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Linearly S-Closed Spaces

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Abstract. We introduce the class of linearly S-closed spaces as a proper subclass of linearly H-closed spaces. This property lies between S-closedness and countable S-closedness. A space is called linearly S-closed if and only if any semi-open chain cover posses a member dense in the space. It is shown that in the class of extremally disconnected spaces the class of linearly H-closed spaces and linearly S-closed spaces coincide. We gave characterizations of these spaces in terms of s-accumulation points of chain filter bases and complete s-accumulation points of families of open subsets. While regular S-closed spaces are compact there is a non compact, regular, linearly S-closed space. It is shown that a Hausdorff, first countable, linearly S-closed space is extremally disconnected. Moreover, in the class of first countable, regular, compact spaces the notions of S-closedness, linearly S-closedness and extremally disconnectedness are equivalent. Some cardinality bounds for this class of spaces are obtained. Several examples are provided to illustrate our results.

1. Introduction and Preliminaries

In 1963, N. Levine [18] defined semi-open sets in a topological space. In this paper we extend the notion of linearly H-closed spaces, introduced by M. Baillif [7] in 2019, by using semi-open sets. The new class of spaces so obtained, which we call linearly S-closed spaces, lies between S-closedness and countable S-closedness.

By 'space' we always mean 'topological space'. An open cover (respectively, semi-open cover) of a space is a cover by open sets (respectively, semi-open sets) in the space. A cover is said to be a chain cover if it is linearly ordered by the inclusion relation. An open chain cover (respectively, semi-open chain cover) is a chain cover consisting of open sets (respectively, semi-open sets) in the space. A topological space X is *linearly H-closed* [7] if and only if any open chain cover of X posses a member dense in X. The class of linearly H-closed spaces lies between H-closed spaces and feebly compact spaces.

Definition 1.1. A space X is *linearly S-closed* if any semi-open chain cover of X has a member dense in X (or equivalently, if any semi-open chain cover has a finite subfamily with a dense union).

A topological space X is *Quasi H-closed (QHC)* [20] (respectively, *S-closed* [23]) if any open cover of X (respectively, semi-open cover of X) has a finite subfamily the union of closures of whose members covers

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X. A space X is *H-closed* [4, p-298] if X is closed in any Hausdorff space containing it as a subspace, which turns out to be equivalent to "any open cover of the space X has a finite subfamily whose union is dense in X". H-closed spaces are trivially linearly H-closed. A Hausdorff *QHC* space as well as a Hausdorff S-closed space is H-closed and therefore linearly H-closed. While regular S-closed spaces are compact there is a linearly S-closed, regular, non compact, extremally disconnected topological space [see Example 4.3]. In Section 2, we give several characterizations of linearly S-closed spaces and investigate basic properties of these spaces. It is pointed out that the new class of spaces lies between S-closed spaces and countably S-closed spaces. Some results on cardinal bounds are also established for the new class of spaces. While a first countable, compact space has cardinality at most continuum [1, Corollary 3.1.30]. We show that, first countable, Hausdorff, linearly S-closed space is finite. Moreover, in this section we deal with mappings of linearly S-closed spaces too. It is shown that the s-continuous image of a linearly S-closed space in any Hausdorff, first countable space is closed. In Section 3, we study and investigate the conditions for which the product is linearly S-closed. We observe that in general the product of linearly S-closed spaces need not be necessarily linearly S-closed. The product of a linearly H-closed space with a H-closed space is linearly H-closed [7, Proposition 2.18]. However, the product of a linearly S-closed space with a S-closed space may fail to be linearly S-closed. Finally in Section 4, we provide several examples to illustrate the results obtained in Section 2 and Section 3.

For a subset A of a space X, we denote the closure of A, interior of A in X by $cl(A)$ (or $\overline{A}$), $int(A)$, respectively. A subset A of a topological space X is,

(i) *semi-open* [18] if and only if there exist an open set $O \subset X$ such that $O \subset A \subset cl(O)$ or equivalently, $A \subset cl(int(A))$.
(ii) *regular semi-open* [8] if and only if there exists a regular open set $O \subset X$ such that $O \subset A \subset cl(O)$.
(iii) *regular open* if $A = int(cl(A))$.
(iv) *regular closed* if $A = cl(int(A))$.
(v) *semi-closed* if $A \supset int(cl(A))$.

In a space X, regular closed sets and regular semi-open sets are semi-open. The closure of a semi-open set as well as the closure of a regular open set is regular closed. The interior of a regular closed set is regular open. The complement of a regular open set, a semi-open set in a space X is regular closed, semi-closed, respectively. The family of semi-open, regular open, regular semi-open and regular closed subsets of X is denoted by $SO(X)$, $RO(X)$, $RSO(X)$ and $RC(X)$ respectively. $RO(X, \tau)$ is a base for a coarser topology τ_s on X known as *semi-regularization topology* on X. Clearly every open set is semi-open. However, a semi-open set may not be open. Union of semi-open sets is semi-open whereas intersection of semi-open sets may fail to be semi-open. Semi-closure and semi-interior of a subset of X is defined analogously to the closure and interior: the *semi-interior* of $A \subset X$ denoted by, $sInt(A)$ is the union of all semi-open sets contained in A and the *semi-closure* of $A \subset X$ (denoted as $sCl(A)$) is the intersection of all semi-closed sets containing A. For any subset $A \subset X$,

$$int(A) \subset sInt(A) \subset A \subset sCl(A) \subset cl(A).$$

The collection of regular closed (respectively, semi-open) subsets of X containing $x \in X$ is denoted by $RC(x)$ (respectively, $SO(x)$). For any $x \in X$, $RC(x) = \{\overline{V} : V \in SO(x)\}$. An *extremally disconnected* [2] (abbreviated as, e.d.) topological space is a topological space in which closure of every open set is open. A subset $A \subset X$ is called *locally dense* if $A \subset int(cl(A))$.

Following are some well known results.

Lemma 1.2. ([12, Lemma 1.1]) *Let (X, τ) be a topological space. Then:*

1. $RC(X, \tau) = RC(X, \tau_s)$.
2. *(X, τ) is extremally disconnected if and only if (X, τ_s) is extremally disconnected.*
3. *For a locally dense subset $A \subset X$*
 $RC(A, \tau|A) = \{F \cap A : F \in RC(X, \tau)\}$.

Note that, a finite topological space is trivially linearly S-closed hence throughout our discussion X is assumed to be an infinite space.

2. Linearly S-closed Spaces

Throughout the paper, spaces will mean the topological spaces on which no separation axiom is assumed unless explicitly stated. The following set-theoretic notions are adopted: κ, λ are regular cardinal numbers and α, β, γ are ordinal numbers. A cardinal number is the set of all ordinals that precede it. Thus $\alpha < \kappa$ and $\alpha \in \kappa$ are the same. Any chain cover posses a subcover indexed by a regular cardinal so that for simplicity we always use such indexing. We use ω for the smallest infinite ordinal and cardinal. ω_1 is the smallest uncountable ordinal and cardinal. The standard example in topology, frequently used in this paper is $\beta\omega$ which is the *Stone* – *Čech* compactification of the integers. We will begin with some basic properties of linearly S-closed spaces and provide some characterizations of the new class of spaces at a later stage. Our first lemma is very basic but plays a key role to establish the next implication diagram, that connects linearly S-closedness to various types of covering properties. The proof of the lemma is very similar to that of [7, Lemma 2.1].

