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L-Convex Quasi-Uniform Spaces

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Abstract. In this paper, L-convex β^*-remotehood system is introduced and some characterizations of both L-convexity and L-convex remotehood system are obtained. Further, β^*-remote mapping is presented and some of its properties are investigated. Based on this, L-convex quasi-uniformity and L-convex quasi-uniformity preserving mapping are introduced. It is proved that L-convex quasi-uniform space and L-convex space are mutually induced. In addition, the category of L-convex spaces and the category of L-convex β^*-remotehood spaces can be embedded into the category of L-convex quasi-uniform spaces.

1. Introduction

In an abstract convex space, a convex structure on a nonempty set is a family of subsets containing the empty set and the largest set and is closed under arbitrary intersections and nested unions. Its theory is called the abstract convex theory which involves many mathematical structures such as lattice, graph, median algebra, metric space, poset and vector space [16].

Convex structure has been extended into fuzzy settings by many ways. Maruyama introduced L-convex structure [3] which has being studied by many scholars [4–6, 24, 31, 34]. Also, Shi and Xiu introduced M-fuzzifying convex structures [13]. Many subsequent studies have been done [7, 17, 22, 23]. Further, Shi and Xiu introduced (L, M)-fuzzy convex structure which is a unified form of L-convex structure and M-fuzzifying convex structure [14]. It characterizations have been studied recently [19, 20]. Now, these fuzzy forms of convex structures have being applied to many fuzzy mathematical structures such as fuzzy topology [2, 17, 18, 20, 25], fuzzy convergence [6, 7, 32, 33] and fuzzy matroid [21, 27].

Uniformity is a topology-like concept which is a convenient tool in interpreting topology. In fuzzy settings, Hutton introduced fuzzy quasi-uniformities by fuzzy uniform operators [1]. Ying introduced M-fuzzifying uniformities and studied relations between M-fuzzifying uniformities and M-fuzzifying topologies [28]. Rodabaugh presented the axiomatic foundations for quasi-uniformities in fuzzy real lines and some other fuzzy settings [8, 9]. Zhang gave a comparison of various uniformities [30]. Shi established the theory of quasi-uniformities in completely distributive lattices and fuzzy sets [10, 11]. Yue extended Shi's quasi-uniformity in a Kubiak-Šostak sense and showed that the category of fuzzy topological spaces can be embedded into the category of fuzzy extension of Shi's quasi-uniform spaces [15, 29].

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As mentioned above, quasi-uniformity is a topology-like structure which is used to interpret topology. Then, is there any convex-like quasi-uniformity which can be used to interpret convex structure? Motivated by this, we present this paper. The arrangement of this paper is as follows. In Section 2, we recall some basic concepts, denotations and results related to L-convex space. In Section 3, we introduce L-convex β^*-remotehood spaces and characterize both L-convex spaces and L-convex remotehood spaces. In section 4, we introduce L-convex quasi-uniformity and study its relations with L-convex space and L-convex β^*-remotehood space. In section 5, we introduce L-convex quasi-uniformity preserving mapping. We find that the category of L-convex spaces and the category of L-convex β^*-remotehood spaces can be embedded into the category of L-convex quasi-uniform spaces.

2. Preliminaries

In this paper, X and Y are nonempty sets. L is a completely distributive lattice. The smallest (resp. largest) element in L is denoted by $\perp$ (resp. $\top$). An element $a \in L$ is called a co-prime, if for all $b, c \in L$, $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. The set of all co-primes in $L \backslash \{\perp\}$ is denoted by $J(L)$. For any $a \in L$, there is an $L_1 \subseteq J(L)$ such that $a = \bigvee_{b \in L_1} b$. A binary relation $\prec$ on L is defined by $a \prec b$ iff for each $L_1 \subseteq L$, $b \leq \bigvee L_1$ implies some $d \in L_1$ with $a \leq d$. The mapping $\beta : L \to 2^L$, defined by $\beta(a) = \{b \in L : b \prec a\}$, satisfies $\beta(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} \beta(a_i)$ for any $\{a_i\}_{i \in I} \subseteq L$. For any $a \in L$, we denote $\beta^*(a) = \beta(a) \cap J(L)$. It is proved that $a = \bigvee \beta(a) = \bigvee \beta^*(a)$, $\beta(a) = \bigcup_{b \in \beta^*(a)} \beta(b)$ and $\beta^*(a) = \bigcup_{b \in \beta^*(a)} \beta^*(b)$ [12].

An L-fuzzy set on X is a mapping $A : X \to L$. The set of all L-fuzzy sets on X is denoted by L^X. The smallest (resp, largest) element in L^X is denoted by $\underline{\perp}$ (resp. $\underline{\top}$). A subset $\{A_i\}_{i \in I} \subseteq L^X$ is called a directed set, denoted by $\{A_i\}_{i \in I}^{dir} \subseteq L^X$, if any pair of indices $i, j \in I$ implies a $k \in I$ such that $A_i \vee A_j \leq A_k$. In this case, we denote $\bigvee_{i \in I} A_i$ by $\bigvee_{i \in I}^{dir} A_i$. An L-fuzzy point x_λ ($\lambda \in L \backslash \{\perp\}$) is an L-fuzzy set defined by $x_\lambda(x) = \lambda$ and $x_\lambda(y) = \perp$ for any $y \in X \backslash \{x\}$. The set of all L-fuzzy points on L^X is denoted by $Pt(L^X)$. Also, we denote $J(L^X) = \{x_\lambda \in Pt(L^X) : \lambda \in J(L)\}$ and $\beta^*(L^X) = \{x_\lambda \in Pt(L^X) : \lambda \in \beta^*(L)\}$.

For $A \in L^X$, we denote $\mathfrak{F}(A) = \{F \in L^X : \exists \varphi \in 2_{fin}^{\beta^*(A)}, F = \bigvee \varphi\}$. It is proved that that (1) $\mathfrak{F}(A)$ is directed; (2) $B \leq A$ iff $\mathfrak{F}(B) \subseteq \mathfrak{F}(A)$ for $A, B \in L^X$; (3) $\beta^*(A) \subseteq \mathfrak{F}(A)$ (4) $\bigvee \mathfrak{F}(A) = A$; (5) $\mathfrak{F}(\bigvee_{i \in I}^{dir} A_i) = \bigcup_{i \in I} \mathfrak{F}(A_i)$ [19].

For a mapping $\varphi : X \to Y$, $\varphi_L^{\to} : L^X \to L^Y$ is defined by $\varphi_L^{\to}(A)(y) = \bigvee \{A(x) : \varphi(x) = y\}$ for $A \in L^X$ and $y \in Y$, and $\varphi_L^{\leftarrow} : L^Y \to L^X$ is defined by $\varphi_L^{\leftarrow}(B)(x) = B(\varphi(x))$ for $B \in L^Y$ and $x \in X$ [12].

Definition 2.1. ([3]) A subset $C \subseteq L^X$ is called an L-convexity on L^X and the pair (X, C) is called an L-convex space, if

(LC1) $\underline{\top}, \underline{\perp} \in C$;
(LC2) $\forall \{A_i\}_{i \in I} \subseteq C$, $\bigwedge_{i \in I} A_i \in C$;
(LC3) $\forall \{A_i\}_{i \in I}^{dir} \subseteq C$, $\bigvee_{i \in I}^{dir} A_i \in C$.

Definition 2.2. ([4]) An operator $co : L^X \to L^X$ is called an L-hull operator on L^X and the pair (X, co) is called an L-hull space, if it satisfies

(LCO1) $co(\underline{\perp}) = \underline{\perp}$;
(LCO2) $A \leq co(A)$;
(LCO3) $co(co(A)) = co(A)$;
(LCO4) $co(\bigvee_{i \in I}^{dir} A_i) = \bigvee_{i \in I} co(A_i)$.

Theorem 2.3. ([19]) *Relations between L-convexities and L-hull operators are as follows.*

(1) If (X, C) is an L-convex space, then the operator $co_C : L^X \to L^X$ defined by $co_C(A) = \bigwedge \{B \in L^X : A \leq B \in C\}$ for any $A \in L^X$, is an L-hull operator on L^X.

(2) If (X, co) is an L-hull space, then the set $C_{co} = \{A \in L^X : co(A) = A\}$ is an L-convexity on L^X.

(3) $co_{C_{co}} = co$ for any L-hull space (X, co).

(4) $C_{co_C} = C$ for any L-convex space (X, co).

Let (X, C_X) and (Y, C_Y) be L-convex spaces. A mapping $\varphi : X \to Y$ is called an L-convexity preserving mapping, if $\varphi_L^{\leftarrow}(A) \in C_X$ for any $A \in C_Y$. The category of L-convex spaces and L-convexity preserving mappings is denoted by L-**CS** [19].

Definition 2.4. ([26]) A set $\mathcal{R} = \{\mathcal{R}_{x_\lambda} : x_\lambda \in J(L^X)\}$ is called an L-convex remotehood system on L^X and the pair $(X, \mathcal{R})$ is called an L-convex remotehood space, where $\mathcal{R}_{x_\lambda} \subseteq L^X$ satisfies

(LCR1) $\underline{\perp} \in \mathcal{R}_{x_\lambda}$;

(LCR2) $A \in \mathcal{R}_{x_\lambda}$ implies $x_\lambda \not\leq A$;

(LCR3) $A \in \mathcal{R}_{x_\lambda}$ iff there is a set $B \in L^X$ such that $x_\lambda \not\leq B \geq A$ and $B \in \mathcal{R}_{y_\mu}$ for any $y_\mu \in J(L^X)$ with $y_\mu \not\leq B$;

(LCR4) $\bigvee_{i\in I}^{dir} A_i \in \mathcal{R}_{x_\lambda}$ iff there is a $\mu \in \beta^*(\lambda)$ such that $A_i \in \mathcal{R}_{x_\mu}$ for all $i \in I$.

Let $(X, \mathcal{R}_X)$ and $(Y, \mathcal{R}_Y)$ be L-convex remotehood spaces. A mapping $\varphi : X \to Y$ is an L-convex remotehood preserving mapping, if $B \in (\mathcal{R}_Y)_{\varphi_L^{\rightarrow}(x_\lambda)}$ implies $\varphi_L^{\leftarrow}(B) \in (\mathcal{R}_X)_{x_\lambda}$ for all $B \in L^Y$ and $x_\lambda \in J(L^X)$. The category of L-convex remotehood spaces and L-convex remotehood preserving mappings is denoted by L-**CRS** [26].

Theorem 2.5. ([26]) *Relations between L-convex spaces and L-convex remotehood spaces are as follows.*

(1) Let $(X, \mathcal{R})$ be an L-convex remotehood space. The set $C_{\mathcal{R}} = \{A \in L^X : \forall x_\lambda \not\leq A,\ A \in \mathcal{R}_{x_\lambda}\}$ is an L-convexity on L^X.

(2) Let (X, C) be an L-convex space. Then the set $\mathcal{R}_C = \{\mathcal{R}^C_{x_\lambda} : x_\lambda \in J(L^X)\}$ is an L-convex remotehood system on L^X, where $\mathcal{R}_{x_\lambda} = \{A \in L^X : \exists B \in C, x_\lambda \not\leq B \geq A\}$.

