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On G-Compactness of Topological Groups with Operations

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Abstract. One can notice that if X is a Hausdorff space, then limits of convergent sequences in X give us a function denoted by $\lim$ from the set of all convergent sequences in X to X. This notion has been extended by Connor and Grosse-Erdmann to an arbitrary linear functional G defined on a subspace of the vector space of real numbers. Following this idea some authors have defined concepts of G-continuity, G-compactness and G-connectedness in topological groups. In this paper we present some results about G-compactness of topological group with operations such as topological groups, topological rings without identity, R-modules, Lie algebras, Jordan algebras and many others.

1. Introduction

Sequential convergence is an important tool in topology and analysis; and therefore one gets into search to find the standard sequential definitions of some concepts such as continuity, compactness and connectedness and the others. In addition to the ordinary convergence of sequences, there exist a wide variety of convergent types which are very important not only in pure mathematics but also in other branches of science involving mathematics especially in information theory, biological science and dynamical systems.

Motivated by an idea introduced in a 1946 American Mathematical Monthly problem [8], a number of authors Posner [36], Iwinski [23], Srinivasan [38], Antoni [3], Antoni and Salat [4], Spigel and Krupnik [39] have studied A-continuity defined by a regular summability matrix A. Some authors Öztürk [40], Savaş and Das [41], Savaş [42], Borsik and Salat [7] have studied A-continuity for methods of almost convergence and for related methods. See also [5] for an introduction to summability matrices and [14] for summability in topological groups. Di Maio and Kočinac [26] defined statistical convergence in topological spaces, introduced statistically sequential spaces and statistically Fréchet spaces, and considered their applications in selection principles theory, function spaces and hyperspaces.

Connor and Grosse-Erdmann [15] have investigated the impact of replacing the convergence sequences on sequential continuity of real functions with G-methods defined on a subspace of the vector space of real sequences. Then Çakallı extended this concept to topological groups and introduced the concept of G-compactness in [13], obtained further results on G-compactness and G- continuity in [11](see also [16] and [12], for some other types of continuities which can not be given by any sequential method) and developed the G-connectedness of topological groups in [10] (see also [9]). Mucuk and Şahan [31] have introduced the

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notions of G-open sets and G-neighbourhoods in topological groups, and investigated more properties of G-continuities.

Lin and Liu in [25] have recently extended the G-methods and several convergent methods on topological groups by introducing the concepts of G-methods, G-submethods and G-topologies on arbitrary sets; and investigated operations on subsets that deal with G-hulls, G-closures, G-kernels and G-interiors. Mucuk and Çakallı [27] recently extended the G-connectedness to the topological groups with operations including topological groups In this paper we present some results about the varieties of G-compactness for topological group with operations including topological groups, topological rings without identity, R-modules, Lie algebras, Jordan algebras and some others. The authors in the paper [1] extend these ideas to a neutrosophic topological space and make some investigations in this direction.

Orzech [34] introduced an algebraic category C called category of groups with operations including groups, rings without identity, R-modules, Lie algebras, Jordan algebras, and many others. The internal category and crossed module in C were studied in [35] and the studies have resumed by the works of Datuashvili [18–21]. Recently some works for topological groups with operations and their internal categories have been carried out in [2, 29, 30, 32, 33].

In this paper we give some results about G-continuity and different kinds of G-compactness such as G-locally compactness, G-countably compactness for topological groups with operations. We also refer the readers to the paper [44] for some discussion about G-compactness.

We acknowledge that an extended abstract including the statements without proofs of a few results of this paper appears in [28] as AIP Conference Proceedings.

2. Preliminaries

Throughout the paper X denotes a Hausdorff topological group with operations, the boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... represent the sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of terms in X; and $s(X)$ and $c(X)$ respectively denote the set of all sequences and the set of all convergent sequences of points in X.