Lemma 2.1. *A space X is linearly S-closed if and only if any infinite semi-open cover of X, has a subfamily of strictly smaller cardinality with a dense union.*

Proof. For any semi-open chain cover $\mathcal{U}$ of X the union of a subfamily of strictly smaller cardinality is contained in some member. Consequently, $\mathcal{U}$ has a member dense in the space X. Conversely, let X be a linearly S-closed space and $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be an infinite semi-open cover of X. For each $\alpha < \kappa$ the sets of the form $V_\alpha = \cup_{\beta<\alpha} U_\beta$ form a semi-open chain cover of X. Since X is linearly S-closed there exists some $\beta < \kappa$ such that V_β is dense in X. Consequently, $\mathcal{U}$ has a subfamily of strictly smaller cardinality with a dense union. □

Recall that a space X is *countably S-closed* [12] if and only if any countable regular closed cover of X has a finite subcover. This turns out to be equivalant to " any countable semi-open cover of X has a finite subfamily, the closures of whose members cover X". A topological space X is called *feebly compact* [4] if every locally finite collection of non empty open subsets of X is finite. This turns out to be equivalant to "any countable open cover has a finite subfamily with a dense union" [4, Theorem 1.11(b)]. The class of countably S-closed spaces lies between S-closed spaces and feebly compact spaces [12]. A *semi-compact* [13] (respectively, *semi-Lindelöf* [14]) space is a topological space for which any semi-open cover has a finite (respectively, countable) subcover. A *countably semi-compact* space (called *semi countably compact* by Dorsett in [13]) is a space for which any countable semi-open cover of X has a finite subcover.

Evidently, we have the following diagram:

$$\begin{tikzcd}
\text{semi-compact} \arrow[r] \arrow[d] & \text{S-closed} \arrow[r] \arrow[d] & \text{Linearly S-closed} \arrow[r] \arrow[d] & \text{Countably S-closed} \arrow[d] \\
\text{compact} \arrow[r] & \text{Quasi H-closed} \arrow[r] & \text{Linearly H-closed} \arrow[r] & \text{Feebly Compact}
\end{tikzcd}$$

Figure(1).

Note that the converses of these implications are false, however. We provide various examples in Section 4 to support our claim.

Example 2.2. A discrete space is linearly S-closed if and only if it is finite.

By a regular closed cover (respectively, regular semi-open cover) of a space X we mean a cover by regular closed sets (respectively, regular semi-open sets) in X. For any subset $A \subset X$, $int(cl(A))$ is regular open [3, p-29, 3(D)]. In our next result we present a variety of characterizations for linearly S-closed spaces.

Theorem 2.3. *For a space* (X, τ), *the following are equivalent:*

(a). (X, τ) *is linearly S-closed.*

(b). *Every regular closed cover of X has a subfamily of strictly smaller cardinality with a dense union.*
(c). *Every regular semi-open chain cover of X has a member dense in X.*
(d). *If* $\{F_\alpha : \alpha < \kappa\}$ *is a decreasing family of nonempty regular closed sets in X, then* $\bigcap\{int(F_\alpha) : \alpha < \kappa\} \neq \phi$.
(e). *If* $\{G_\alpha : \alpha < \kappa\}$ *is a decreasing family of nonempty regular open sets in X then* $\bigcap\{G_\alpha : \alpha < \kappa\} \neq \phi$.

Proof. a) ⇒ b) is trivial. Regular closed sets are semi-open, now use Lemma 2.1.

For b) ⇒ a) Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be any semi-open cover of X. Since closure of a semi-open set is regular closed, the collection $\mathcal{V} = \{cl(U_\alpha) : \alpha < \kappa\}$ form a regular closed cover of X. Therefore, $\mathcal{V}$ has a subfamily $\mathcal{V}' = \{cl(U_{\alpha(i)}) : \alpha(i) \in I \subset \kappa\}$ where $|I| < \kappa$ such that $\bigcup_{\alpha(i)\in I}(cl(U_{\alpha(i)}))$ is dense in X. This implies that $cl(\bigcup_{\alpha(i)\in I} U_{\alpha(i)}) \supset \bigcup_{\alpha(i)\in I}(cl(U_{\alpha(i)}))$ is dense in X. Since $(\bigcup_{\alpha(i)\in I} U_{\alpha(i)})$ is dense in $cl(\bigcup_{\alpha(i)\in I} U_{\alpha(i)})$ and $cl(\bigcup_{\alpha(i)\in I} U_{\alpha(i)})$ is dense in X. Therefore $(\bigcup_{\alpha(i)\in I} U_{\alpha(i)})$ where $|I| < \kappa$ is dense in X.

a) ⇒ c) is trivial. Regular semi-open sets are semi-open.

c) ⇒ a) Assume that the space X is not linearly S-closed. Then there exists a semi-open chain cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ having no member dense in X. Since $int(cl(U_\alpha)) \subset int(cl(U_\alpha)) \cup U_\alpha \subset cl(int(cl(U_\alpha)))$ for each $\alpha < \kappa$. Therefore, the collection $\{int(cl(U_\alpha)) \cup U_\alpha : \alpha < \kappa\}$ form a regular semi-open chain cover of X having no member dense in X.

a) ⇒ d) Let $\{F_\alpha : \alpha < \kappa\}$ be a decreasing family of nonempty regular closed sets in X such that $\bigcap\{int(F_\alpha) : \alpha < \kappa\} = \phi$, then the collection $\{X \setminus int(F_\alpha) : \alpha < \kappa\}$ form a semi-open chain cover of X and hence there exist a $\beta < \kappa$ such that $cl(X \setminus int(F_\beta)) = X$ which implies that $X \setminus int(F_\beta) = X$. Therefore, $int(F_\beta) = \phi$, a contradiction.

d) ⇒ e) Let $\{G_\alpha : \alpha < \kappa\}$ be a decreasing family of nonempty regular open sets in X then the collection $\{F_\alpha = cl(G_\alpha) : \alpha < \kappa\}$ be a decreasing family of nonempty regular closed sets in X and hence $\bigcap\{G_\alpha : \alpha < \kappa\} = \bigcap\{int(F_\alpha) : \alpha < \kappa\} \neq \phi$.

e) ⇒ a) Suppose to the contrary that, X is not linearly S-closed and $\{U_\alpha : \alpha < \kappa\}$ is a semi-open chain cover having no member dense in X. Then $\{X \setminus cl(U_\alpha) : \alpha < \kappa\}$ is a decreasing family of nonempty regular open sets such that $\bigcap\{X \setminus cl(U_\alpha) : \alpha < \kappa\} = \phi$, a contradiction. □

We now focus on some fundamental properties of linearly S-closed spaces. To begin with recall that, a topological property R is called *semi-regular* if, (X, τ) has property R if and only if (X, τ_s) has property R.