*(3) L-**CRS** is isomorphic to L-**CS**.*

3. L-convex β^*-remotehood space

In an L-topological space, the supremum of two L-fuzzy sets is an L-remotehood of an L-fuzzy point iff each of them is an L-remotehood of that point [12]. However, as described in (LCR3) of Definition 2.4, in an L-convex space, the supremum of a directed subset of L-fuzzy sets is an L-convex remotehood of an L-fuzzy point can not imply that each L-fuzzy set is an L-convex remotehood of that point. To solve this problem, we introduce the notion of L-convex β^*-remotehood space by which we characterize L-convex spaces and L-convex remotehood spaces. For this, we firstly present the following lemma.

Lemma 3.1. *For any $x_\lambda \in \beta^*(L^X)$, we denote*

$$\beta^*_\lambda(L) = \{\eta \in \beta^*(L) : \lambda \in \beta^*(\eta)\}$$

and

$$\psi_{x_\lambda}(L^X) = \{A \in L^X : \forall \eta \in \beta^*_\lambda(L),\ x_\eta \not\leq A\}.$$

For all $A, B \in L^X$ and any $\{A_i\}_{i\in I} \subseteq L^X$, it follows that

(1) $B \leq A \in \psi_{x_\lambda}(L^X)$ implies $B \in \psi_{x_\lambda}(L^X)$;

(2) $\bigvee_{i\in I} A_i \in \psi_{x_\lambda}(L^X)$ iff $A_i \in \psi_{x_\lambda}(L^X)$ for any $i \in I$;

(3) $\psi_{x_\lambda}(L^X) = \bigcap_{\eta\in\beta^*_\lambda(L)} \psi_{x_\eta}(L^X)$;

(4) $B \leq A$ iff $A \in \psi_{x_\lambda}(L^X)$ implies $B \in \psi_{x_\lambda}(L^X)$ for any $x_\lambda \in \beta^*(L^X)$.

Proof. (1). It is clear.

(2). Clearly, $\bigvee_{i\in I} A_i \in \psi_{x_\lambda}(L^X)$ implies $A_i \in \psi_{x_\lambda}(L^X)$ for any $i \in I$. Conversely, let $A_i \in \psi_{x_\lambda}(L^X)$ for any $i \in I$. Suppose that $\bigvee_{i\in I} A_i \notin \psi_{x_\lambda}(L^X)$. Then there is an $\eta \in \beta^*_\lambda(L)$ such that $x_\eta \leq \bigvee_{i\in I} A_i$. Since $\eta \in \beta^*_\lambda(L)$, there is a $\theta \in \beta^*(\eta)$ such that $\theta \in \beta^*_\lambda(L)$. Further, since $x_\theta \prec x_\eta \leq \bigvee_{i\in I} A_i$, there is a $j \in I$ such that $x_\theta \leq A_j$. But this contradicts $A_j \in \psi_{x_\lambda}(L^X)$. Therefore $\bigvee_{i\in I} A_i \in \psi_{x_\lambda}(L^X)$.

(3). Clearly, $\psi_{x_\lambda}(L^X) \subseteq \bigcap_{\eta\in\beta^*_\lambda(L)} \psi_{x_\eta}(L^X)$. Conversely, let $A \in \bigcap_{\eta\in\beta^*_\lambda(L)} \psi_{x_\eta}(L^X)$. For any $\theta \in \beta^*_\lambda(L)$, there is a $\delta \in \beta^*(\theta)$ such that $\delta \in \beta^*_\lambda(L)$. Thus $A \in \psi_{x_\delta}(L^X)$ and $x_\theta \not\leq A$. Hence $A \in \psi_{x_\lambda}(L^X)$. Therefore $\bigcap_{\eta\in\beta^*_\lambda(L)} \psi_{x_\eta}(L^X) \subseteq \psi_{x_\lambda}(L^X)$.

(4). The necessity is clear. For the sufficiency, assume that $A \in \psi_{x_\lambda}(L^X)$ implies $B \in \psi_{x_\lambda}(L^X)$ for any $x_\lambda \in \beta^*(L^X)$. Suppose that $B \nleq A$. There is a $y_\mu \in \beta^*(L^X)$ such that $y_\mu \prec B$ and $y_\mu \nleq A$. Then $A \in \psi_{y_\mu}(L^X)$ which implies $B \in \psi_{y_\mu}(L^X)$. Since $y_\mu \prec B$, there is an $\eta \in \beta^*_\mu(L)$ such that $y_\eta \leq B$. But this contradicts $B \in \psi_{y_\mu}(L^X)$. Therefore $B \leq A$. □

Definition 3.2. A set $\hat{\mathcal{R}} = \{\hat{\mathcal{R}}_{x_\lambda} : x_\lambda \in \beta^*(L^X)\}$ is called an L-convex β^*-remotehood system on L^X and the pair $(X, \hat{\mathcal{R}})$ is called an L-convex β^*-remotehood space, if for any $x_\lambda \in \beta^*(L^X)$ the set $\hat{\mathcal{R}}_{x_\lambda} \subseteq L^X$ satisfies

(LCBR1) $\underline{\perp} \in \hat{\mathcal{R}}_{x_\lambda}$;

(LCBR2) $A \in \hat{\mathcal{R}}_{x_\lambda}$ iff any $\eta \in \beta^*_\lambda(L)$ implies a set $B \in \psi_{x_\eta}(L^X)$ such that $A \leq B \in \hat{\mathcal{R}}_{y_\mu}$ for any $y_\mu \in \beta^*(L^X)$ with $B \in \psi_{y_\mu}(L^X)$;

(LCBR3) $\bigvee^{dir}_{i\in I} A_i \in \hat{\mathcal{R}}_{x_\lambda}$ iff $A_i \in \hat{\mathcal{R}}_{x_\lambda}$ for all $i \in I$.

Let $(X, \hat{\mathcal{R}}_X)$ and $(Y, \hat{\mathcal{R}}_Y)$ be L-convex β^*-remotehood spaces. A mapping $\varphi : X \to Y$ is called an L-convex β^*-remotehood preserving mapping, if $B \in (\hat{\mathcal{R}}_Y)_{\varphi^{\to}_L(x_\lambda)}$ implies $\varphi^{\leftarrow}_L(B) \in (\hat{\mathcal{R}}_X)_{x_\lambda}$ for all $x_\lambda \in \beta^*(L^X)$ and $B \in L^Y$. The category of L-convex β^*-remotehood spaces and L-convex β^*-remotehood preserving mappings is denoted by L-**CBRS**.

Lemma 3.3. *Let $(X, \hat{\mathcal{R}})$ be an L-convex β^*-remotehood space. For any $x_\lambda \in \beta^*(L^X)$, it is true that*

(1) $A \leq B \in \hat{\mathcal{R}}_{x_\lambda}$ implies $A \in \hat{\mathcal{R}}_{x_\lambda}$;

(2) $\hat{\mathcal{R}}_{x_\lambda} = \bigcap_{\eta\in\beta^*_\lambda(L)} \hat{\mathcal{R}}_{x_\eta}$.

Proof. (1). It directly follows from (LCBR2).

(2). Clearly, $\hat{\mathcal{R}}_{x_\lambda} \subseteq \bigcap_{\eta\in\beta^*_\lambda(L)} \hat{\mathcal{R}}_{x_\eta}$ by (LCBR2). Conversely, if $A \in \bigcap_{\eta\in\beta^*_\lambda(L)} \hat{\mathcal{R}}_{x_\eta}$, then $A \in \hat{\mathcal{R}}_{x_\eta}$ for any $\eta \in \beta^*_\lambda(L)$. By (LCBR2), for any $\theta \in \beta^*_\eta(L)$ there is a set $D_\theta \in \psi_{x_\theta}(L^X)$ such that $A \leq D_\theta \in \hat{\mathcal{R}}_{y_\mu}$ for any $y_\mu \in \beta^*(L^X)$ with $D_\theta \in \psi_{y_\mu}(L^X)$.

Let $D = \bigwedge_{\eta\in\beta^*_\lambda(L)} \bigwedge_{\theta\in\beta^*_\eta(L)} D_\theta$. For any $\eta \in \beta^*_\lambda(L)$, we say that $D \in \psi_{x_\eta}(L^X)$. Otherwise, $x_\theta \leq D$ for some $\theta \in \beta^*_\eta(L)$. There is a $\mu \in \beta^*(\theta)$ such that $\mu \in \beta^*_\eta(L)$. Hence $x_\theta \leq D \leq D_\mu \in \psi_{x_\mu}(L^X)$. It is a contradiction. Therefore $D \in \psi_{x_\eta}(L^X)$.

Further, let $z_\delta \in \beta^*(L^X)$ with $D \in \psi_{z_\delta}(L^X)$. To prove that $D \in \hat{\mathcal{R}}_{z_\delta}$, let $\nu \in \beta^*_\delta(L)$. Then $z_\nu \nleq D$. Thus there are $\eta \in \beta^*_\lambda(L)$ and $\theta \in \beta^*_\eta(L)$ such that $z_\nu \nleq D_\theta$. Hence $D_\theta \in \psi_{z_\nu}(L^X)$ and $D_\theta \in \hat{\mathcal{R}}_{z_\nu}$ by the assumption. Therefore $D \in \hat{\mathcal{R}}_{z_\delta}$ by (LCBR2).

Now, for $\eta \in \beta^*_\lambda(L)$, it follows that $A \leq D \in \psi_{x_\eta}(L^X)$ and $D \in \hat{\mathcal{R}}_{z_\delta}$ for any $z_\delta \in \beta^*(L^X)$ with $D \in \psi_{z_\delta}(L^X)$. So $A \in \hat{\mathcal{R}}_{x_\lambda}$ by (LCBR2). Therefore $\bigcap_{\eta\in\beta^*_\lambda(L)} \hat{\mathcal{R}}_{x_\eta} \subseteq \hat{\mathcal{R}}_{x_\lambda}$. □

We study relations between L-convex spaces and L-convex β^*-remotehood spaces.

Theorem 3.4. *Let $(X, \hat{\mathcal{R}})$ be an L-convex β^*-remotehood space. Then the set*

$$C_{\hat{\mathcal{R}}} = \{A \in L^X : \forall x_\lambda \in \beta^*(L^X), A \in \psi_{x_\lambda}(L^X) \text{ implies } A \in \hat{\mathcal{R}}_{x_\lambda}\}$$

is an L-convexity on L^X.

Proof. (LC1). By (LCBR1), we know that $\underline{\perp} \in \hat{\mathcal{R}}_{x_\lambda}$ for any $x_\lambda \in \beta^*(L^X)$. Thus $\underline{\perp} \in C_{\hat{\mathcal{R}}}$. Also, $\underline{\top} \in C_{\hat{\mathcal{R}}}$ is trivial.

(LC2). Let $\{A_i\}_{i\in I} \subseteq C_{\hat{\mathcal{R}}}$ and let $x_\lambda \in \beta^*(L^X)$ with $\bigwedge_{i\in I} A_i \in \psi_{x_\lambda}(L^X)$. For any $\eta \in \beta^*_\lambda(L)$, we have $x_\eta \nleq \bigwedge_{i\in I} A_i$. So there is a $j \in I$ such that $x_\eta \nleq A_j$. Thus $A_j \in \psi_{x_\eta}(L^X)$ which implies that $A_i \in \hat{\mathcal{R}}_{x_\eta}$. Hence $\bigwedge_{i\in I} A_i \in \hat{\mathcal{R}}_{x_\eta}$ by (1) of Lemma 3.3. As a result, by (2) of Lemma 3.3, it follows that

$$\bigwedge_{i\in I} A_i \in \bigcap_{\eta\in\beta^*_\lambda(L)} \hat{\mathcal{R}}_{x_\eta} = \hat{\mathcal{R}}_{x_\lambda}$$

Therefore $\bigwedge_{i\in I} A_i \in C_{\hat{\mathcal{R}}}$.