By a G-method of sequential convergence for X, we mean a morphism defined on a subgroup with multiple operations which is to be denoted $c_G(X)$ of $s(X)$ into X. A sequence $\mathbf{x} = (x_n)$ is said to be *G-convergent* to ℓ if $\mathbf{x} \in c_G(X)$ and $G(\mathbf{x}) = \ell$. In particular, $\lim$ function defined on $c(X)$ is a G-method with $G = \lim$.

A method G is called *regular* if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$; and called *subsequential* if, whenever a sequence $\mathbf{x}$ is G-convergent to ł, then there is a subsequence $\mathbf{y}$ of $\mathbf{x}$ with $\lim \mathbf{y} = \ell$ [15] and G is said to *preserve the G-convergence of subsequences* if, whenever a sequence $\mathbf{x}$ is G-convergent to ł then any subsequence of $\mathbf{x}$ is also G-convergent to the same point ℓ.

A map $f\colon X \to X$ is called *G-continuous* if $G(f(\mathbf{x})) = f(G(\mathbf{x}))$ for $\mathbf{x} \in c_G(X)$ [11].

The notion of regularity introduced above coincides with the classical notion of regularity for summability matrices (see [5] for an introduction to regular summability matrices and see [45] for a general view of sequences of reals or complex).

For a subset $A \subseteq X$, a point $\ell \in X$ is said to be in the *G-hull* of A whenever there exists a sequence $\mathbf{x} = (x_n)$ in A with $G(\mathbf{x}) = \ell$ and the G-hull of A is denoted by $\overline{A}^G$ in [15] but by the notations in [25], we write $[A]_G$; and say that A is *G-closed* if $[A]_G \subseteq A$. If G is a regular method, then $A \subseteq [A]_G$, and hence A is G-closed if and only if $[A]_G = A$. Even for regular methods $[[A]_G]_G = [A]_G$ is not always true and the union of any two G-closed subsets of X need not also be a G-closed subset of X [11, Counterexample 1]. A subset $U \subseteq X$ is called *G-open* if $X \setminus U$ is G-closed. If $B \subseteq A \subseteq X$ and $a \in A$, then we say a is in the *G-hull of B* in A if there is a sequence $\mathbf{x} = (x_n)$ of points in B such that $G(\mathbf{x}) = a$. A subset F of A is called *G-closed* in A if there exists a G-closed subset K of X such that $F = K \cap A$. We say that a subset U of A is *G-open* in A if $A \backslash U$ is G-closed in A. Here note that a subset U of A is G-open in A if and only if there exists a G-open subset V of X such that $U = A \cap V$. The union of any G-open subsets of X is G-open. A subset V is a *G-neighborhood* of a if there exists a U-sequential open subset of X with $a \in U$ such that $U \subseteq V$. The union of G-open subsets of A is called *G-interior* of A and denoted by $A^{\circ G}$ [31].

We remark that as it is stated in [25, Remark 2.2] since the definition of G-method already involves sequences the term 'sequentially' in G-sequentially closed sets seems redundant, so they choose the terminology of G-closed sets. By the same idea we use the similar terminology G-open sets, G-continuity, G-connectedness, G-compactness and etc.

The idea of the definition of a category of groups with operations comes from Higgins [22] and Orzech [34]; and the definition below is from Porter [35] and Datuashvili [17, p.21], which is adapted from Orzech [34].

Let C be a category of groups with a set of operations Ω and with a set E of identities such that E includes the group laws, and the following conditions hold: If Ω_i is the set of i-ary operations in Ω, then

1. $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

2. The group operations written additively $0, -$ and $+$ are the elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega'_2 = \Omega_2 \backslash \{+\}$, $\Omega'_1 = \Omega_1 \backslash \{-\}$ and assume that if $\star \in \Omega'_2$, then $\star^\circ$ defined by $x \star^\circ y = y \star x$ is also in Ω'_2. Also assume that $\Omega_0 = \{0\}$.