Proposition 2.4. *The property being linearly S-closed is semi-regular.*

Proof. Let (X, τ) be a linearly S-closed space. By using Lemma 1.2, any τ_s-regular closed cover of X is a τ-regular closed cover of X and hence has a subfamily of strictly smaller cardinality with a union dense in (X, τ). Since τ_s is coarser then τ, the union of the subfamily is dense in (X, τ_s) too. Hence by using Theorem 2.3(b), (X, τ_s) is linearly S-closed.

Conversely, suppose that (X, τ) is not linearly S-closed. Then there exist a decreasing family $\{F_\alpha : \alpha < \kappa\}$ of nonempty τ-regular closed sets in X such that $\bigcap\{int_\tau(F_\alpha) : \alpha < \kappa\} = \phi$. Since $int_{\tau_s}(F_\alpha) \subset int_\tau(F_\alpha)$ for each $\alpha < \kappa$, we have $\bigcap\{int_{\tau_s}(F_\alpha) : \alpha < \kappa\} = \phi$. Hence, by using Lemma 1.2 and Theorem 2.3(d), (X, τ_s) is not linearly S-closed. □

A topological space (X, τ) with a topological property R is *maximal* R if no topology τ' stronger than τ on X has property R. A *submaximal* space is a topological space in which every dense subset is open. If R is a semi-regular property then maximal R spaces are submaximal [9, Theorem 2]. Therefore,

Proposition 2.5. *A maximal linearly S-closed space is submaximal.*

A topological property P is said to be *contagious* if a space (X, τ) has property P whenever a dense subspace of (X, τ) has property P.

Proposition 2.6. *The property, being linearly S-closed is contagious.*

Proof. Let X be a topological space and D be a dense linearly S-closed subspace of X. For any semi-open chain cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of X, the family $\{U_\alpha \cap D : \alpha < \kappa\}$ is a semi-open chain cover of D and hence has a member dense in D. Since D is dense in X, the member is dense in X also. Consequently, $\mathcal{U}$ has a member dense in X. □

The property being linearly S-closed is not hereditary. Moreover, it is not hereditary with respect to open and dense or closed subsets of the space. For example $\omega \subset \beta\omega$ is open and dense in $\beta\omega$. Since $\beta\omega$ is S-closed [23, Corollary to Theorem 5] therefore it is linearly S-closed too. But ω is an infinite, countable, discrete space and hence can not be linearly S-closed. Also $\beta\omega \setminus \omega$ is closed in $\beta\omega$ but not linearly S-closed because it is not countably S-closed [12, Example 4.5]. Our next result is,

Proposition 2.7. *Let X be a linearly S-closed space. Then:*

(a) *A regular open subset of X is linearly S-closed.*
(b) *A regular closed subset of X is linearly S-closed.*
(c) *If $(A, \tau|A)$ is a linearly S-closed subspace of a topological space (X, τ) (X need not be linearly S-closed), and if $A \subset B \subset cl(A)$ then $(B, \tau|B)$ is linearly S-closed.*
(d) *The closure of a linearly S-closed subspace of a topological space is linearly S-closed.*
(e) *If O is an open set in X, then $cl(O)$ is linearly S-closed.*

Proof. (a) Let $A \subset X$ be a regular open subset in X and $\mathcal{G} = \{G_\alpha : \alpha < \kappa\}$ be any decreasing family of non empty regular open sets in A. Then $\mathcal{G}$ is a decreasing family of non empty regular open sets in X also. Since X is linearly S-closed, $\bigcap\{G_\alpha : \alpha < \kappa\} \neq \phi$. Hence, by using Theorem 2.3(e) A is linearly S-closed.

(b) Let F be a regular closed subset of X then $int(F)$ is regular open in X and dense in F. From (a), $int(F)$ is linearly S-closed. Hence, by using Proposition 2.6, F is linearly S-closed.

(c) By the hypothesis, A is a dense linearly S-closed subspace of $(B, \tau|B)$. Therefore, by using Proposition 2.6 $(B, \tau|B)$ is linearly S-closed.

(d) Clearly (d) follows by (c).

(e) In a topological space X, if O is open then $cl(O)$ is regular closed and hence the result follows by (b). □

Recall that, a mapping $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is called:

(a) *semi-continuous* [18] if the preimage of every open set in Y is semi-open in X.
(b) *s-continuous* if the preimage of every semi-open set in Y is open in X.
(c) *irresolute* [10] if the preimage of every semi-open set in Y is semi-open in X.
(d) *semi-homeomorphism* if f is an irresolute, bijection and image of a semi-open set in X is semi-open in Y.

In our next result we have explored the behavior of the new class of spaces under various types of mappings between topological spaces.

Proposition 2.8. *The following statements are true:*

(a) *An irresolute image of a linearly S-closed space is linearly S-closed.*
(b) *An s-continuous image of a linearly H-closed space is linearly S-closed.*
(c) *A semi-continuous (in particular continuous) image of a linearly S-closed space is linearly H-closed.*

Proof. The proof of each statement is trivial and that directly follows by applying the definitions of the mappings involved, thus omitted. □

Corollary 2.9. *An open, continuous image of a linearly S-closed space is linearly S-closed.*

Proof. Since open and continuous mappings are irresolute [10, Theorem 1.2]. Therefore, the result follows by Proposition 2.8(a). □

A property which is preserved by semi-homeomorphism is known as *semi-topological* property. Since every homeomorphism is a semi-homeomorphism as well, a semi-topological property is a topological property. Thus being linearly S-closed is a semi-topological property as well as a topological property. Recall that a continuous mapping $f : X \to Y$ is *perfect* [1, p-182] if X is Hausdorff, f is closed and all fibers $f^{\leftarrow}(y)$ of f are compact subsets of X. Compactness and local compactness are inverse invariants of perfect maps [1, Theorem 3.7.24, p-188]. Linearly H-closedness is inverse invariant of perfect, open maps [6, Theorem 2.4]. We have found a similar result for linearly S-closed spaces too. A subset $A \subset X$ is called *semi-compact subset* [22] relative to X if every cover of A by semi-open sets in X has a finite subcover. A mapping $f : X \to Y$ is called *s-perfect* [21] if image of each semi-closed set $A \subset X$ is semi-closed in Y and all fibers of f, $f^{\leftarrow}(y)$ are semi-compact relative to X. The proof of following theorem is very similar to that of [6, Theorem 2.4].