(LC3). Let $\{A_i\}_{i\in I}^{dir} \subseteq C_{\hat{\mathcal{R}}}$ and let $x_\lambda \in \beta^*(L^X)$ with $\bigvee_{i\in I}^{dir} A_i \in \psi_{x_\lambda}(L^X)$. Then $A_i \in \psi_{x_\lambda}(L^X)$ and $A_i \in \hat{\mathcal{R}}_{x_\lambda}$ for any $i \in I$. Hence $\bigvee_{i\in I}^{dir} A_i \in \hat{\mathcal{R}}_{x_\lambda}$ by (LCBR3). Therefore $\bigvee_{i\in I}^{dir} A_i \in C_{\hat{\mathcal{R}}}$. $\square$

Theorem 3.5. *Let $(X, \hat{\mathcal{R}}_X)$ and $(Y, \hat{\mathcal{R}}_Y)$ be L-convex β^*-remotehood spaces. If $\varphi : X \to Y$ is an L-convex β^*-remotehood preserving mapping, then $\varphi : (X, C_{\hat{\mathcal{R}}_X}) \to (Y, C_{\hat{\mathcal{R}}_Y})$ is an L-convexity preserving mapping.*

Proof. Let $B \in C_{\hat{\mathcal{R}}_Y}$. To prove that $\varphi_L^{\leftarrow}(B) \in C_{\hat{\mathcal{R}}_X}$, let $x_\lambda \in \beta^*(L^X)$ with $\varphi_L^{\leftarrow}(B) \in \psi_{x_\lambda}(L^X)$. Then $\varphi_L^{\rightarrow}(x_\lambda) \in \beta^*(L^Y)$ and $B \in \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^Y)$. Thus $B \in (\hat{\mathcal{R}}_Y)_{\varphi_L^{\rightarrow}(x_\lambda)}$ followed by $\varphi_L^{\leftarrow}(B) \in (\hat{\mathcal{R}}_X)_{x_\lambda}$. Hence $\varphi_L^{\leftarrow}(B) \in C_{\hat{\mathcal{R}}_X}$. Therefore φ is an L-convexity preserving mapping. $\square$

Theorem 3.6. *Let (X, C) be an L-convex space. For any $x_\lambda \in \beta^*(L^X)$, define*

$$\hat{\mathcal{R}}_{x_\lambda}^{C} = \{A \in L^X : \exists B \in C \cap \psi_{x_\lambda}(L^X),\ A \leq B\}.$$

Then $\hat{\mathcal{R}}_C = \{\hat{\mathcal{R}}_{x_\lambda}^{C} : x_\lambda \in \beta^*(L^X)\}$ is an L-convex β^*-remotehood system.

Proof. (LCBR1). It is clear.

(LCBR2). If $A \in \hat{\mathcal{R}}_{x_\lambda}^{C}$, then there is a set $B \in C \cap \psi_{x_\lambda}(L^X)$ such that $A \leq B$. Thus $B \in \psi_{x_\eta}(L^X)$ for any $\eta \in \beta_\lambda^*(L)$. In addition, for any $y_\mu \in \beta^*(L^X)$ with $B \in \psi_{y_\mu}(L^X)$, it is clear that $B \in C \cap \psi_{y_\mu}(L^X)$. This shows that $B \in \hat{\mathcal{R}}_{y_\mu}^{C}$. Hence the necessity of (LCBR2) holds for $\hat{\mathcal{R}}_{x_\lambda}^{C}$.

Conversely, assume that any $\eta \in \beta_\lambda^*(L)$ implies some $B_\eta \in \psi_{x_\eta}(L^X)$ such that $A \leq B_\eta \in \hat{\mathcal{R}}_{y_\mu}^{C}$ for any $y_\mu \in \beta^*(L^X)$ with $B_\eta \in \psi_{y_\mu}(L^X)$. Let $\eta \in \beta_\lambda^*(L)$. Then $B_\eta \in \hat{\mathcal{R}}_{x_\eta}^{C}$ by the assumption. Thus there is a set $E_\eta \in C \cap \psi_{x_\eta}(L^X)$ such that $B_\eta \leq E_\eta$. Further, let $E = \bigwedge_{\eta\in\beta_\lambda^*(L)} E_\eta$. It follows that $A \leq \bigwedge_{\eta\in\beta_\lambda^*(L)} B_\eta \leq E$ and

$$E \in C \cap \bigcap_{\eta\in\beta_\lambda^*(L)} \psi_{x_\eta}(L^X) = C \cap \psi_{x_\lambda}(L^X).$$

Hence $A \in \hat{\mathcal{R}}_{x_\lambda}^{C}$. Therefore the sufficiency of (LCBR2) holds for $\hat{\mathcal{R}}_{x_\lambda}^{C}$.

(LCBR3). Let $\{A_i\}_{i\in I}^{dir} \subseteq L^X$. If $\bigvee_{i\in I}^{dir} A_i \in \hat{\mathcal{R}}_{x_\lambda}^{C}$ then $A_i \in \hat{\mathcal{R}}_{x_\lambda}^{C}$ for any $i \in I$. Conversely, let $A_i \in \hat{\mathcal{R}}_{x_\lambda}^{C}$ for any $i \in I$. There is a set $B_i \in C \cap \psi_{x_\lambda}(L^X)$ such that $A_i \leq B_i$ for any $i \in I$.

Let $E_i = \bigwedge\{D_i \in C \cap \psi_{x_\lambda}(L^X) : A_i \leq D_i\}$ for any $i \in I$. We have $A_i \leq E_i$ and $E_i \in C \cap \psi_{x_\lambda}(L^X)$. Since $\{A_i\}_{i\in I}^{dir}$ is directed, the set $\{E_i\}_{i\in I}$ is also directed. Thus $\bigvee_{i\in I}^{dir} E_i \in C \cap \psi_{x_\lambda}(L^X)$ and $\bigvee_{i\in I}^{dir} A_i \leq \bigvee_{i\in I}^{dir} E_i$. Therefore $\bigvee_{i\in I}^{dir} A_i \in \hat{\mathcal{R}}_{x_\lambda}^{C}$.

In conclusion, $\hat{\mathcal{R}}_C$ is an L-convex β^*-remotehood system. $\square$

Theorem 3.7. *Let (X, C_X) and (Y, C_Y) be L-convex spaces. If $\varphi : X \to Y$ is an L-convexity preserving mapping, then $\varphi : (X, \hat{\mathcal{R}}_{C_X}) \to (Y, \hat{\mathcal{R}}_{C_Y})$ is an L-convex β^*-remotehood preserving mapping.*

Proof. Let $x_\lambda \in \beta^*(L^X)$ and $A \in \hat{\mathcal{R}}_{\varphi_L^{\rightarrow}(x_\lambda)}^{C_Y}$. Then there is a set $B \in C_Y \cap \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^Y)$ such that $A \leq B$. Thus $\varphi_L^{\leftarrow}(B) \in C_X \cap \psi_{x_\lambda}(L^X)$ and $\varphi_L^{\leftarrow}(A) \leq \varphi_L^{\leftarrow}(B)$. Hence $\varphi_L^{\leftarrow}(B) \in \hat{\mathcal{R}}_{x_\lambda}^{C_X}$. Therefore φ is an L-convex β^*-remotehood preserving mapping. $\square$

Theorem 3.8. *$\hat{\mathcal{R}}_{C_{\hat{\mathcal{R}}}} = \hat{\mathcal{R}}$ for any L-convex β^*-remotehood space $(X, \hat{\mathcal{R}})$.*

Proof. Let $x_\lambda \in \beta^*(L^X)$ and $A \in \hat{\mathcal{R}}_{x_\lambda}^{C_{\hat{\mathcal{R}}}}$. There is a set $B \in C_{\hat{\mathcal{R}}} \cap \psi_{x_\lambda}(L^X)$ such that $A \leq B$. Further, $B \in C_{\hat{\mathcal{R}}}$ implies $B \in \hat{\mathcal{R}}_{y_\mu}$ for any $y_\mu \in \beta^*(L^X)$ with $B \in \psi_{y_\mu}(L^X)$. In particular, $B \in \hat{\mathcal{R}}_{x_\lambda}$. Hence $A \in \hat{\mathcal{R}}_{x_\lambda}$. Therefore $\hat{\mathcal{R}}_{x_\lambda}^{C_{\hat{\mathcal{R}}}} \subseteq \hat{\mathcal{R}}_{x_\lambda}$.

Conversely, let $A \in \hat{\mathcal{R}}_{x_\lambda}$. By (LCBR2), any $\eta \in \beta_\lambda^*(L)$ implies some $B_\eta \in \psi_{x_\eta}(L^X)$ such that $A \leq B_\eta \in \hat{\mathcal{R}}_{y_\mu}$ for any $y_\mu \in \beta^*(L^X)$ with $B_\eta \in \psi_{y_\mu}(L^X)$.

Let $B = \bigwedge_{\eta\in\beta_\lambda^*(L)} B_\eta$. To prove that $B \in C_{\hat{\mathcal{R}}}$, it is sufficient to prove that $B \in \hat{\mathcal{R}}_{z_\delta}$ for any $z_\delta \in \beta^*(L^X)$ with $B \in \psi_{z_\delta}(L^X)$. Indeed, for any $\theta \in \beta_\delta^*(L)$, it is clear that $z_\theta \not\leq B$. Then there is an $\eta \in \beta_\lambda^*(L)$ such that $z_\theta \not\leq B_\eta$.

Thus $B_\eta \in \psi_{z_\theta}(L^X)$ and so $B_\eta \in \hat{\mathcal{R}}_{z_\theta}$ by the assumption. Hence $B \in \hat{\mathcal{R}}_{z_\theta}$. Therefore $B \in \bigcap_{\theta\in\beta^*_\delta(L)} \hat{\mathcal{R}}_{z_\theta} = \hat{\mathcal{R}}_{z_\delta}$ by (3) of Lemma 3.3. This implies that $B \in C_{\hat{\mathcal{R}}}$. So

$$A \leq B \in C_{\hat{\mathcal{R}}} \cap \bigcap_{\eta\in\beta^*_\lambda(L)} \psi_{x_\eta}(L^X) = C_{\hat{\mathcal{R}}} \cap \psi_{x_\lambda}(L^X).$$

Thus $\hat{\mathcal{R}}_{x_\lambda} \subseteq \hat{\mathcal{R}}^{C_{\hat{\mathcal{R}}}}_{x_\lambda}$.

