcal{M}}(Y, Z)$ is admissible (resp. upper admissible, lower admissible) if and only if its dual pair $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ forms an admissible pair (resp. upper admissible pair, lower admissible pair).*

Proof. The proof is left for the readers. □

Now, in the following we discuss the above theorem in the light of the following result.

Let $\mathbb{T}^+(\mathfrak{T})$ and $\mathbb{T}^-(\mathfrak{T})$ be the topologies generated by $\mathcal{S}^+(\mathfrak{T})$ and $\mathcal{S}^-(\mathfrak{T})$ as discussed in Example 3.8 on $O_{\mathbb{Z}}^+(\mathbb{R})$ and $O_{\mathbb{Z}}^-(\mathbb{R})$ respectively. Then the pair $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ is the dual of the open-open topology generated by $\mathcal{S}_{\tau,\mu}^{\mathcal{M}}$ on $C_{\mathcal{M}}(\mathbb{R}, \mathbb{Z})$. We show that $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ is an upper admissible pair in this case.

Proposition 3.19. *Let $Y = \mathbb{R}$ and $Z = \mathbb{Z}$ be the set of real numbers and integers equipped with topologies τ and μ respectively. Let the pair $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ be the topologies generated by $\mathcal{S}^+(\mathfrak{T})$ and $\mathcal{S}^-(\mathfrak{T})$ as discussed in Example 3.8 on $O_{\mathbb{Z}}^+(\mathbb{R})$ and $O_{\mathbb{Z}}^-(\mathbb{R})$ respectively. Then $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ forms an upper admissible pair.*

Proof. Let $Y = \mathbb{R}$ and $Z = \mathbb{Z}$ be the set of real numbers and integers respectively. Let the pair $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ be the topologies discussed in Example 3.8. Here, we have to show that $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ forms an upper admissible pair. For this, let (X, μ_1) be any topological space. We have to show that the continuity with respect to first variable of the map $\overline{G}\colon X \times \mu \to O^+_{\mathbb{Z}}(\mathbb{R}) \cap O^-_{\mathbb{Z}}(\mathbb{R})$ implies the continuity of the map $G^*\colon X \to C_{\mathcal{M}}(\mathbb{R}, \mathbb{Z})$.

Let $x \in X$, and $\Big((U_1, V_\alpha(n_1))^+, V_\beta(m)\Big) \cap \Big((U_2, V_\alpha(n_2))^-, V_\beta(m)\Big) \in (\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ be a subbasic open neighbourhood of $G^*(x)$. That is, $G^*(x) \in \Big((U_1, V_\alpha(n_1))^+, V_\beta(m)\Big)$ and $G^*(x) \in \Big((U_2, V_\alpha(n_2))^-, V_\beta(m)\Big)$. Thus, we have $[G * (x)]^+(V_\beta(m)) \in (U_1, V_\alpha(n_1))^+$ and $[G * (x)]^-(V_\beta(m)) \in (U_2, V_\alpha(n_2))^-$. Therefore, we have $\overline{G_{V_\beta(m)}}(x) \in (U_1, V_\alpha(n_1))^+$ and $\overline{G_{V_\beta(m)}}(x) \in (U_2, V_\alpha(n_2))^-$, that is, $\overline{G_{V_\beta(m)}}(x) \in (U_1, V_\alpha(n_1))^+ \cap (U_2, V_\alpha(n_2))^-$. Since the map $\overline{G}\colon X \times \mu \to O^+_{\mathbb{Z}}(\mathbb{R}) \cap O^-_{\mathbb{Z}}(\mathbb{R})$ is given to be continuous with respect to the first variable and $(U_1, V_\alpha(n_1))^+ \cap (U_2, V_\alpha(n_2))^-$ is a subbasic open neighbourhood of $\overline{G_{V_\beta(m)}}(x)$, there exists an open neighbourhood V of x such that $\overline{G_{V_\beta(m)}}(V) \subseteq (U_1, V_\alpha(n_1))^+ \cap (U_2, V_\alpha(n_2))^-$. Now, for $y \in V$, we have $\overline{G_{V_\beta(m)}}(y) \in (U_1, V_\alpha(n_1))^+ \cap (U_2, V_\alpha(n_2))^-$. Hence, we have $G^*(V) \subseteq \Big((U_1, V_\alpha(n_1))^+, V_\beta(m)\Big) \cap \Big((U_2, V_\alpha(n_2))^-, V_\beta(m)\Big)$. Therefore, G^* is continuous. Hence the result. □

In the subspace topology of the function space $C(Y, Z)$, the above results reduce to the Corollary 3.6, 3.8, 3.10, 3.15 of [6].

Remark 3.20. In the present study, we have not investigated the relationship between a topology $\mathfrak{T}$ on $C_{\mathcal{M}}(Y, Z)$ and the dual $\mathfrak{T}(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$ of $(\mathbb{T}^+(\mathfrak{T}), \mathbb{T}^-(\mathfrak{T}))$. Similarly, for the pair $(\mathbb{T}^+, \mathbb{T}^-)$ on $(O^+_Z(Y), O^-_Z(Y))$, its relationship with the dual of the dual topology $\mathfrak{T}(\mathbb{T}^+, \mathbb{T}^-)$ needs to be investigated further. The same result may be taken up as further continuation of the above work.

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