Theorem 2.10. *The property of being linearly S-closed space is an inverse invariant of s-perfect, open maps.*

Proof. Suppose that $f : X \to Y$ is a surjective, s-perfect, open map from a space X to a linearly S-closed space Y. Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be any semi-open chain cover of X, then for each α, $V_\alpha = Y \setminus f[X \setminus U_\alpha]$ is a semi-open set in Y. We claim that, the collection $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ form a semi-open chain cover of Y. To prove our claim, for any $y \in Y$ since the fiber of y under f, $\{f^{\leftarrow}(y)\}$ is semi-compact relative to X the chain cover $\mathcal{U}$ of $\{f^{\leftarrow}(y)\}$ by semi-open sets in X has a finite sub-cover. Hence there exists a $\gamma < \kappa$ such that $\{f^{\leftarrow}(y)\} \subset U_\alpha$ for all $\gamma < \alpha < \kappa$ which implies $y \notin f(X \setminus U_\alpha)$ which further implies that $y \in V_\alpha$ for each $\gamma < \alpha < \kappa$. Since Y is linearly S-closed the semi-open chain cover $\mathcal{V}$ of Y has a member dense in Y. Therefore there exist a $\beta < \kappa$ such that $V_\beta = Y \setminus f[X \setminus U_\beta]$ is dense in Y. Now we shall show that U_β is dense in X. Assume to contrary that U_β is not dense in X. Since f is open $f(X \setminus \overline{U_\beta})$ is a nonempty open subset of Y and is disjoint from V_β. This gives a contradiction. □

A point $x \in X$ is an *adherent point* of a filter (filter base) $\mathcal{F} = \{F_\alpha\}$ if for each open set $O \ni x$ and each $F_\alpha \in \mathcal{F}$, $F_\alpha \cap O \neq \phi$. A filter base $\mathcal{F} = \{F_\alpha\}$ *s-converges* [23] to a point $x \in X$ if for each $V \in SO(x)$ there exists a $F_\alpha \in \mathcal{F}$ such that $F_\alpha \subset cl(V)$ or equivalently, if for each $V \in RC(x)$ there exists a $F_\alpha \in \mathcal{F}$ such that $F_\alpha \subset V$. A filter base $\mathcal{F} = \{F_\alpha\}$ *s-accumulates* [23] to a point $x \in X$ (i.e. $x \in \theta\text{-}ad_s(\mathcal{F})$) if for each $V \in SO(x)$ and each $F_\alpha \in \mathcal{F}$, $F_\alpha \cap cl(V) \neq \phi$ or equivalently, if for each $V \in RC(x)$ and each $F_\alpha \in \mathcal{F}$, $F_\alpha \cap V \neq \phi$.

A space X is H-closed if and only if every open filter base (i.e. the filter base consisting of open sets only) on X has an adherent point in X [4, Proposition 4.8(b)]. A space X is S-closed if and only if every filter base on X s-accumulates to some point in X [23, Theorem 2]. A chain filter base is a filter base linearly ordered by set inclusion relation. An open chain filter base is a chain filter base consisting of open sets only. A space X is linearly H-closed if and only if any open chain filter base on X has an adherent point [7, Lemma 2.2]. We have found a similar characterization for linearly S-closed spaces as follows,

Theorem 2.11. *A space (X, τ) is linearly S-closed if and only if any open chain filter-base on X s-accumulate to a point in X.*

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha < \kappa\}$ be an open chain filter base on X that does not s-accumulate to any point in X. Then for each $x \in X$ there exists a $V(x) \in SO(x)$ and an $F_{\alpha(x)} \in \mathcal{F}$ such that $F_{\alpha(x)} \cap \overline{V(x)} = \phi$ which implies $x \notin sCl(F_{\alpha(x)})$. Hence $\bigcap\{sCl(F_\alpha) : \alpha < \kappa\} = \phi$. Therefore the collection $\{X \setminus sCl(F_\alpha) : \alpha < \kappa\}$ form a semi-open chain cover of X having no member dense in X. As if there exist a $\beta < \kappa$ such that $X \setminus sCl(F_\beta)$ is dense in X i.e. $cl(X \setminus sCl(F_\beta)) = X$ then $int(sCl(F_\beta)) = \phi$ which implies that $F_\beta = \phi$ because $F_\beta \subset int(sCl(F_\beta))$. This contradicts the necessary assumption $F_\beta \neq \phi$ to be a member of the filter base.

Conversely, suppose that X is not linearly S-closed and $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ is a semi-open chain cover of X having no member dense in X. Then $\mathcal{F} = \{X \setminus \overline{U_\alpha} : \alpha < \kappa\}$ is an open chain filter base on X and hence s-accumulate to a point $x_0 \in X$. But there exist a $\beta < \kappa$ such that x_0 is contained in a semi-open set U_β and $\overline{U_\beta} \cap X \setminus \overline{U_\beta} = \phi$ which contradicts the fact that $\mathcal{F}$ s-accumulate to x_0. □

Theorem 2.12. *A topological space X is linearly S-closed if and only if any open chain filter base on X, having at most one s-accumulation point, is s-convergent.*

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha < \kappa\}$ be an open chain filter base on X with at most one s-accumulation point. Since X is linearly S-closed, $\mathcal{F}$ has an s-accumulation point $x_0 \in X$ and by the assumption this s-accumulation point is unique. Our claim is that $\mathcal{F}$, s-converges to x_0. To prove our claim suppose that $\mathcal{F}$ does not s-converge to x_0. Then for some $V \in SO(x_0)$ and $\gamma \in \kappa$, $F_\alpha \not\subset cl(V)$ for all $\gamma < \alpha < \kappa$. This implies $F_\alpha \cap (X \setminus cl(V)) \neq \phi$ for all $\gamma < \alpha < \kappa$. Therefore, the collection $\mathcal{F}^* = \{F_\alpha \setminus cl(V) : F_\alpha \in \mathcal{F}\}$ form an open chain filter base in X which (by construction) does not s-accumulate to any point in V. This gives $\phi \neq \theta\text{-}ad_s(\mathcal{F}^*) \subset \{x_0\} \setminus V$, a contradiction.

Conversely, suppose that X is not linearly S-closed and $\mathcal{F} = \{F_\alpha : \alpha < \kappa\}$ is an open chain filter base on X having no s-accumulation point. By hypothesis $\mathcal{F}$, s-converges to some point $x_0 \in X$. Therefore, for any $V \in SO(x_0)$ there exists some $\beta_V < \kappa$ such that $F_{\beta_V} \subset cl(V)$. Hence, for each $F_\alpha \in \mathcal{F}$; $F_\alpha \cap cl(V) \neq \phi$. Consequently, $\mathcal{F}$ s-accumulates to $x_0 \in X$, a contradiction. □

Corollary 2.13. *An open chain filter base with unique s-accumulation point in a linearly S-closed space is s-convergent.*

Linearly S-closedness is linked to other compact like covering properties as shown in Figure 1. We are now interested to find under what conditions the reverse implications follows too. We will begin with the following result.

Theorem 2.14. *An extremally disconnected topological space is linearly S-closed if and only if it is linearly H-closed.*

Proof. A linearly S-closed space is linearly H-closed follows directly by definition.