In conclusion, $\hat{\mathcal{R}}^{C_{\hat{\mathcal{R}}}}_{x_\lambda} = \hat{\mathcal{R}}_{x_\lambda}$ for any $x_\lambda \in \beta^*(L^X)$. Therefore $\hat{\mathcal{R}} = \hat{\mathcal{R}}_{C_{\hat{\mathcal{R}}}}$. □

Theorem 3.9. *$C_{\hat{\mathcal{R}}_C} = C$ for any L-convex space (X, C).*

Proof. Let $A \in C_{\hat{\mathcal{R}}_C}$. For any $x_\lambda \in \beta^*(\lambda)$ with $A \in \psi_{x_\lambda}(L^X)$, it follows that $A \in \hat{\mathcal{R}}^C_{x_\lambda}$. Thus there is a set $B_{x_\lambda} \in C \cap \psi_{x_\lambda}(L^X)$ such that $A \leq B_{x_\lambda}$. Let $B = \bigwedge_{A\in\psi_{x_\lambda}(L^X)} B_{x_\lambda}$. Then $A \leq B \in C$. On the other hand, for any $y_\eta \nleq A$, there is a $\mu \in \beta^*(\eta)$ such that $y_\mu \nleq A$. Thus $A \in \psi_{y_\mu}(L^X)$ and $B_{y_\mu} \in \psi_{y_\mu}(L^X) \cap C$. Hence $y_\eta \nleq B_{y_\mu} \geq B$ followed by $y_\mu \nleq B$. This implies that $B \leq A$. Hence $A = B \in C$. Therefore $C_{\hat{\mathcal{R}}_C} \subseteq C$.

Conversely, let $A \in C$. For any $x_\lambda \in \beta^*(L^X)$ with $A \in \psi_{x_\lambda}(L^X)$, it is clear that $A \in C \cap \psi_{x_\lambda}(L^X)$. This directly implies that $A \in \hat{\mathcal{R}}^C_{x_\lambda}$ and $A \in C_{\hat{\mathcal{R}}_C}$. Thus $C \subseteq C_{\hat{\mathcal{R}}_C}$. This shows that $C_{\hat{\mathcal{R}}_C} = C$. □

Based on Theorems 3.4 and 3.5, we define a functor: $\mathbb{F} : L$-**CBRS** $\to L$-**CS** by

$$\mathbb{F}((X, \hat{\mathcal{R}})) = (X, C_{\hat{\mathcal{R}}}), \qquad \mathbb{F}(\varphi) = \varphi.$$

By Theorems 3.4–3.9, $\mathbb{F}$ is isomorphic. Thus we have the following conclusion.

Theorem 3.10. *L-**CBRS** is isomorphic to L-**CS**.*

Based on Theorems 2.5 and 3.6, the following example shows that L-convex β^*-remotehood system and L-convex remotehood system are different. Although their differences may seem trivial, the difference of β^*-remotehood systems provides some necessary conveniences in defining L-fuzzy β^*-remote mappings and L-convex quasi-uniforms in the next section.

Example 3.11. Let $X = \{x\}$ and $L = [0, 1]$. It is clear that $C = \{\underline{\perp}, x_{\frac{1}{2}}, \underline{\top}\}$ is an L-convexity on L^X. In addition, it is easy to check the following results.

(1) $\mathcal{R}_C = \{\mathcal{R}^C_{x_\lambda} : 0 < \lambda < 1\}$ is an L-convex remotehood system, where

$$\mathcal{R}^C_{x_\lambda} = \begin{cases} \{\underline{\perp}\}, & 0 < \lambda \leq \frac{1}{2}, \\ \{\underline{\perp}, x_{\frac{1}{2}}\}, & \frac{1}{2} < \lambda < 1; \end{cases}$$

(2) $\hat{\mathcal{R}}_C = \{\hat{\mathcal{R}}^C_{x_\lambda} : 0 < \lambda < 1\}$ is an L-convex β^*-remotehood system, where

$$\hat{\mathcal{R}}^C_{x_\lambda} = \begin{cases} \{\underline{\perp}\}, & 0 < \lambda < \frac{1}{2}, \\ \{\underline{\perp}, x_{\frac{1}{2}}\}, & \frac{1}{2} \leq \lambda < 1. \end{cases}$$

Therefore $\hat{\mathcal{R}}^C$ and $\mathcal{R}^C$ are different.

The following theorem gives a direct relation between L-convex β^*-remotehood systems and L-convex remotehood systems.

Theorem 3.12. *(1) Let $(X, \hat{\mathcal{R}})$ be an L-convex β^*-remotehood space. Define $\mathcal{R}^{\hat{\mathcal{R}}}_{x_\lambda} = \bigcup_{\mu\in\beta^*(\lambda)} \hat{\mathcal{R}}_{x_\lambda}$ for any $x_\lambda \in J(L^X)$. Then $\mathcal{R}_{\hat{\mathcal{R}}} = \{\mathcal{R}^{\hat{\mathcal{R}}}_{x_\lambda} : x_\lambda \in J(L^X)\}$ is an L-convex remotehood system.*

(2) Let $(X, \mathcal{R})$ be an L-convex remotehood space. Define $\hat{\mathcal{R}}^{\mathcal{R}}_{x_\lambda} = \bigcap_{\lambda<\eta\in J(L)} \mathcal{R}_{x_\eta}$ for any $x_\lambda \in \beta^*(L^X)$. Then $\hat{\mathcal{R}}_{\mathcal{R}} = \{\hat{\mathcal{R}}^{\mathcal{R}}_{x_\lambda} : x_\lambda \in \beta^*(L^X)\}$ is an L-convex β^*-remotehood system.

(3) $\hat{\mathcal{R}}_{\mathcal{R}_{\hat{\mathcal{R}}}} = \hat{\mathcal{R}}$ for any L-convex β^*-remotehood space $(X, \hat{\mathcal{R}})$.

(4) $\mathcal{R}_{\hat{\mathcal{R}}_{\mathcal{R}}} = \mathcal{R}$ for any L-convex remotehood space $(X, \mathcal{R})$.

*(5) L-**CBRS** is isomorphic to L-**CRS**.*

4. *L*-convex quasi-uniform space

In this section, we introduce the notion of L-convex quasi-uniform space and study its relations with L-convex spaces and L-convex β^*-remotehood spaces. For this, we introduce L-fuzzy β^*-remote mappings as follows.

A mapping $f : \beta^*(L^X) \to L^X$ is called an L-fuzzy β^*-remote mapping, if $f(x_\lambda) \in \psi_{x_\lambda}(L^X)$ for any $x_\lambda \in \beta^*(L^X)$. The set of all L-fuzzy β^*-remote mappings is denoted by $\Re(L^X)$. For $f, g \in \Re(L^X)$, we denote $f \leq g$ provided that $f(x_\lambda) \leq g(x_\lambda)$ for any $x_\lambda \in \beta^*(L^X)$. Clearly, the mapping $f_0 : \beta^*(L^X) \to L^X$, defined by $f_0(x_\lambda) = \underline{\perp}$ for any $x_\lambda \in \beta^*(L^X)$, is the smallest L-fuzzy β^*-remote mapping.

For all $f, g \in \Re(L^X)$, $\{f_i\}_{i\in I} \subseteq \Re(L^X)$ and $x_\lambda \in \beta^*(L^X)$, we further define

(1) $(\bigvee_{i\in I} f_i)(x_\lambda) = \bigvee_{i\in I} f_i(x_\lambda)$;

(2) $(\bigwedge_{i\in I} f_i)(x_\lambda) = \bigwedge_{i\in I} f_i(x_\lambda)$;

(3) $(f \diamond g)(x_\lambda) = \bigwedge\{f(y_\mu) : g(x_\lambda) \in \psi_{y_\mu}(L^X)\}$.

A subset $\{f_i\}_{i\in I} \subseteq \Re(L^X)$ is called directed if any pair of indices $i, j \in I$ implies a $k \in I$ such that $f_i \vee f_j \leq f_k$. In this case, $\bigvee_{i\in I} f_i$ is denoted by $\bigvee^{dir}_{i\in I} f_i$.

Lemma 4.1. *Let $f, g, h \in \Re(L^X)$ and $\{f_i\}_{i\in I} \subseteq \Re(L^X)$. We have*

(1) $f \diamond g \in \Re(L^X)$ and $f \diamond g \leq f \wedge g$;

(2) $\bigvee_{i\in I} f_i \in \Re(L^X)$ and $\bigwedge_{i\in I} f_i \in \Re(L^X)$;

(3) $(f \diamond g) \diamond h = f \diamond (g \diamond h)$.

Proof. (1). For any $x_\lambda \in \beta^*(L^X)$, we have $g(x_\lambda) \in \psi_{x_\lambda}(L^X)$ and so $(f \diamond g)(x_\lambda) \leq f(x_\lambda)$. Thus $f \diamond g \leq f$ which implies that $f \diamond g \in \Re(L^X)$.

Suppose that $(f \diamond g)(x_\lambda) \nleq g(x_\lambda)$. Then there is a $y_\mu \in \beta^*(L^X)$ such that $y_\mu \prec (f \diamond g)(x_\lambda)$ and $y_\mu \nleq g(x_\lambda)$. By $y_\mu \nleq g(x_\lambda)$, there is an $\eta \in \beta^*(\mu)$ such that $y_\eta \nleq g(x_\lambda)$. So $g(x_\lambda) \in \psi_{y_\eta}(L^X)$ and

$$y_\mu \leq (f \diamond g)(x_\lambda) \leq f(y_\eta) \in \psi_{y_\eta}(L^X).$$

It is a contradiction. Hence $(f \diamond g)(x_\lambda) \leq g(x_\lambda)$. Therefore $f \diamond g \leq f \wedge g$.

(2). It directly follows from (1) and (2) of Lemma 3.1.

(3). Let $x_\lambda, y_\mu \in \beta^*(L^X)$. We have

$$\begin{aligned} y_\mu \nleq [(f \diamond g) \diamond h](x_\lambda) \quad &\Leftrightarrow \quad \exists h(x_\lambda) \in \psi_{z_\eta}(L^X),\ y_\mu \nleq (f \diamond g)(z_\eta) \\ &\Leftrightarrow \quad \exists h(x_\lambda) \in \psi_{z_\eta}(L^X),\ \exists g(z_\eta) \in \psi_{w_\theta}(L^X), y_\mu \nleq f(w_\theta) \\ &\Leftrightarrow \quad \exists (g \diamond h)(x_\lambda) \in \psi_{w_\theta}(L^X),\ y_\mu \nleq f(w_\theta) \\ &\Leftrightarrow \quad y_\mu \nleq [f \diamond (g \diamond h)](x_\lambda). \end{aligned}$$

Thus $[(f \diamond g) \diamond h](x_\lambda) = [f \diamond (g \diamond h)](x_\lambda)$. Therefore $[(f \diamond g) \diamond h] = [f \diamond (g \diamond h)]$. □

Definition 4.2. A subset $\mathcal{U} \subseteq \Re(L^X)$ is called an L-convex quasi-uniformity on L^X and the pair $(X, \mathcal{U})$ is called an L-convex quasi-uniform space, if

(LCQU1) $f_0 \in \mathcal{U}$;

(LCQU2) $f \in \mathcal{U}$ iff there is a $g \in \mathcal{U}$ such that $f \leq g \diamond g$;

(LCQU3) $\{f_i\}^{dir}_{i\in I} \subseteq \mathcal{U}$ implies $\bigvee^{dir}_{i\in I} f_i \in \mathcal{U}$.

Let $\mathcal{U}$ be an L-convex quasi-uniformity on L^X. A subset $\mathcal{B} \subseteq \mathcal{U}$ is called an L-convex quasi-uniform base of $\mathcal{U}$ if any $f \in \mathcal{U}$ implies a $g \in \mathcal{B}$ such that $f \leq g$. A subset $\Phi \subseteq \mathcal{U}$ is called an L-convex quasi-uniform subbase of $\mathcal{U}$ if $\mathcal{B}_\Phi = \{\bigvee^{dir}_{i\in I} f_i : \{f_i\}^{dir}_{i\in I} \subseteq \Phi\}$ is an L-convex quasi-uniform base of $\mathcal{U}$. Clearly, an L-convex quasi-uniform base is an L-convex quasi-uniform subbase.