3. For each $\star \in \Omega'_2$, E includes the identity $x \star (y + z) = x \star y + x \star z$.

4. For each $\omega \in \Omega'_1$ and $\star \in \Omega'_2$, E includes the identities $\omega(x + y) = \omega(x) + \omega(y)$ and $\omega(x) \star y = \omega(x \star y)$.

Then the category C satisfying the conditions (1)-(4) is called a *category of groups with operations*.

From now on C will be a category of groups with operations.

A *morphism* between any two objects of C is a group homomorphism, which preserves the operations of Ω'_1 and Ω'_2.

The set Ω_0 contains exactly one element, the group identity; hence for instance the category of associative rings with unit is not a category of groups with operations. The categories of groups, rings generally without identity, R-modules, associative, associative commutative, Lie, Leibniz, alternative algebras are examples of categories of groups with operations.

The subobject in the category C can be defined as follows.

Definition 2.1. Let X be a group with operations, i.e., an object of C. A subgroup A of X is called a *subgroup with operations* subject to the the following conditions:

1. $a \star b \in A$ for $a, b \in A$ and $\star \in \Omega'_2$;
2. $\omega(a) \in A$ for $a \in A$ and $\omega \in \Omega'_1$.

The normal subobject in the category C is defined as follows.

Definition 2.2. ([34, Definition 1.7]) Let X be an object in C and A be a subgroup with operations of X. A is called a *normal subgroup with operations or ideal* if

1. $(A, +)$ is a normal subgroup of $(X, +)$;
2. $x \star a \in A$ for $x \in X$, $a \in A$ and $\star \in \Omega'_2$.

The category of topological groups with operations are defined in [2, pp. 228] (see also [32, Definition 3.4]) as follows:

A category $\mathsf{Top}^{\mathsf{C}}$ of topological groups with a set Ω of continuous operations and with a set E of identities such that E includes the group laws such that the axioms (1)-(4) above are satisfied, is called a *category of topological groups with operations*.

A *morphism* between any two objects of $\mathsf{Top}^{\mathsf{C}}$ is a continuous group homomorphism, which preserves the operations in Ω'_1 and Ω'_2.

The categories of topological groups, topological rings and topological R-modules are examples of categories of topological groups with operations.

In the rest of the paper $\mathsf{Top}^{\mathsf{C}}$ will denote the category of topological groups with operations and X will denote an object of $\mathsf{Top}^{\mathsf{C}}$; and G will be a regular sequential method unless otherwise is stated.

3. *G*-compact topological groups with operations

We remark that the results about G-locally compactness of this section are new even in topological group case.

Recall from [13, Definition 1] that a subset A of X is called *G-compact* whenever any sequence $\mathbf{x} = (x_n)$ of points in A has a subsequence $\mathbf{y} = (x_{n_k})$ with $G(\mathbf{y}) = \ell \in A$.

We now give the following example for G-compact subsets of $\mathbb{R}$ in which $\mathbb{R}$ can be thought as a topological group with operations since it is a topological group, topological ring and $\mathbb{R}$-module on itself.

Example 3.1. Let G be a method on $\mathbb{R}$ defined by $G(\mathbf{x}) = \lim \frac{x_n + x_{n+1}}{2}$ for the sequences $\mathbf{x} = (x_n)$ in $\mathbb{R}$ whenever these limits exit. The method G is regular since $G(\mathbf{x}) = \lim \frac{x_n + x_{n+1}}{2} = \lim \mathbf{x}$ for any convergent sequence $\mathbf{x} = (x_n)$ but it is neither subsequential nor preserves the convergences of the subsequences. For example the sequence $\mathbf{x} = (1, 3, 1, 3, \dots)$ is G-convergent to the point 2 but neither there is a subsequence $\mathbf{y}$ with $\lim \mathbf{y} = 2$ nor all subsequences G-converges to the same point 2.