For the reverse implication, if X is extremally disconnected then closure of an open set in X is open. Also the interior of a semi-open set is dense in it. Given a semi-open chain cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of a linearly H-closed space X, the collection $\mathcal{V} = \{cl(int(U_\alpha)) : U_\alpha \in \mathcal{U}, \alpha < \kappa\}$ form an open chain cover of X and hence posses a member dense in X. Consequently, $\mathcal{U}$ has a member dense in X. □

Following is a corollary to [8, Corollary 1]

Corollary 2.15. *An extremally disconnected QHC space is linearly S-closed.*

Proposition 2.16. *A linearly S-closed, semi-Lindelöf space is S-closed.*

Proof. Given a semi-open cover $\mathcal{U}$ of a linearly S-closed space X, semi-*Lindelöfness* of X implies that $\mathcal{U}$ has a countable subcover. Since X is linearly S-closed, the subcover has a finite subfamily the closures of whose members covers X. □

Proposition 2.17. *A countably S-closed, regular, Lindelöf space is compact.*

Proof. Let X be a countably S-closed, regular, *Lindelöf* space and $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be an open cover of X. For each $x \in U_\alpha \in \mathcal{U}$ there exist an open set $V_\alpha(x)$ such that $x \in V_\alpha(x) \subset cl(V_\alpha(x)) \subset U_\alpha$. The collection $\mathcal{V} = \{V_\alpha(x) : x \in X\}$ form an open cover of X and hence has a countable subcover. Moreover, X is countably S-closed therefore the countable subcover has a finite subfamily the closures of whose members covers X. Consequently, $\mathcal{U}$ has a finite subcover. □

Corollary 2.18. *A countably S-closed, regular, semi-Lindelöf space is compact.*

Corollary 2.19. *A linearly S-closed, regular, Lindelöf space is compact.*

Remark 2.20. There exist a regular, linearly S-closed space which is non compact (see for instance, [Example 4.3]). Therefore, the hypothesis of being *Lindelöf* can not be entirely dropped in the above corollary.

Corollary 2.21. *A regular, linearly S-closed space of countable cardinality is compact.*

A topological space X is called *s-regular* if for each semi-open set $U \subset X$ and each $x \in U$ there exist a semi-open set $V \subset X$ such that $x \in V \subset cl(V) \subset U$.

Proposition 2.22. *A s-regular, semi-Lindelöf, countably S-closed space is semi-compact.*

Proof. Let X be a s-regular semi-*Lindelöf*, countably S-closed space and $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be a semi-open cover of X, then for each $x \in U_\alpha$, $\alpha < \kappa$ there exists a semi-open set $V_{\alpha,x}$ such that $x \in V_{\alpha,x} \subset cl(V_{\alpha,x}) \subset U_\alpha$. The collection $\mathcal{V} = \{V_{\alpha,x} : \alpha < \kappa\}$ form a semi-open cover of X. Since X is semi-*Lindelöf*, $\mathcal{V}$ has a countable subcover. Countably S-closedness of X implies that the subcover has a finite subfamily, the closures of whose members covers X. Consequently, $\mathcal{U}$ has a finite subcover. □

Corollary 2.23. *A s-regular, semi-Lindelöf, linearly S-closed space is semi-compact.*

A topological space X is *locally linearly S-closed* (respectively, *locally S-closed* [25]) if each point of X has a linearly S-closed (respectively, S-closed) open neighbourhood. Clearly a locally S-closed space is locally linearly S-closed. A linearly S-closed space is locally linearly S-closed but the converse is not true. An infinite set with discrete topology is locally linearly S-closed but not linearly S-closed. In our next result we provide a necessary and sufficient condition for a locally linearly S-closed space to be linearly S-closed. In order to prove our next theorem we need a lemma which is stated as follows:

Lemma 2.24. *If a space X is a finite union of open linearly S-closed subspaces then it is linearly S-closed.*

Proof. Suppose that $X = Y_1 \cup Y_2 \cup Y_3 .... \cup Y_n$ where each Y_i, $i = 1, 2, ..., n$; $n \in \mathbb{N}$ is an open linearly S-closed space in its subspace topology. Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be a semi-open chain cover (of infinite regular cardinality κ) of X. Then for each $1 \leq i \leq n$ the family $\mathcal{U}_i = \{U_\alpha \cap Y_i : U_\alpha \in \mathcal{U}, \alpha < \kappa\}$ form a semi-open chain cover of Y_i in its subspace topology. Therefore, for each $1 \leq i \leq n$ the semi-open chain cover $\mathcal{U}_i$ has a finite subfamily $\mathcal{V}_i$ whose union is dense in Y_i. Further suppose that for each $1 \leq i \leq n$, $J_i = \{\alpha < \kappa : U_\alpha \cap Y_i \in \mathcal{V}_i\}$. Clearly for each i, $|J_i| < \omega$ and $Y_i = cl(\cup_{\alpha \in J_i}(U_\alpha \cap Y_i))$. This implies that $X = \bigcup_{i=1}^{n} Y_i = \bigcup_{i=1}^{n} cl(\cup_{\alpha \in J_i}(U_\alpha \cap Y_i)) \subset \bigcup_{i=1}^{n} cl(\cup_{\alpha \in J_i}(U_\alpha)) \subset cl(\bigcup_{i=1}^{n}(\cup_{\alpha \in J_i} U_\alpha))$. Let $J = \bigcup_{i=1}^{n} J_i$. Since $|J| < \omega$ and $X = cl(\bigcup_{\alpha \in J} U_\alpha)$, the space X is linearly S-closed. □

Remark 2.25. If X is a countable union of linearly S-closed subspaces then X need not be linearly S-closed. A countable discrete space X is a countable union of singleton subspaces which are linearly S-closed in their subspace topology. But X is not linearly S-closed.

Corollary 2.26. *A topological space (X, τ) is linearly S-closed if it is the union of a finite number of clopen linearly S-closed subspaces of X.*

Corollary 2.27. *A finite sum of linearly S-closed spaces is linearly S-closed.*

Theorem 2.28. *A QHC space is linearly S-closed if and only if locally linearly S-closed.*

Proof. The necessary part is trivially true.