Now, we construct an L-convexity by an L-convex quasi-uniformity.

Lemma 4.3. *Let* $(X,\mathcal{U})$ *be an L-convex quasi-uniform space. For any* $A \in L^X$,

$$A_{\mathcal{U}} = \bigvee\{x_\lambda \in \beta^*(L^X) : \forall f \in \mathcal{U},\ A \nleq f(x_\lambda)\}.$$

For all $x_\lambda \in \beta^*(L^X)$, $f \in \mathcal{U}$ *and* $\{A_i\}_{i\in I}^{dir} \subseteq L^X$, *we have*

(1) $A \le f(x_\lambda)$ *implies* $A_{\mathcal{U}} \le g(x_\lambda)$ *for any* $g \in \mathcal{U}$ *with* $f \le g \diamond g$;

(2) $A_{\mathcal{U}} \in \psi_{x_\lambda}(L^X)$ *iff any* $\eta \in \beta^*_\lambda(L)$ *implies some* $g \in \mathcal{U}$ *such that* $A \le g(x_\eta)$;

(3) $f(x_\lambda)_{\mathcal{U}} \in \psi_{x_\lambda}(L^X)$;

(4) $A \le A_{\mathcal{U}}$;

(5) $(\bigvee_{i\in I}^{dir} A_i)_{\mathcal{U}} = \bigvee_{i\in I}^{dir}(A_i)_{\mathcal{U}}$.

Proof. (1). Let $A \le f(x_\lambda)$ and let $g \in \mathcal{U}$ with $f \le g \diamond g$. Suppose that $A_{\mathcal{U}} \nleq g(x_\lambda)$. There is a $y_\mu \in \beta^*(L^X)$ such that $y_\mu \nleq g(x_\lambda)$ and $A \nleq h(y_\mu)$ for any $h \in \mathcal{U}$. In particular, we have $A \nleq g(y_\mu)$. Also, since $g(x_\lambda) \in \psi_{y_\mu}(L^X)$ by $y_\mu \nleq g(x_\lambda)$, we have

$$A \le f(x_\lambda) \le (g \diamond g)(x_\lambda) \le g(y_\mu).$$

It is a contradiction. So $A_{\mathcal{U}} \le g(x_\lambda)$.

(2). Let $A_{\mathcal{U}} \in \psi_{x_\lambda}(L^X)$ and let $\eta \in \beta^*_\lambda(L)$. Suppose that $A \nleq g(x_\eta)$ for any $g \in \mathcal{U}$. Then $x_\eta \le A_{\mathcal{U}}$ and thus $A_{\mathcal{U}} \notin \psi_{x_\lambda}(L^X)$. But it is a contradiction. So there must be some $g \in \mathcal{U}$ such that $A \le g(x_\eta)$.

Conversely, assume that any $\eta \in \beta^*_\lambda(L)$ implies some $g \in \mathcal{U}$ such that $A \le g(x_\eta)$. If $A_{\mathcal{U}} \notin \psi_{x_\lambda}(L^X)$, then there is a $\delta \in \beta^*_\lambda(L)$ such that $x_\delta \le A_{\mathcal{U}}$. By $\delta \in \beta^*_\lambda(L)$, there is a $\mu \in \beta^*_\lambda(L)$ such that $\mu \in \beta^*(\delta)$. Thus $x_\mu \prec A_{\mathcal{U}}$ which implies an $x_\eta \in \beta^*(L^X)$ such that $x_\mu \prec x_\eta$ and $A \nleq g(x_\eta)$ for any $g \in \mathcal{U}$. Hence $\eta \in \beta^*_\lambda(L)$ and $A \nleq g(x_\eta)$ for any $g \in \mathcal{U}$. But this contradicts the assumption. Therefore $A_{\mathcal{U}} \in \psi_{x_\lambda}(L^X)$.

(3). By (LCQU2), there is a $g \in \mathcal{U}$ such that $f \le g\diamond g$. For any $\eta \in \beta^*_\lambda(L)$, we have $g(x_\lambda) \in \psi_{x_\lambda}(L^X) \subseteq \psi_{x_\eta}(L^X)$. Thus $f(x_\lambda) \le (g \diamond g)(x_\lambda) \le g(x_\eta)$. Hence $f(x_\lambda)_{\mathcal{U}} \in \psi_{x_\lambda}(L^X)$ follows from (2).

(4). For any $y_\mu \in \beta^*(A)$, there is an $\eta \in \beta^*_\mu(L)$ such that $y_\eta \in \beta^*(A)$. In addition, $g(y_\mu) \in \psi_{y_\mu}(L^X)$ for any $g \in \mathcal{U}$. Thus $y_\eta \nleq g(y_\mu)$ followed by $A \nleq g(y_\mu)$. Hence $y_\mu \le A_{\mathcal{U}}$. Therefore $A = \bigvee_{y_\mu\in\beta^*(A)} y_\mu \le A_{\mathcal{U}}$.

(5). By the definition, $\bigvee_{i\in I}^{dir}(A_i)_{\mathcal{U}} \le (\bigvee_{i\in I}^{dir} A_i)_{\mathcal{U}}$. Conversely, let $y_\mu \in \beta^*(L^X)$ with $y_\mu \nleq \bigvee_{i\in I}^{dir}(A_i)_{\mathcal{U}}$. Then there is an $\eta \in \beta^*(\mu)$ such that $y_\eta \nleq \bigvee_{i\in I}^{dir}(A_i)_{\mathcal{U}}$. Thus $y_\eta \nleq (A_i)_{\mathcal{U}}$ for any $i \in I$. So there is an $f_i \in \mathcal{U}$ such that $A_i \le f_i(y_\eta)$.

Let $\Xi_i = \{f_i \in \mathcal{U} : A_i \le f_i(y_\eta)\}$ for any $i \in I$. Then $\bigwedge_{f_i\in\Xi_i} f_i \in \mathcal{U}$ by (LCQU2). Since $\{A_i\}_{i\in I}^{dir} \subseteq L^X$, the set $\{\bigwedge_{f_i\in\Xi_i} f_i\}_{i\in I} \subseteq \mathcal{U}$ is directed. Let $f = \bigvee_{i\in I}^{dir} \bigwedge_{f_i\in\Xi_i} f_i$. Then $f \in \mathcal{U}$ by (LCQU3). In addition, $\bigvee_{i\in I}^{dir} A_i \le f(y_\eta)$. Further, by (LCQU2), there is a $g \in \mathcal{U}$ such that $f \le g \diamond g$.

Suppose that $y_\mu \le (\bigvee_{i\in I}^{dir} A_i)_{\mathcal{U}}$. Then $y_\eta \prec (\bigvee_{i\in I}^{dir} A_i)_{\mathcal{U}}$. Thus there is a $y_\delta \in \beta^*(L^X)$ such that $y_\eta \prec y_\delta$ and $\bigvee_{i\in I}^{dir} A_i \nleq h(y_\delta)$ for any $h \in \mathcal{U}$. In particular, we have $\bigvee_{i\in I}^{dir} A_i \nleq g(y_\delta)$. Since $g(y_\eta) \in \psi_{y_\eta}(L^X)$, it follows that $g(y_\eta) \in \psi_{y_\delta}(L^X)$ and thus

$$\bigvee_{i\in I}^{dir} A_i \le f(y_\eta) \le (g \diamond g)(y_\eta) \le g(y_\delta).$$

It is a contradiction. Hence $y_\mu \nleq (\bigvee_{i\in I}^{dir} A_i)_{\mathcal{U}}$. Therefore $(\bigvee_{i\in I}^{dir} A_i)_{\mathcal{U}} \le \bigvee_{i\in I}^{dir}(A_i)_{\mathcal{U}}$. $\square$

Theorem 4.4. *Let* $(X,\mathcal{U})$ *be an L-convex quasi-uniform space. Define an operator* $co_{\mathcal{U}} : L^X \to L^X$ *by*

$$\forall A \in L^X,\ \ co_{\mathcal{U}}(A) = A_{\mathcal{U}}.$$

Then $co_{\mathcal{U}}$ *is an L-hull operator of some L-convexity denoted by* $\mathcal{C}_{\mathcal{U}}$.

Proof. We prove that $co_{\mathcal{U}}$ satisfies (LCO1)–(LCO4).

Indeed, (LCO1) is clear. In addition, (LCO2) and (LCO4) directly follow from (4) and (5) of Lemma 4.3.

(LCO3). Let $x_\lambda \in \beta^*(L^X)$ with $co_{\mathcal{U}}(A) \in \psi_{x_\lambda}(L^X)$. By (2) of Lemma 4.3, any $\eta \in \beta^*_\lambda(L)$ implies some $f \in \mathcal{U}$ such that $A \le f(x_\eta)$. Further, by (LCQU3), there are $g,h \in \mathcal{U}$ such that $f \le g \diamond g$ and $g \le h \diamond h$.

Thus $co_{\mathcal{U}}(A) \leq g(x_\lambda)$ and $co_{\mathcal{U}}(co_{\mathcal{U}}(A)) \leq h(x_\lambda)$ by (1) of Lemma 4.3. Hence $co_{\mathcal{U}}(co_{\mathcal{U}}(A)) \in \psi_{x_\lambda}(L^X)$. By the arbitrariness of $x_\lambda \in \beta^*(L^X)$, we conclude from (4) of Lemma 3.1 that $co_{\mathcal{U}}(co_{\mathcal{U}}(A)) \leq co_{\mathcal{U}}(A)$. So $co_{\mathcal{U}}(co_{\mathcal{U}}(A)) = co_{\mathcal{U}}(A)$ as desired.

Therefore $co_{\mathcal{U}}$ is an L-hull operator. □

Next, we construct an L-convex quasi-uniform space from an L-convex space.

Lemma 4.5. *Let $(X, \mathcal{C})$ be an L-convex space. For any $A \in L^X$, define a mapping $f_A : \beta^*(L^X) \to L^X$ by*

$$f_A(x_\lambda) = \bigvee\{B \in \mathcal{C} \cap \psi_{x_\lambda}(L^X) : B \leq A\}$$

for any $x_\lambda \in \beta^*(L^X)$. For all $A \in L^X$, $\{A_i\}_{i\in I}^{dir} \subseteq L^X$ and $\{B_i\}_{i\in I} \subseteq L^X$, the follows statements are valid.

(1) $f_A \in \Re(L^X)$;

(2) $\{A_i\}_{i\in I}^{dir} \subseteq \mathcal{C}$ implies $\bigvee_{i\in I} f_{A_i} = f_{\bigvee_{i\in I}^{dir} A_i}$;

(3) $\bigwedge_{i\in I} f_{B_i} = f_{\bigwedge_{i\in I} B_i}$;

(4) $f_A \diamond f_A = f_A$.

Proof. (1). It directly follows from (2) of Lemma 3.1.