Let A be a finite subset of $\mathbb{R}$ and $\mathbf{x} = (x_n)$ be a sequence of the terms in A. Then there are some terms repeating infinitely. For a infinitely repeating term x_{n_0} the constant subsequence $\mathbf{y} = (x_{n_0})$ is G-converging to $x_{n_0} \in A$. Hence A is G-compact and therefore all finite subsets of $\mathbb{R}$ are G-compact.

The following examples are true not only for topological groups with operations but also for sets.

Example 3.2. Let X be a set and $c \in X$ a certain element, and let G be a method on X defined by $G(\mathbf{x}) = c$ for all sequences in X. Clearly the method G is not regular. For example for any convergence sequence $\mathbf{x}$ with $\lim \mathbf{x} = ł$ which differs c, one has $G(\mathbf{x}) \neq \lim \mathbf{x}$. It preserves the convergences but it is not subsequential.

Let A be any subset of X with $c \in A$ and $\mathbf{x} = (x_n)$ be a sequence of the terms in A. Then all subsequences of $\mathbf{x} = (x_n)$ G-converge to c. Therefore the subset A is G-compact whenever $c \in A$. In particular X becomes G-compact.

Example 3.3. Let G be a method on a subset X defined by $G(\mathbf{x}) = x_1$ for all sequences $\mathbf{x} = (x_n)$ in X. The method G is not regular since for example for the sequence $\mathbf{x} = (\frac{1}{n})$ one has that $G(\mathbf{x}) = x_1 = 1$ but $\lim \mathbf{x} = 0$. It is also neither subsequential nor it preserves the convergences of the subsequences.

Let A be a subset X and $\mathbf{x} = (x_n)$ be a sequence of the terms in A. Then any subsequence of $\mathbf{x} = (x_n)$ G-convergences to the point x_1 in A. Hence all subsets of X are G-compact.

In the following theorem, we prove that the product of two G-compact subsets is also a G-compact.

Theorem 3.4. *Let G be a method preserving the G-convergence of subsequences. Then the product of two G-compact subsets of X is also G-compact.*

Proof. Let A and B be G-compact subsets of X and $\mathbf{x}$ be a sequence of points in $A \times B$. By the G-compactness of A, we can choose a subsequence $\mathbf{y}$ of $\mathbf{x}$ such that $G(\pi_1(\mathbf{y})) = u \in A$ and by the G-compactness of B choose a subsequence $\mathbf{z}$ of $\mathbf{y}$ such that $G(\pi_2(\mathbf{z})) = v \in B$. Since G preserves the G-convergences of subsequences we have $G(\pi_1(\mathbf{z})) = G(\pi_1(\mathbf{y})) = u$ and hence

$$\begin{aligned} G(\mathbf{z}) &= (G(\pi_1(\mathbf{z})), G(\pi_2(\mathbf{z}))) \\ &= (u, v) \in A \times B. \end{aligned}$$

This proves that $A \times B$ is G-compact. □

Theorem 3.5. *If G is a method preserving the G-convergence of subsequences and X is G-compact, then any G-closed subset of $X \times X$ is also G-compact.*

Proof. If X is G-compact, then by Theorem 3.4 $X \times X$ is G-compact. If A is a G-closed subset of $X \times X$ and $\mathbf{x}$ is a sequence of points in A, then by the G-compactness of X we can choose a subsequence $\mathbf{y}$ of $\mathbf{x}$ such that $G(\pi_1(\mathbf{y})) = a$ and choose a subsequence $\mathbf{z}$ of $\mathbf{y}$ such that $G(\pi_2(\mathbf{z})) = b$. Since G preserves the G-convergence of subsequences we have $G(\pi_1(\mathbf{z})) = G(\pi_1(\mathbf{y}))$ and hence

$$\begin{aligned} G(\mathbf{z}) &= (G(\pi_1(\mathbf{z})), G(\pi_2(\mathbf{z}))) \\ &= (a, b). \end{aligned}$$

Since A is G-closed, $(a, b) \in A$ which proves that A is G-compact. □

Corollary 3.6. *If X is G-compact, then the following are satisfied:*

1. If $f\colon X \to X$ is a morphism of groups with operations and G-continuous, then $A = \{(x, y) \mid f(x) = f(y)\}$ is a G-compact subgroup with operation of $X \times X$.