To prove sufficient part, since X is locally linearly S-closed, for each $x \in X$ there exist an open neighbourhood U_x of x which is linearly S-closed. Then the collection $\mathcal{U} = \{U_x : x \in X\}$ form an open cover of X. Since X is QHC the open cover $\mathcal{U}$ has a finite subfamily $\{U_{x_i} : 1 \leq i \leq n\}$ such that $X = \bigcup_{i=1}^{n} cl(U_{x_i}) = cl(\bigcup_{i=1}^{n} U_{x_i})$. By Lemma 2.24, $\bigcup_{i=1}^{n} U_{x_i}$ is linearly S-closed and hence by Proposition 2.7(d) $X = cl(\bigcup_{i=1}^{n} U_{x_i})$ is linearly S-closed. □

Corollary 2.29. *A locally S-closed, QHC space is linearly S-closed.*

Proposition 2.30. *Let X be a topological space and suppose that $X = \bigcup_{i=1}^{n} Y_i$ where each Y_i is locally dense, linearly S-closed subspace of X. Then X is linearly S-closed.*

Proof. Use Theorem 2.3 and Lemma 1.2, the proof is quite similar to that of Lemma 2.24 and hence will not be given here. □

Proposition 2.31. *A space (X, τ) is linearly S-closed if every dense subspace is QHC.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be a semi-open chain cover of X. Then $D = \bigcup_{\alpha<\kappa}(int(U_\alpha))$ is dense in X and hence QHC. Thus the open cover $\{int(U_\alpha) : \alpha < \kappa\}$ of D has a finite subfamily, the closures of whose members cover D. Consequently, $\mathcal{U}$ has a finite subfamily, the closures of whose members cover X. □

A topological space X is called *lob-space* [11] if every point of X has a linearly ordered local base (linearly ordered by reverse subset inclusion). A topological space is extremally disconnected if and only if every regular open set is clopen. An extremally disconnected, compact space is S-closed [23, Theorem 5] and hence linearly S-closed. In [23] Thompson proves that a regular S-closed space is extremally disconnected. A similar result holds for linearly S-closed spaces too.

Theorem 2.32. *If X is a Hausdorff, linearly S-closed, lob-space then X is extremally disconnected.*

Proof. Suppose that X is not extremally disconnected, then there exists a regular open set $O \subset X$ such that $(cl(O) \setminus O)$ *and* $(X \setminus cl(O))$ both are non-empty. Let $x_0 \in (cl(O) \setminus O)$, then for every neighborhood V of x_0, $V \cap O \neq \phi$. Suppose that $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ is an open neighborhood chain filter-base at x_0 in X. Then the family $\mathcal{U} = \{V_\alpha \cap O : \alpha < \kappa\}$ form an open chain filter base in O. By using Proposition 2.7(a), O is linearly S-closed and hence the chain filter base $\mathcal{U}$ s-accumulates to some point $p \in O$. Since $p \neq x_0$ and X is Hausdorff, there exist disjoint open sets $W_0 \ni x_0$ and $W_1 \ni p$ in X. By construction, the filter base $\mathcal{V}$ converges to x_0. Therefore, there exist a $\gamma < \kappa$ such that $(V_\alpha \cap O) \subset W_0$, for all $\gamma < \alpha < \kappa$, which implies that $(V_\alpha \cap O) \cap cl(W_1) = \phi$, for all $\gamma < \alpha < \kappa$, which contradicts the fact that $\mathcal{U}$ s-accumulates to p. □

Corollary 2.33. *A regular, linearly S-closed, lob space is extremally disconnected.*

Corollary 2.34. *Regular, linearly S-closed, lob space is zero-dimensional.*

Proof. Extremally disconnected, regular spaces are zero-dimensional [2, p-303]. □

Since first countable spaces are lob spaces. We have,

Corollary 2.35. *A first countable, Hausdorff, linearly S-closed space is extremally disconnected.*

A compact, first countable space has cardinality at most continuum [1, p-132]. Thompson in [23, Theorem 3] shows that S-closed, first countable, regular spaces are finite. In our next results we tried to find some bounds on the cardinality of linearly S-closed spaces. We will begin with the following result,

Theorem 2.36. *A first countable, Hausdorff, linearly S-closed space is finite always.*

Proof. First countable, Hausdorff, extremally disconnected space is discrete [2, p-301] and hence the theorem follows by using Corollary 2.35. □

Remark 2.37. However, the result in Theorem 2.36 is false for non Hausdorff spaces because a countably infinite space with co-finite topology is an extremally disconnected, non Hausdorff, second countable, linearly S-closed space. Also the assumption of first countability can not be entirely dropped here as $\beta\omega$ is an infinite Hausdorff, linearly S-closed space which is not first countable.

Corollary 2.38. *Linearly S-closed, metrizable space is finite.*

Proof. Metrizable spaces are first countable and regular always. □

Corollary 2.39. *Each infinite, linearly S-closed, regular, Lindelöf space is uncountable.*

Proof. By using Corollary 2.19, an infinite, linearly S-closed, regular, *Lindelöf* space is compact and a compact, regular space of countable cardinality is metrizable. The general result follows by Corollary 2.38. □

Corollary 2.40. *First countable, compact, linearly S-closed spaces are finite.*

Theorem 2.41. *If X is a first countable, regular, compact space. Then the following are equivalent:*

(a). *X is S-closed.*
(b). *X is extremally disconnected.*
(c). *X is linearly S-closed.*
(d). *X is finite.*

Proof. (a)↔ (b) is followed by [23, Theorem 7].
(a)→ (c) is obvious.
(c)→ (b) is followed by Corollary 2.35.
(c)→ (d) is followed by Theorem 2.36.
(d)→ (c) is trivial. □

Thompson in [24] gave a characterization of S-closed spaces using the class of irresolute functions. A Hausdorff space X is S-closed if and only if the irresolute image of X in any Hausdorff space is closed.

Theorem 2.42. *An s-continuous image of a linearly H-closed space in any Hausdorff, first countable space is closed.*

Proof. By using Proposition 2.8(b) and Theorem 2.36, the s-continuous image of a linearly H-closed space in any Hausdorff, first countable space is finite and hence closed. □

Theorem 2.43. *An irresolute image of a linearly S-closed space in any Hausdorff, first countable space is closed.*

Proof. By using Proposition 2.8(a) and Theorem 2.36, the irresolute image of a linearly S-closed space in any Hausdorff, first countable space is finite and hence closed. □

We have already seen in Figure(1) that linearly S-closed → countably S-closed. We are now interested to find a set of conditions under which the reverse implication is true. For the purpose, a set of conditions is given in the following Lemma. To state the lemma we need some more definitions that are stated here. Given an infinite regular cardinal κ, a space X is *initially κ-semi Lindelöf (respectively, initially κ-linearly semi Lindelöf)* if and only if any semi-open cover (respectively, semi-open chain cover) of cardinality $\leq \kappa$ has a countable sub-cover. Note that every space is initially ω-semi *Lindelöf* as well as initially ω-linearly semi *Lindelöf*. The *weak semi Lindelöf* number $wsL(X)$ of a space X is the least cardinal κ such that any semi-open cover of X has a subfamily of cardinality $\leq \kappa$ whose union is dense in X. The *weak Lindelöf* number [7] (denoted as, $wL(X)$) of a space X is the least cardinal κ such that any open cover of X has a subfamily of cardinality $\leq \kappa$ whose union is dense in X. By definition, $wL(X) \leq wsL(X)$. A topological space for which $wsL(X) = \omega$ is known as *weakly semi-Lindelöf* space. Note that if $Y \subset X$ is dense in X, then $wsL(X) \leq wsL(Y)$ and if Y is countably S-closed then so is X. A similar discussion about linearly H-closed spaces can be found in [7].