(2). Let $x_\lambda \in \beta^*(L^X)$. For any $y_\mu \in \beta^*(L^X)$ with $y_\mu \prec \bigvee_{i\in I} f_{A_i}(x_\lambda)$, there is an $i \in I$ such that $y_\mu \prec f_{A_i}(x_\lambda)$. Thus there is a set $B \leq A_i$ such that $y_\mu \prec B \in \mathcal{C} \cap \psi_{x_\lambda}(L^X)$. Hence $B \leq A_i \leq \bigvee_{i\in I}^{dir} A_i$ which implies that $y_\mu \leq f_{\bigvee_{i\in I}^{dir} A_i}(x_\lambda)$. Therefore $\bigvee_{i\in I} f_{A_i}(x_\lambda) \leq f_{\bigvee_{i\in I}^{dir} A_i}(x_\lambda)$.

Conversely, let $z_\eta \in \beta^*(L^X)$ with $z_\eta \prec f_{\bigvee_{i\in I}^{dir} A_i}(x_\lambda)$. Then there is a set $D \leq \bigvee_{i\in I}^{dir} A_i$ such that $z_\eta \prec D \in \mathcal{C} \cap \psi_{x_\lambda}(L^X)$. By (LCO4), it follows that $z_\eta \prec D = co(D) = \bigvee_{F\in\mathfrak{F}(D)} co(F)$. Thus there is an $F \in \mathfrak{F}(D)$ such that $z_\eta \prec co(F) \leq D \leq \bigvee_{i\in I}^{dir} A_i$. So $co(F) \in \mathcal{C} \cap \psi_{x_\lambda}(L^X)$ by (1) of Lemma 3.1. Further, since $F \in \mathfrak{F}(D) \subseteq \mathfrak{F}(\bigvee_{i\in I}^{dir} A_i) = \bigcup_{i\in I} \mathfrak{F}(A_i)$, there is an $i \in I$ such that $F \in \mathfrak{F}(A_i)$. Hence $co(F) \leq A_i$ and $z_\eta \leq co(F) \leq f_{A_i}(x_\lambda) \leq \bigvee_{i\in I} f_{A_i}(x_\lambda)$. Therefore $f_{\bigvee_{i\in I}^{dir} A_i}(x_\lambda) \leq \bigvee_{i\in I} f_{A_i}(x_\lambda)$.

Now, $\bigvee_{i\in I} f_{A_i}(x_\lambda) = f_{\bigvee_{i\in I}^{dir} A_i}(x_\lambda)$ for any $x_\lambda \in \beta^*(L^X)$. Thus $\bigvee_{i\in I} f_{A_i} = f_{\bigvee_{i\in I}^{dir} A_i}$.

(3). Let $x_\lambda \in \beta^*(L^X)$ and let $y_\mu \in \beta^*(L^X)$ with $y_\mu \prec \bigwedge_{i\in I} f_{B_i}(x_\lambda)$. Then $y_\mu \prec f_{B_i}(x_\lambda)$ for any $i \in I$. Thus there is a set $D_i \leq B_i$ such that $y_\mu \prec D_i \in \mathcal{C} \cap \psi_{x_\lambda}(L^X)$. Thus $y_\mu \leq \bigwedge_{i\in I} D_i \in \mathcal{C} \cap \psi_{x_\lambda}(L^X)$ and $\bigwedge_{i\in I} D_i \leq \bigwedge_{i\in I} B_i$. Hence $y_\mu \leq f_{\bigwedge_{i\in I} B_i}(x_\lambda)$. Therefore $\bigwedge_{i\in I} f_{B_i}(x_\lambda) \leq f_{\bigwedge_{i\in I} B_i}(x_\lambda)$.

Conversely, let $z_\eta \in \beta^*(L^X)$ with $z_\eta \prec f_{\bigwedge_{i\in I} B_i}(x_\lambda)$. Then there is a set $B \leq \bigwedge_{i\in I} B_i$ such that $z_\eta \prec B \in \mathcal{C} \cap \psi_{x_\lambda}(L^X)$. Thus $B \leq B_i$ and $z_\eta \leq f_{B_i}(x_\lambda)$ for any $i \in I$. Hence $z_\eta \leq \bigwedge_{i\in I} f_{B_i}(x_\lambda)$. Therefore $f_{\bigwedge_{i\in I} B_i}(x_\lambda) \leq \bigwedge_{i\in I} f_{B_i}(x_\lambda)$.

So $\bigwedge_{i\in I} f_{B_i}(x_\lambda) = f_{\bigwedge_{i\in I} B_i}(x_\lambda)$ for any $x_\lambda \in \beta^*(L^X)$. That is, $\bigwedge_{i\in I} f_{B_i} = f_{\bigwedge_{i\in I} B_i}$.

(4). Clearly, $f_A \diamond f_A \leq f_A$. Suppose that $f_A \nleq f_A \diamond f_A$. There is an $x_\lambda \in \beta^*(L^X)$ such that $f_A(x_\lambda) \nleq (f_A \diamond f_A)(x_\lambda)$. Thus there is a $z_\eta \in \beta^*(L^X)$ such that $z_\eta \prec f_A(x_\lambda)$ and $z_\eta \nleq (f_A \diamond f_A)(x_\lambda)$.

By $z_\eta \nleq (f_A \diamond f_A)(x_\lambda)$, there is a $y_\mu \in \beta^*(L^X)$ such that $f_A(x_\lambda) \in \psi_{y_\mu}(L^X)$ and $z_\eta \nleq f_A(y_\mu)$. Further, by $z_\eta \prec f_A(x_\lambda)$, there is a set $B \leq A$ such that $z_\eta \prec B \leq f_A(x_\lambda)$ and $B \in \mathcal{C} \cap \psi_{x_\lambda}(L^X)$. Hence $B \in \mathcal{C} \cap \psi_{y_\mu}(L^X)$ followed by $z_\eta \leq B \leq f_A(y_\mu)$. It is a contradiction. So $f_A \leq f_A \diamond f_A$ must hold. Therefore $f_A \diamond f_A = f_A$. □

Theorem 4.6. *Let $(X, \mathcal{C})$ be an L-convex space. Define a set*

$$\mathcal{U}_{\mathcal{C}} = \{f \in \Re(L^X) : \exists A \in \mathcal{C},\ \forall x_\lambda \in \beta^*(L^X), f(x_\lambda) \leq f_A(x_\lambda) \in \mathcal{C}\}.$$

Then $\mathcal{U}_{\mathcal{C}}$ is an L-convex quasi-uniformity.

Proof. We prove that $\mathcal{U}_{\mathcal{C}}$ satisfies (LCQU1)–(LCQU3).

(LCQU1). It is clear that $\underline{\perp} \in \mathcal{C}$ and $f_0(x_\lambda) = f_{\underline{\perp}}(x_\lambda) = \underline{\perp} \in \mathcal{C}$ for any $x_\lambda \in \beta^*(L^X)$. Thus $f_0 \in \mathcal{U}_{\mathcal{C}}$.

(LCQU2). If $f \in \mathcal{U}_{\mathcal{C}}$, then there is a set $A \in \mathcal{C}$ such that $f(x_\lambda) \leq f_A(x_\lambda) \in \mathcal{C}$ for any $x_\lambda \in \beta^*(L^X)$. Since $f_A \in \mathcal{U}_{\mathcal{C}}$ and $f_A \diamond f_A = f_A$ by (4) of Lemma 4.5, we find that $f \leq f_A \diamond f_A$. Therefore (LCQU2) holds for $\mathcal{U}_{\mathcal{C}}$.

(LCQU3). Let $\{f_i\}_{i\in I}^{dir} \subseteq \mathcal{U}_{\mathcal{C}}$. For any $i \in I$, there is a set $A_i \in \mathcal{C}$ such that $f_i(x_\lambda) \leq f_{A_i}(x_\lambda) \in \mathcal{C}$ for any $x_\lambda \in \beta^*(L^X)$. For any $i \in I$, we denote

$$\Xi_i = \{D_i \in \mathcal{C} : \forall x_\lambda{\in}\beta^*(L^X), f_i(x_\lambda){\leq}f_{D_i}(x_\lambda) \in \mathcal{C}\}.$$

Then $A_i \in \Xi_i$ and $\bigwedge_{D_i\in\Xi_i} D_i \in C$ by (LC2). Also, by (3) of Lemma 4.5, for any $x_\lambda \in \beta^*(L^X)$, it follows that

$$f_i(x_\lambda) \leq f_{\bigwedge_{D_i\in\Xi_i} D_i}(x_\lambda) = \bigwedge_{D_i\in\Xi_i} f_{D_i}(x_\lambda) \in C.$$

Since $\{f_i\}_{i\in I}^{dir} \subseteq \Re(L^X)$, the set $\{\bigwedge_{D_i\in\Xi_i} D_i\}_{i\in I} \subseteq C$ is also directed. Thus $\bigvee_{i\in I}^{dir} \bigwedge_{D_i\in\Xi_i} D_i \in C$ by (LC3). In addition, by (2) of Lemma 4.5,

$$\bigvee_{i\in I}^{dir} f_i(x_\lambda) \leq f_{\bigvee_{i\in I}^{dir} \bigwedge_{D_i\in\Xi_i} D_i}(x_\lambda) = \bigvee_{i\in I}^{dir} f_{\bigwedge_{D_i\in\Xi_i} D_i}(x_\lambda) \in C.$$

Hence $\bigvee_{i\in I}^{dir} f_i \in \mathcal{U}_C$. $\square$

Corollary 4.7. *For any L-convex space (X, C), the set $\mathcal{B} = \{f_A : A \in C,\ \forall x_\lambda \in \beta^*(L^X),\ f_A(x_\lambda) \in C\}$ is an L-convex quasi-uniform base of $\mathcal{U}_C$.*

Theorem 4.8. *$C_{\mathcal{U}_C} = C$ for any L-convex space (X, C).*

Proof. Let $B \in C_{\mathcal{U}_C}$ and let $y_\mu \in \beta^*(L^X)$ with $co_{\mathcal{U}_C}(B) = B \in \psi_{y_\mu}(L^X)$. By (2) of Lemma 4.3, for any $\eta \in \beta^*_\mu(L)$, there is an $f \in \mathcal{U}_C$ such that $B \leq f(y_\eta)$. Further, by $f \in \mathcal{U}_C$, there is a set $A \in C$ such that $f(x_\lambda) \leq f_A(x_\lambda) \in C$ for any $x_\lambda \in \beta^*(L^X)$. In particular, $B \leq f(y_\eta) \leq f_A(y_\eta) \in C \cap \psi_{y_\eta}(L^X)$. Thus

$$co(B) \leq co(f_A(y_\eta)) = f_A(y_\eta) \in C \cap \psi_{y_\eta}(L^X).$$

Further, by the arbitrariness of $\eta \in \beta^*_\mu(L)$ and (3) of Lemma 3.1, we have

$$co(B) \in \bigcap_{\eta\in\beta^*_\mu(L)} \psi_{y_\eta}(L^X) = \psi_{y_\mu}(L^X).$$

Hence $co(B) \leq B$ by (5) of Lemma 3.1. So $B = co(B) \in C$. Therefore $C_{\mathcal{U}_C} \subseteq C$.