2. $\Delta X = \{(x, x) \mid x \in X\}$ is a G-compact subgroup with operations of $X \times X$.

3. For the G-continuous morphisms $f, g\colon X \to X$ of groups with operations, $A = \{x \in X \mid f(x) = g(x)\}$ is a G-compact subgroup with operations of X.

Proof. The proofs of (1) and (2) are obtained as a result of Theorems 3.4 and 3.5; and the proof of (3) is obtained as a result of [13, Theorem 1]. □

The following theorem is proved in [13, Theorem 2] in the case where G is a regular subsequential method.

Theorem 3.7. *If G is a method preserving the G-convergence of subsequences, then any G-compact subset of X is G-closed.*

Proof. Let A be a G-compact subset of X and $\mathbf{x}$ a sequence of the points in A with $G(\mathbf{x}) = u$. Since A is G-compact, there is a subsequence $\mathbf{y}$ of $\mathbf{x}$ such that $G(\mathbf{y}) = v \in A$. Since G preserves the G-convergence of subsequences, $G(\mathbf{x}) = G(\mathbf{y})$ and hence $u \in A$. Hence A is G-closed. □

Theorem 3.8. *If G is a method preserving the G-convergence of subsequences, then any G-compact subset of $X \times X$ is G-closed.*

Proof. Let A be a G-compact subset of $X \times X$ and $\mathbf{x}$ a sequence of points in A with $G(\mathbf{x}) = (a, b)$. By the G-compactness of A, choose a subsequence $\mathbf{y}$ of $\mathbf{x}$ such that $G(\mathbf{y}) = (u, v) \in A$. Since the method G preserves the G-convergence of subsequences we have $G(\mathbf{x}) = G(\mathbf{y})$ and $(a, b) \in A$. Hence A is G-closed. □

As a result of Theorems 3.5 and 3.8 we can state the following corollary.

Corollary 3.9. *If X is G-compact and G is a method preserving the G-convergence of subsequences, then a subset of $X \times X$ is G-compact if and only if it is G-closed.*

Theorem 3.10. *If G is a method preserving the convergence of subsequences and X is G-compact, then any G-continuous map $f\colon X \to X$ is G-closed.*

Proof. Let A be a G-closed subset of X. Since X is G-compact by [13, Thorem 1], A is G-compact. Since f is G-continuous by [13, Theorem 7], $f(A)$ is G-Compact. Hence by Theorem 3.7, $f(A)$ is G-closed. □

Theorem 3.11. *Let G be a method preserving the convergence of subsequences. If A is a G-compact subgroup with operations of X and $f\colon X \to X$ is G-continuous, then the graph set $B = \{(a, f(a)) \mid a \in A\}$ is a G-compact subgroup with operations of $X \times X$.*

Proof. We know by [27, Theorem 2.19] that B is a subgroup with operations of $X \times X$. Hence we need just to prove that B is G-compact. Let $\mathbf{x}$ be a sequence of points in B. Since A is G-compact subset and f is G-continuous by [13, Theorem 7], the image $f(A)$ is G-compact. As similar to the proof of Theorem 3.4, by the G-compactness of A, choose a subsequence $\mathbf{y}$ of $\mathbf{x}$ such that $G(\pi_1(\mathbf{y})) = u \in A$ and by the G-compactness of $f(A)$ choose a subsequence $\mathbf{z}$ of $\mathbf{y}$ such that $G(\pi_2(\mathbf{z})) = v \in f(A)$. Since G preserves the G-convergences of subsequences we have $G(\pi_1(\mathbf{z})) = G(\pi_1(\mathbf{y})) = u$ and hence