Lemma 2.44. *Let $Y \subset X$ be dense in X, and κ be an infinite cardinal. Assume as well that $wsL(X) \leq \kappa$, Y is both initially κ-linearly semi Lindelöf and countably S-closed. Then X is linearly S-closed.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha < \lambda\}$ (where λ is a regular cardinal) be an infinite semi-open chain cover of X then the collection $\mathfrak{U}_Y = \{U_\alpha \cap Y : \alpha < \lambda\}$ form a semi-open chain cover of Y. Assume first that $\lambda \leq \kappa$. Since Y is initially κ-linearly semi *Lindelöf*, $\mathfrak{U}_Y$ has a countable subcover, which has a finite subfamily, the closures of whose members cover Y (being countably S-closed). Consequently, $\mathcal{U}$ has a member dense in X. Now suppose $\lambda > \kappa$, since $wsL(X) \leq \kappa$ the semi-open chain cover $\mathcal{U}$ has a subfamily of cardinality $\leq \kappa < \lambda$ whose union is dense in X and by regularity of λ this union is contained in some U_α. □

Proposition 2.45. *A countably S-closed, semi-Lindelöf space is linearly S-closed.*

Proof. Let X be a countably S-closed, semi-*Lindelöf* space. Given a semi-open chain cover of X, semi-*Lindelöfness* gives a countable subcover and countable S-closedness gives a finite subfamily of the subcover, the closures of whose members cover X. □

Corollary 2.46. *A space containing a dense countably S-closed, semi-Lindelöf space is linearly S-closed.*

A family of pairwise disjoint nonempty open sets in a topological space X is known as *cellular family*. The *cellularity* of a space X (denoted by $c(X)$) is defined as the supremum of the cardinalities of the cellular families in X. A space X satisfies the *countable chain condition* (abbreviated as CCC) (respectively, *finite chain condition* (abbreviated as FCC)) if and only if the cellularity of X is at most ω (respectively, finite). A semi-compact space, semi-Lindelöf space always satisfies FCC and CCC, respectively [15, Proposition 2.2]. A space X is countably S-closed if and only if for any countable decreasing family of nonempty regular open sets in X, the intersection of of the members of the family is nonempty [12, Theorem 2.2(6)].

Proposition 2.47. *A countably S-closed space satisfying countable chain condition is linearly S-closed.*

Proof. Assume the contrary, that X is not linearly S-closed. Since X is countably S-closed. It has a decreasing family of regular open sets $\{B_\alpha : \alpha < \kappa\}$ of uncountable regular cardinality κ such that $\bigcap\{B_\alpha : \alpha < \kappa\} = \phi$. We may assume that $B_{\alpha+1}$ is strictly contained in B_α, then the family $\{(B_\alpha \setminus B_{\alpha+1}) : \alpha < \kappa\}$ form an uncountable family of mutually disjoint non-empty semi-open sets in X. This implies $\{int(B_\alpha \setminus B_{\alpha+1}) : \alpha < \kappa\}$ is an uncountable family of mutually disjoint nonempty open sets in X, a contradiction. □

Corollary 2.48. *A CCC, feebly compact, extremally disconnected space is linearly S-closed.*

Proof. Feebly compact, extremally disconnected spaces are countably S-closed. □

Corollary 2.49. *A separable, countably S-closed space is linearly S-closed.*

Recall that a point $x \in X$ is called a *complete accumulation point* of a subset $A \subset X$ if for each neighbourhood U of x, $|A \cap U| = |U|$. In [17] the authors says that, a point $x \in X$ is a *complete accumulation point of a family $\mathcal{O}$* of regular infinite cardinality κ of nonempty open subsets of X if for each neighbourhood V of x, $|\{O \in \mathcal{O} : O \cap V \neq \phi\}| = \kappa$. A topological space X is *weakly linearly Lindelöf* [17] if and only if any family of non empty open subsets of regular uncountable cardinality in X has a complete accumulation point in X. A space X is linearly H-closed if and only if any family of regular infinite cardinality of mutually disjoint non-empty open sets in X has a complete accumulation point in X [6, Theorem 2.11]. This discussion motivates us to give a new characterization for the class of spaces we introduced. To state the characterization, we need a definition, stated as follows:

Definition 2.50. A point $x \in X$ is called a *complete s-accumulation point* of a family $\mathcal{A}$ of regular infinite cardinality κ of subsets of X if for each $V \in SO(x)$, $|\{A \in \mathcal{A} : A \cap cl(V) \neq \phi\}| = \kappa$.

Recall that, a point $x \in X$ is in *θ-semiclosure* of $A \subset X$ (i.e. $x \in \theta\text{-}cl_s(A)$) if and only if for each $V \in SO(x)$, $cl(V) \cap A \neq \phi$. A is *θ-semiclosed* if and only if $A = \theta\text{-}cl_s(A)$. Following is a well known result,

Lemma 2.51. ([16, Corollary 1]) *A regular open subset of a space X is θ-semi closed.*

Theorem 2.52. *A space X is linearly S-closed if and only if every family of regular infinite cardinality of mutually disjoint nonempty open subsets of X has a complete s-accumulation point in X.*

Proof. Let X be a linearly S-closed space and $\mathcal{O} = \{O_\alpha : \alpha < \kappa\}$ be any family of mutually disjoint nonempty open subsets of X, where κ is a regular (infinite) cardinal. Let $B_\alpha = \bigcup\{O_\beta : \alpha \leq \beta < \kappa\}$ then the family $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ is an open chain filter base on X. Since X is linearly S-closed, the chain filter base $\mathcal{B}$ has an s-accumulation point $x_0 \in X$. Clearly for each $V \in SO(x_0)$, $cl(V)$ meets κ-many elements of $\mathcal{O}$ and hence x_0 is a complete s-accumulation point of $\mathcal{O}$.

Conversely, assume that X is not linearly S-closed. Then there exist a decreasing family of nonempty regular open sets $\{U_\alpha : \alpha < \kappa\}$ of regular infinite cardinality κ such that $\bigcap\{U_\alpha : \alpha < \kappa\} = \phi$. This implies that for any $x \in X$ there exists some $\beta(x) < \kappa$ such that $x \notin U_{\beta(x)}$. Consequently from Lemma 2.51, $x \notin \theta\text{-}cl_s(U_{\beta(x)})$ and hence $x \notin \theta\text{-}cl_s(U_\alpha \setminus (U_{\alpha+1}))$ for each $\alpha > \beta(x)$. Since κ-many sets of the form $(U_\alpha \setminus U_{\alpha+1})$ are non empty, the collection $\{(U_\alpha \setminus U_{\alpha+1}) : \alpha < \kappa\}$ is a mutually disjoint family of regular infinite cardinality κ of nonempty semi-open sets in X having no complete s-accumulation point in X. Therefore the collection $\{int(U_\alpha \setminus U_{\alpha+1}) : \alpha < \kappa\}$ is a mutually disjoint family of regular infinite cardinality κ of nonempty open sets of X which has no complete s-accumulation point in X. $\square$

3. Product

The product of a linearly S-closed space and a S-closed space need not be linearly S-closed. Moreover, linearly S-closedness is not a productive property, as shown in the following example.