Conversely, for any $D \in C$, it is clear that $co_{\mathcal{U}_C}(D) \in C$ by $C_{\mathcal{U}_C} \subseteq C$. Also, by (3) of Lemma 4.3, for any $x_\lambda \in \beta^*(L^X)$, we have $co_{\mathcal{U}_C}(f_{co_{\mathcal{U}_C}(D)}(x_\lambda)) \in \psi_{x_\lambda}(L^X)$. Further, by $f_{co_{\mathcal{U}_C}(D)}(x_\lambda) \leq co_{\mathcal{U}_C}(D)$, it follows that

$$co_{\mathcal{U}_C}(f_{co_{\mathcal{U}_C}(D)}(x_\lambda)) \leq co_{\mathcal{U}_C}(co_{\mathcal{U}_C}(D)) = co_{\mathcal{U}_C}(D).$$

Thus $co_{\mathcal{U}_C}(f_{co_{\mathcal{U}_C}(D)}(x_\lambda)) \leq f_{co_{\mathcal{U}_C}(D)}(x_\lambda)$ and

$$f_{co_{\mathcal{U}_C}(D)}(x_\lambda) = co_{\mathcal{U}_C}(f_{co_{\mathcal{U}_C}(D)}(x_\lambda)) \in C_{\mathcal{U}_C} \subseteq C.$$

Therefore $f_{co_{\mathcal{U}_C}(D)} \in \mathcal{U}_C$.

To prove that $D \in C_{\mathcal{U}_C}$, let $x_\lambda \in \beta^*(L^X)$ with $D \in \psi_{x_\lambda}(L^X)$. We have

$$D \leq f_{co_{\mathcal{U}_C}(D)}(x_\lambda) = co_{\mathcal{U}_C}(f_{co_{\mathcal{U}_C}(D)}(x_\lambda))$$

and $co_{\mathcal{U}_C}(D) \leq f_{co_{\mathcal{U}_C}(D)}(x_\lambda) \in \psi_{x_\lambda}(L^X)$. This implies $co_{\mathcal{U}_C}(D) \in \psi_{x_\lambda}(L^X)$. Thus $co_{\mathcal{U}_C}(D) \leq D$ by (4) of Lemma 3.1. Hence $D = co_{\mathcal{U}_C}(D) \in C_{\mathcal{U}_C}$. Therefore $C \subseteq C_{\mathcal{U}_C}$. In conclusion, $C_{\mathcal{U}_C} = C$, as desired. $\square$

Example 4.9. Let $X = \{x\}$ and $L = \{\perp, a, b, \top\}$ be the diamond lattice, where a and b are incomparable. Then $\beta^*(L^X) = \{x_a, x_b\}$.

(1) Let $\mathcal{U} = \{f \in \Re(L^X) : f \leq f_1\}$, where $f_1(x_a) = x_b$ and $f_1(x_b) = x_a$. Since $f_1 \diamond f_1 = f_1$, it is easy to check that $\mathcal{U}$ is an L-convex quasi-uniformity on L^X. In addition, $C_{\mathcal{U}} = L^X$ and $\mathcal{U} = \mathcal{U}_{C_{\mathcal{U}}}$.

(2) Let $C = \{\underline{\perp}, x_a, \underline{\top}\}$. Then C is an L-convexity on L^X. Further, it is easy to check that $f_{x_a}(x_a) = f_{\underline{\top}}(x_a) = \underline{\perp} \in C$ and $f_{x_a}(x_b) = f_{\underline{\top}}(x_b) = x_a \in C$. Thus $\mathcal{U}_C = \{f \in \mathcal{R}(L^X) : f \leq f_{\underline{\top}}\} = \{f_0, f_{\underline{\top}}\}$ is an L-convex quasi-uniformity on L^X satisfying $C = C_{\mathcal{U}_C}$.

Remark 4.10. If an L-convex quasi-uniform space $(X, \mathcal{U})$ is induced by an L-convex space (X, C) (i.e., $\mathcal{U} = \mathcal{U}_C$), then it directly follows from Theorem 4.8 that $\mathcal{U} = \mathcal{U}_{C_\mathcal{U}}$. However, in general, given an L-convex quasi-uniformity $\mathcal{U}$ on L^X, it is possible that $\mathcal{U}_{C_\mathcal{U}} \neq \mathcal{U}$. For example, let $X = \{x\}$ and $L = \{\bot, a, b, \top\}$ be a diamond lattice. Define two mappings $f_1, f_2 : \beta^*(L^X) \to L^X$ by $f_1(x_b) = x_a$, $f_2(x_a) = x_b$ and $f_1(x_a) = f_2(x_b) = \underline{\bot}$. Then $f_1, f_2 \in \Re(L^X)$ satisfying $f_1 = f_1 \diamond f_1$ and $f_2 = f_2 \diamond f_2$. Thus the set $\mathcal{U} = \{f_0, f_1, f_2\}$ is an L-convex quasi-uniformity on L^X. In addition, $C_\mathcal{U} = L^X$ and $\mathcal{U}_{C_\mathcal{U}} = \{f \in \Re(L^X) : f \leq f_{\underline{\top}}\} = \{f_0, f_1, f_2, f_{\underline{\top}}\}$, where $f_{\underline{\top}} \in \Re(L^X)$ satisfies $f_{\underline{\top}}(x_a) = x_b$ and $f_{\underline{\top}}(x_b) = x_a$. Clearly, $\mathcal{U}_{C_\mathcal{U}} \neq \mathcal{U}$.

Now, we discuss relations between L-convex quasi-uniformities and L-convex β^*-remotehood systems. For this, we first construct an L-convex β^*-remotehood system from an L-convex quasi-uniformity.

Theorem 4.11. *Let $(X, \mathcal{U})$ be an L-convex quasi-uniform space. For $x_\lambda \in \beta^*(L^X)$, define a set*

$$\hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda} = \{A \in L^X : \forall \eta \in \beta^*_\lambda(L),\ \exists f \in \mathcal{U},\ A \leq f(x_\eta)\}.$$

Then $\hat{\mathcal{R}}_\mathcal{U} = \{\hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda} : x_\lambda \in \beta^*(L^X)\}$ is an L-convex β^*-remotehood system.

Proof. Let $x_\lambda \in \beta^*(L^X)$ and $A \in L^X$. Then $A \in \hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}$ iff $A_\mathcal{U} \in \psi_{x_\lambda}(L^X)$ by (2) of Lemma 4.3. Next, we prove that $\hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}$ satisfies (LCBR1)–(LCBR3).

(LCBR1). It is clear that $\underline{\bot} \in \hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}$ since $f_0 \in \mathcal{U}$ and $\underline{\bot} \leq f_0(x_\eta)$ for any $\eta \in \beta^*_\lambda(L)$.

(LCBR2). Let $A \in \hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}$. For any $\eta \in \beta^*_\lambda(L)$, we have $A \leq A_\mathcal{U} \in \psi_{x_\lambda}(L^X) \subseteq \psi_{x_\eta}(L^X)$. Further, let $y_\mu \in \beta^*(L^X)$ with $A_\mathcal{U} \in \psi_{y_\mu}(L^X)$. For any $\theta \in \beta^*_\mu(L)$, it is clear that

$$y_\theta \nleq A_\mathcal{U} = co_\mathcal{U}(A) = co_\mathcal{U}(co_\mathcal{U}(A)) = (A_\mathcal{U})_\mathcal{U}.$$

Thus, by the definition of $(A_\mathcal{U})_\mathcal{U}$, there is a $g \in \mathcal{U}$ such that $A_\mathcal{U} \leq g(y_\theta)$. This implies that $A_\mathcal{U} \in \hat{\mathcal{R}}^{\mathcal{U}}_{y_\mu}$. Hence the necessity of (LCBR2) holds for $\hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}$.

Assume that any $\eta \in \beta^*_\lambda(L)$ implies some $B_\eta \in \psi_{x_\eta}(L^X)$ such that $A \leq B_\eta \in \hat{\mathcal{R}}^{\mathcal{U}}_{y_\mu}$ for any $y_\mu \in \beta^*(L^X)$ with $B_\eta \in \psi_{y_\mu}(L^X)$. Let $\eta \in \beta^*_\lambda(L)$. Then $B_\eta \in \hat{\mathcal{R}}^{\mathcal{U}}_{x_\eta}$ by the assumption. Thus $A \leq B_\eta \leq (B_\eta)_\mathcal{U} \in \psi_{x_\eta}(L^X)$. Further, let $B = \bigwedge_{\eta \in \beta^*_\lambda(L)} B_\eta$. Then $A \leq B \leq B_\mathcal{U}$ and

$$B_\mathcal{U} \leq \bigwedge_{\eta \in \beta^*_\lambda(L)} (B_\eta)_\mathcal{U} \in \bigcap_{\eta \in \beta^*_\lambda(L)} \psi_{x_\eta}(L^X) = \psi_{x_\lambda}(L^X).$$

This shows that $A_\mathcal{U} \leq B_\mathcal{U} \in \psi_{x_\lambda}(L^X)$ and $A_\mathcal{U} \in \psi_{x_\lambda}(L^X)$. Hence $A \in \hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}$. Therefore the sufficiency of (LCBR2) holds for $\hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}$.

(LCBR3). Let $\{A_i\}^{dir}_{i \in I} \subseteq L^X$. By (5) of Lemma 4.3, it follows that

$$\begin{aligned}
\bigvee^{dir}_{i \in I} A_i \in \hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda} \quad &\Leftrightarrow \quad (\bigvee^{dir}_{i \in I} A_i)_\mathcal{U} \in \psi_{x_\lambda}(L^X) \\
&\Leftrightarrow \quad \bigvee^{dir}_{i \in I} (A_i)_\mathcal{U} \in \psi_{x_\lambda}(L^X) \\
&\Leftrightarrow \quad \forall i \in I,\ (A_i)_\mathcal{U} \in \psi_{x_\lambda}(L^X) \\
&\Leftrightarrow \quad \forall i \in I,\ A_i \in \hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}.
\end{aligned}$$

So (LCBR3) holds for $\hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}$.

Therefore $\hat{\mathcal{R}}_\mathcal{U}$ is an L-convex β^*-remotehood system. □

Theorem 4.12. *Let $(X, \mathcal{U})$ be an L-convex quasi-uniform space. Then $C_\mathcal{U} = C_{\hat{\mathcal{R}}_\mathcal{U}}$ and $\hat{\mathcal{R}}_\mathcal{U} = \hat{\mathcal{R}}_{C_\mathcal{U}}$.*

Proof. Firstly, we check that $\mathcal{C}_{\mathcal{U}} = \mathcal{C}_{\hat{\mathcal{R}}_{\mathcal{U}}}$.

If $A \in \mathcal{C}_{\mathcal{U}}$ then $A = co_{\mathcal{U}}(A)$. For any $x_\lambda \in \beta^*(L^X)$ with $A \in \psi_{x_\lambda}(L^X)$, we have $co_{\mathcal{U}}(A) = A_{\mathcal{U}} \in \psi_{x_\lambda}(L^X)$. Thus $A \in \hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}$ and $A \in \mathcal{C}_{\hat{\mathcal{R}}_{\mathcal{U}}}$. Therefore $\mathcal{C}_{\mathcal{U}} \subseteq \mathcal{C}_{\hat{\mathcal{R}}_{\mathcal{U}}}$.