$$G(\mathbf{z}) = (G(\pi_1(\mathbf{z})), G(\pi_2(\mathbf{z}))) = (u, v).$$

Since $\mathbf{z}$ is a sequence in B, $f(\pi_1(\mathbf{z})) = \pi_2(\mathbf{z})$. Hence the G-continuity of f and $u = G(\pi_1(\mathbf{z}))$ imply that

$$\begin{aligned} f(u) &= f(G(\pi_1(\mathbf{z}))) \\ &= G(f(\pi_1(\mathbf{z}))) \\ &= G(\pi_2(\mathbf{z})) \\ &= v \end{aligned}$$

and that $(u, v) \in B$. This proves that B is G-compact. □

We can give the definition of G-locally compactness for topological groups with operations as follows:

Definition 3.12. A topological group with operations X is called *G-locally compact* if every point of X has a fundamental system of G-compact neighbourhoods.

The proofs of the following theorems appear in [28] and therefore they are omitted.

Theorem 3.13. ([28, Theorem 5]) *If X is G-locally compact, then any G-closed subset of X is also G-locally compact.*

Theorem 3.14. ([28, Theorem 6]) *If X is a G-locally compact and G is a method which preserves the G-convergence of subsequences, then $X \times X$ is G-locally compact.*

We recall form [13, Definition 2] that a point $x \in X$ is called a *G-accumulation point* of A if there is a sequence $\mathbf{a} = (a_n)$ of points in $A \backslash \{x\}$ such that $G(\mathbf{a}) = x$ and that from [13, Definition 2] a subset A of X is called *G-countably compact* if any infinite subset of A has at least one G-accumulation point in A.

The following example can be restated for topological groups with operations.

Example 3.15. Let G be a method on an infinite subset X defined by $G(\mathbf{x}) = x_1$ for all sequences $\mathbf{x} = (x_n)$ in X. Let B be a infinite subset of A and let $\mathbf{x} = (x_n)$ be a sequence in $B \setminus \{x\}$. Then the sequence $\mathbf{x} = (x_n)$ is G-converging to $x_1 \in B$. That means x_1 is a G-accumulation point of B. Therefore any infinite subset A of X is G-countably compact.

We now prove the following result.

Theorem 3.16. *The finite product of G-countably compact subsets of X is also G-countably compact.*

Proof. Let A and B be G-countably compact subsets of X. If $U \subseteq A \times B$ is an infinite subset, then at least one of the subsets $\pi_1(U) \subseteq A$ or $\pi_2(U) \subseteq B$ is infinite. Suppose that $\pi_1(U) \subseteq A$ is infinite. Since A is G-countably compact, $\pi_1(U)$ has at least one G-accumulation point a in A. Hence there is a sequence $\mathbf{a}$ of points in $\pi_1(U) \setminus \{a\}$ such that $G(\mathbf{a}) = a$. Then we have a sequence $\mathbf{x} = (\mathbf{a}, \mathbf{b})$ of points in $U \backslash (a, b)$ such that $G(\mathbf{x}) = (a, b)$ where $b \in \pi_2(U)$ and $\mathbf{b}$ is the constant sequence $\mathbf{b} = (b, b, \dots)$. □

Theorem 3.17. *If X is G-countably compact, then any G-closed subset of $X \times X$ is G-countably compact.*

Proof. Let X be G-countably compact, $A \subseteq X \times X$ a G-closed subset and B an infinite subset of X. Since by Theorem 3.16 $X \times X$ is G-countably compact, B has a G-accumulation point $x \in X$ and hence there is a sequence $\mathbf{a}$ of points in $B \backslash \{x\}$ such that $G(\mathbf{a}) = x$. Since A is G-sequentailly closed, $x \in A$. Hence A is G-countably compact. □

Corollary 3.18. *If X is G-countably compact, then we have the following:*

1. If $f: X \to X$ is a morphism of groups with operations and G-continuous, then $A = \{(x, y) \mid f(x) = f(y)\}$ is a G-countably compact subgroup with operation $X \times X$.