Example 3.1. From [23] $\beta\omega$ is a regular, compact, extremally disconnected, S-closed space, therefore linearly S-closed. However $\beta\omega \times \beta\omega$ is not countably S-closed [12, Example 4.4] and hence not linearly S-closed.

Unlike for linearly S-closed spaces, the product of a linearly H-closed space and a H-closed space is linearly H-closed [6, Theorem 4.1]. From Example 3.1, we see that, in general the product of a compact space and a linearly S-closed space is not linearly S-closed. This consideration leads us to investigate the conditions under which the product is linearly S-closed. We will begin with the following result,

Theorem 3.2. *If X is a separable space, Y is linearly S-closed and $X \times Y$ is countably S-closed then $X \times Y$ is linearly S-closed.*

Proof. Suppose to the contrary $X \times Y$ is not linearly S-closed and that $D = \{d_n : n \in \omega\}$ is a countable dense subset of X. Since $X \times Y$ is countably S-closed there is a decreasing family $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ (of uncountable regular cardinality κ) of regular open sets in $X \times Y$, such that $\bigcap \mathcal{B} = \phi$. Since D is dense in X, $\pi_X(B_\alpha) \cap D \neq \phi$, for all $\alpha < \kappa$. Thus for each $\alpha < \kappa$, B_α must intersect at least one of the subspaces $\{d_m\} \times Y$ of $X \times Y$ where $d_m \in D$. For any $n \in \omega$ let us denote $I_n = \{\alpha < \kappa : B_\alpha \cap (\{d_n\} \times Y) \neq \phi\}$. Since κ is a regular cardinal $I_m \subset \kappa$ is cofinal in κ for some $m \in \omega$ and hence the collection $\{B_\alpha \cap (\{d_m\} \times Y) : \alpha \in I_m\}$ form a decreasing family of uncountable regular cardinality of nonempty regular open sets in $\{d_m\} \times Y$ such that $\bigcap\{B_\alpha \cap (\{d_m\} \times Y) : \alpha \in I_m\} = \phi$, which implies that $\{d_m\} \times Y$ is not linearly S-closed. Since $\{d_m\} \times Y$ is homeomorphic to Y therefore, Y is not linearly S-closed, a contradiction. $\square$

Corollary 3.3. *If X is separable, Y is linearly S-closed and $X \times Y$ is a feebly compact, extremally disconnected space then $X \times Y$ is linearly S-closed.*

Theorem 3.4. *If $(\Pi_{\mathfrak{U}} X_\alpha, \Pi_{\mathfrak{U}} \tau_\alpha)$ ($\mathfrak{U}$ is an index set) is linearly S-closed then (X_α, τ_α) is linearly S-closed for each $\alpha \in \mathfrak{U}$.*

Proof. Let $\{U_{\beta_\alpha} : \beta_\alpha \in \Lambda\}$, $\alpha \in \mathfrak{U}$ where Λ is an index set, be a semi-open chain cover of (X_α, τ_α), then $\{\pi^{-1}(U_{\beta_\alpha}) : \beta_\alpha \in \Lambda\}$ is a semi-open chain cover of $\Pi_{\mathfrak{U}} X_\alpha$. Thus there is a $\beta_{\alpha(k)} \in \Lambda$ such that $\pi^{-1}(U_{\beta_\alpha(k)})$ is dense in $\Pi_{\mathfrak{U}} X_\alpha$. Since $cl_{\Pi_{\mathfrak{U}} \tau_\alpha}(\Pi_{\mathfrak{U}} A_\alpha) = \Pi_{\mathfrak{U}} cl_{\tau_\alpha}(A_\alpha)$ we have $U_{\beta_\alpha(k)}$ is dense in X_α. $\square$

4. Examples

Example 4.1. The one point compactification of a discrete space of infinite cardinality is not countably S-closed [12, Example 4.3(iii)] and hence not linearly S-closed.

Example 4.2. The *Katětov* expansion $k\omega$ [4, see p-311, and p-450] of positive integers is an example of a non compact, extremally disconnected, space which is S-closed and hence linearly S-closed.

Example 4.3. The space $X = \beta\omega \setminus \{p\}$, where $p \in \beta\omega \setminus \omega$ is a regular, countably compact thus feebly compact, non compact, extremally disconnected, separable (hence CCC), countably S-closed space [12, Example 4.1]. Since a CCC, feebly compact space is linearly H-closed [6, Corollary 2.7]. Hence, by using Theorem 2.14, X is a linearly S-closed space which is not S-closed [23, Corollary].

Example 4.4. For any discrete space D of infinite cardinality, its *Stone − Čech* compactification βD is a compact, Hausdorff, extremally disconnected, linearly H-closed space and therefore by using Theorem 2.14, βD is linearly S-closed.

Example 4.5. ([12, Example 4.2])Suppose that (Y, τ) is a topological space such that $Y \setminus \{p\}$ is a countably S-closed subspace for some non-isolated point $p \in Y$. Let Y_1 and Y_2 denote two disjoint copies of $Y \setminus \{p\}$. For any subset $A \subset Y$ denote the corresponding subsets of Y_1 and Y_2 by A_1 and A_2, respectively. Let $X = Y_1 \cup Y_2 \cup \{p\}$. Define a topology σ on X in the following way. For any $x \in X$, if $x \in Y_1$ ($x \in Y_2$, respectively) then the basic open neighbourhoods of x in σ are of the form U_1 (U_2, respectively) where U is an open subset of $Y \setminus \{p\}$ and the basic open neighbourhoods of p in σ are of the form $(V \setminus \{p\})_1 \cup (V \setminus \{p\})_2 \cup \{p\}$ where V is an open neighbourhood p in (Y, τ). It is easy to see that Y_1 and Y_2 are regular open sets in (X, σ) and homeomorphic to $Y \setminus \{p\}$. Since neither of Y_1 and Y_2 is closed in (X, σ) it is not extremally disconnected. Although it is countably S-closed.

In particular, if we take $Y = \beta\omega$, $p \in \beta\omega \setminus \omega$. Then the resulting space is a Hausdorff, separable, compact, countably S-closed space. It is not S-closed because a Hausdorff S-closed space is extremally disconnected [23, Theorem 7]. By using Corollary 2.49, X is linearly S-closed.

Example 4.6. Following are the examples of non compact, linearly H-closed spaces, which are not linearly S-closed, due to Theorem 2.36.

1. Bell [5, Example 1] constructed an infinite, first countable, countably compact, Tychonoff space which is linearly H-closed [7, Example 2.9].
2. The space Ψ due to J. Isbell and S. Mrówka [19] is an infinite, first countable, locally compact, Tychonoff space and shown to be linearly H-closed in [7, Example 2.9].

Example 4.7. ω_1 the space of all countable ordinals is an infinite, first countable, regular, countably compact, thus feebly compact space. Since first countable, Hausdorff, linearly S-closed spaces are finit always, ω_1 is not linearly S-closed. Moreover, the 'canonical" cover by initial segments α ($\alpha < \omega_1$) is a semi-open chain cover of ω_1 having no member dense in it.

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