2. $\Delta X = \{(x, x) \mid x \in X\}$ is a G-countably compact subgroup with operations of $X \times X$.

3. For the G-continuous morphisms $f, g: X \to X$ of groups with operations, $A = \{x \in X \mid f(x) = g(x)\}$ is a G-countably compact subgroup with operations of X.

Proof. The proofs of (1) and (2) are the results of Theorems 3.16 and 3.17.

The proof of (3) is obtained by the fact that a G-closed subset of X is G-countably compact whenever X is G-countably compact. □

Theorem 3.19. *If A is a G-countably compact subgroup with operation of X and $f: X \to X$ is a G-continuous, then the graph set $B = \{(a, f(a)) \mid a \in A\}$ is a G-countably compact subgroup with operations.*

Proof. If $U \subseteq B$ is an infinite subset, then, say, $\pi_1(U)$ is an infinite subset of A and since A is a G-countably compact subset, there exists at least a G-accumulation point x in A. Hence there is a sequence $\mathbf{a} = (a_n)$ of points in $\pi_1(U)\backslash\{x\}$ such that $G(\mathbf{a}) = x$. Then for a constant sequence $\mathbf{b} = (y, y, \dots)$ with $y = f(x)$, $\mathbf{x} = (\mathbf{a}, \mathbf{b})$ is a sequence of the points in $U\backslash\{x, y\}$ such that $G(\mathbf{x}) = (x, y) \in B$. Hence (x, y) is a G-accumulation point of U in B and hence B becomes G-countably compact. □

Finally we can state the following corollary.

Corollary 3.20. *Let K_0 be the G-connected component of $0 \in X$. Then we have the following:*

1. If X is G-compact then K_0 is a G-compact subgroup with operations.

2. If X is G-locally compact, then K_0 is a G-locally compact subgroup with operations.

3. If X is G-countably compact, then K_0 is a G-countable compact subgroup with operations.

Proof. Since by [27, Theorem 3.3], K_0 is G-closed subgroup with operations of X,

1. follows from the fact that a G-closed subset of X is G-compact whenever X is G-compact [13, Theorem 1];

2. is a result of Theorems 3.13;

3. follows from the fact that a G-closed subset of X is G-countably compact whenever X is G-countably compact. □

4. Conclusion

In this paper we consider different kinds of G-compactness for a category of topological groups with operations which include topological groups. Some of the results are even new in topological group case.

To generalize the results of this paper to more general case of topological $\mathbb{T}$ algebras, we first recall a fact on semi-abelian categories: The notion of semi-abelian category as proposed in [24] (see also [37] and [43]) has typical categorical properties such as possessing finite products, coproducts, a zero object and hence kernels, pullbacks of monomorphisms and coequalizers of kernel pairs. Groups, rings, algebras and all abelian categories are semi-abelian, say.

In [6] for a certain algebraic theory the term 'algebraic model' is used for the objects of the semi-abelian category. Let $\mathbb{T}$ be an algebraic theory whose category is semi-abelian. A *topological model* of $\mathbb{T}$ is a model of the theory of $\mathbb{T}$ with a topology which makes all the operations of the theory continuous. The category $\mathsf{Top}^{\mathbb{T}}$, for a semi-abelian theory $\mathbb{T}$, is generally no longer semi-abelian because it is not Bar exact. But in [6] the category $\mathsf{Top}^{\mathbb{T}}$ of the topological models $\mathbb{T}$ is studied and some classical results in topological groups is generalized to this category $\mathsf{Top}^{\mathbb{T}}$. For example when $\mathbb{T}$ is the theory of groups, then $\mathsf{Top}^{\mathbb{T}}$ becomes the category of topological groups and we obtain the results for topological groups.

Hence the methods of the paper [6] could be be useful to deal with $\mathsf{Top}^{\mathbb{T}}$ and obtain more general results for topological $\mathbb{T}$ algebras.

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