Conversely, let $A \in \mathcal{C}_{\hat{\mathcal{R}}_{\mathcal{U}}}$. Then $A \in \psi_{x_\lambda}(L^X)$ implies $A \in \hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}$ for all $x_\lambda \in \beta^*(L^X)$. So $A \in \psi_{x_\lambda}(L^X)$ implies $co_{\mathcal{U}}(A) = A_{\mathcal{U}} \in \psi_{x_\lambda}(L^X)$ for any $x_\lambda \in \beta^*(L^X)$. Thus $co_{\mathcal{U}}(A) \leq A$ by (4) of Lemma 3.1. Hence $co_{\mathcal{U}}(A) = A$ which implies $A \in \mathcal{C}_{\mathcal{U}}$. Therefore $\mathcal{C}_{\hat{\mathcal{R}}_{\mathcal{U}}} \subseteq \mathcal{C}_{\mathcal{U}}$. Therefore $\mathcal{C}_{\hat{\mathcal{R}}_{\mathcal{U}}} = \mathcal{C}_{\mathcal{U}}$.

Next, to verify that $\hat{\mathcal{R}}_{\mathcal{U}} = \hat{\mathcal{R}}_{\mathcal{C}_{\mathcal{U}}}$, let $x_\lambda \in \beta^*(L^X)$. We verify that $\hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda} = \hat{\mathcal{R}}^{\mathcal{C}_{\mathcal{U}}}_{x_\lambda}$.

Let $A \in L^X$. By (2) of Theorem 4.4, it follows that

$$\begin{aligned} A \in \hat{\mathcal{R}}^{\mathcal{C}_{\mathcal{U}}}_{x_\lambda} &\Leftrightarrow \exists B \in \mathcal{C}_{\mathcal{U}} \cap \psi_{x_\lambda}(L^X),\ A \leq B \\ &\Leftrightarrow \exists B \in \mathcal{C}_{\mathcal{U}},\ \forall \eta \in \beta^*_\lambda(L),\ x_\eta \nleq B = co_{\mathcal{U}}(B) \geq A \\ &\Rightarrow \exists B \in \mathcal{C}_{\mathcal{U}},\ \forall \eta \in \beta^*_\lambda(L),\ \exists f \in \mathcal{U}, A \leq B \leq f(x_\eta) \\ &\Rightarrow \forall \eta \in \beta^*_\lambda(L),\ \exists f \in \mathcal{U},\ A \leq f(x_\eta) \\ &\Leftrightarrow A \in \hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}. \end{aligned}$$

Thus $\hat{\mathcal{R}}^{\mathcal{C}_{\mathcal{U}}}_{x_\lambda} \subseteq \hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}$.

Conversely, let $A \in \hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda}$. For any $\eta \in \beta^*_\lambda(L)$, there is an $f_\eta \in \mathcal{U}$ such that $A \leq f_\eta(x_\eta) \in \psi_{x_\eta}(L^X)$. Thus $A \leq \bigwedge_{\eta \in \beta^*_\lambda(L)} f_\eta(x_\eta) \in \psi_{x_\lambda}(L^X)$. Hence $A \in \psi_{x_\lambda}(L^X)$ and $co_{\mathcal{U}}(A) = A_{\mathcal{U}} \in \psi_{x_\lambda}(L^X)$. This implies that $A \leq co_{\mathcal{U}}(A) \in \mathcal{C}_{\mathcal{U}} \cap \psi_{x_\lambda}(L^X)$. So $A \in \hat{\mathcal{R}}^{\mathcal{C}_{\mathcal{U}}}_{x_\lambda}$. Therefore $\hat{\mathcal{R}}^{\mathcal{U}}_{x_\lambda} \subseteq \hat{\mathcal{R}}^{\mathcal{C}_{\mathcal{U}}}_{x_\lambda}$. □

Also, an L-convex quasi-uniformity can be directly constructed from an L-convex β^*-remotehood system.

Theorem 4.13. *Let $(X, \hat{\mathcal{R}})$ be an L-convex β^*-remotehood space. Define a set*

$$\mathcal{U}_{\hat{\mathcal{R}}} = \{f \in \Re(L^X) : \forall x_\lambda \in \beta^*(L^X),\ \exists B \in \hat{\mathcal{R}}_{x_\lambda},\ f(x_\lambda) \leq B\}.$$

Then $\mathcal{U}_{\hat{\mathcal{R}}}$ is an L-convex quasi-uniformity satisfying $\mathcal{U}_{\hat{\mathcal{R}}} = \mathcal{U}_{\mathcal{C}_{\hat{\mathcal{R}}}}$ and $\hat{\mathcal{R}}_{\mathcal{U}_{\hat{\mathcal{R}}}} = \hat{\mathcal{R}}$.

Proof. By Theorems 3.4 and 4.6, $\mathcal{U}_{\mathcal{C}_{\hat{\mathcal{R}}}}$ is an L-convex quasi-uniformity. To prove that $\mathcal{U}_{\hat{\mathcal{R}}}$ is an L-convex quasi-uniformity, it is sufficient to prove that $\mathcal{U}_{\hat{\mathcal{R}}} = \mathcal{U}_{\mathcal{C}_{\hat{\mathcal{R}}}}$.

Let $f \in \mathcal{U}_{\hat{\mathcal{R}}}$. Then any $x_\lambda \in \beta^*(L^X)$ implies a set $B \in \hat{\mathcal{R}}_{x_\lambda}$ such that $f(x_\lambda) \leq B$. Further, by $B \in \hat{\mathcal{R}}_{x_\lambda}$ and (LCBR2), any $\eta \in \beta^*_\lambda(L)$ implies some $E_\eta \in \psi_{x_\eta}(L^X)$ such that $B \leq E_\eta \in \hat{\mathcal{R}}_{y_\mu}$ for any $y_\mu \in \beta^*(L^X)$ with $E_\eta \in \psi_{y_\mu}(L^X)$.

Let $E = \bigwedge_{\eta \in \beta^*_\lambda(L)} E_\eta$. By (3) of Lemma 3.1, it follows that

$$B \leq E = \bigwedge_{\eta \in \beta^*_\lambda(L)} E_\eta \in \bigcap_{\eta \in \beta^*_\lambda(L)} \psi_{x_\eta}(L^X) = \psi_{x_\lambda}(L^X).$$

Further, to prove that $E \in \mathcal{C}_{\hat{\mathcal{R}}}$, let $y_\mu \in \beta^*(L^X)$ with $E \in \psi_{y_\mu}(L^X)$. It is sufficient to prove that $E \in \hat{\mathcal{R}}_{y_\mu}$.

For any $\theta \in \beta^*_\mu(L)$, it is clear that $y_\theta \nleq E$. Then there is an $\eta \in \beta^*_\lambda(L)$ such that $y_\theta \nleq E_\eta$. Thus $E_\eta \in \psi_{y_\theta}(L^X)$ followed by $E \leq E_\eta \in \hat{\mathcal{R}}_{y_\theta}$. So $E \in \hat{\mathcal{R}}_{y_\theta}$ by (1) of Lemma 3.3 and $E \in \bigcap_{\theta \in \beta^*_\mu(L)} \hat{\mathcal{R}}_{y_\theta} = \hat{\mathcal{R}}_{y_\mu}$ by (2) of Lemma 3.3. Therefore $E \in \mathcal{C}_{\hat{\mathcal{R}}}$.

Now, we obtain that $E \in \mathcal{C}_{\hat{\mathcal{R}}} \cap \psi_{x_\lambda}$ for any $x_\lambda \in \beta^*(L^X)$. This implies that $f(x_\lambda) \leq f_E(x_\lambda) = E \in \mathcal{C}_{\hat{\mathcal{R}}}$ for any $x_\lambda \in \beta^*(L^X)$. Thus $f \in \mathcal{U}_{\mathcal{C}_{\hat{\mathcal{R}}}}$. Therefore $\mathcal{U}_{\hat{\mathcal{R}}} \subseteq \mathcal{U}_{\mathcal{C}_{\hat{\mathcal{R}}}}$.

Conversely, let $f \in \mathcal{U}_{\mathcal{C}_{\hat{\mathcal{R}}}}$. Then there is a set $A \in \mathcal{C}_{\hat{\mathcal{R}}}$ such that $f(x_\lambda) \leq f_A(x_\lambda) \in \mathcal{C}_{\hat{\mathcal{R}}}$ for any $x_\lambda \in \beta^*(L^X)$. Notice that $f_A(x_\lambda) \in \psi_{x_\lambda}(L^X)$ by (1) of Lemma 4.5. Thus $f_A(x_\lambda) \in \mathcal{C}_{\hat{\mathcal{R}}}$ implies that $f_A(x_\lambda) \in \hat{\mathcal{R}}_{x_\lambda}$. Hence $f \in \mathcal{U}_{\hat{\mathcal{R}}}$. Therefore $\mathcal{U}_{\mathcal{C}_{\hat{\mathcal{R}}}} \subseteq \mathcal{U}_{\hat{\mathcal{R}}}$.

In conclusion, $\mathcal{U}_{\hat{\mathcal{R}}} = \mathcal{U}_{\mathcal{C}_{\hat{\mathcal{R}}}}$. That is, $\mathcal{U}_{\hat{\mathcal{R}}}$ is an L-convex quasi-uniformity on L^X.

To prove that $\hat{\mathcal{R}}_{\mathcal{U}_{\hat{\mathcal{R}}}} = \hat{\mathcal{R}}$, we need to prove that $\hat{\mathcal{R}}^{\mathcal{U}_{\hat{\mathcal{R}}}}_{x_\lambda} = \hat{\mathcal{R}}_{x_\lambda}$ for any $x_\lambda \in \beta^*(L^X)$.

Let $A \in \hat{\mathcal{R}}^{\mathcal{U}_{\hat{\mathcal{R}}}}_{x_\lambda}$. Then, for any $\eta \in \beta^*_\lambda(L)$, there is an $f \in \mathcal{U}_{\hat{\mathcal{R}}}$ such that $A \le f(x_\eta)$. Further, by $f \in \mathcal{U}_{\hat{\mathcal{R}}}$ and $x_\eta \in \beta^*(L^X)$, there is a set $B_\eta \in \hat{\mathcal{R}}_{x_\eta}$ such that $f(x_\eta) \le B_\eta$. Thus $A \le f(x_\eta) \le B_\eta \in \hat{\mathcal{R}}_{x_\eta}$ which implies that $A \in \hat{\mathcal{R}}_{x_\eta}$. Hence $A \in \bigcap_{\eta\in\beta^*_\lambda(L)} \hat{\mathcal{R}}_{x_\eta} = \hat{\mathcal{R}}_{x_\lambda}$. Therefore $\hat{\mathcal{R}}^{\mathcal{U}_{\hat{\mathcal{R}}}}_{x_\lambda} \subseteq \hat{\mathcal{R}}_{x_\lambda}$.

Conversely, let $A \in \hat{\mathcal{R}}_{x_\lambda}$. By (LCBR2), any $\theta \in \beta^*_\lambda(L)$ implies a set $B_\theta \in \psi_{x_\theta}(L^X)$ such that $A \le B_\theta \in \hat{\mathcal{R}}_{y_\mu}$ for any $y_\mu \in \beta^*(L^X)$ with $B_\theta \in \psi_{y_\mu}(L^X)$.

Let $E = \bigwedge_{\theta\in\beta^*_\lambda(L)} B_\theta$. By the proof of Theorem 4.13, we have $E \in \mathcal{C}_{\hat{\mathcal{R}}} \cap \psi_{x_\lambda}(L^X)$. Define an L-fuzzy β^*-remote mapping $f_{x_\lambda} : \beta^*(L^X) \to L^X$ by: for any $y_\mu \in \beta^*(L^X)$,

$$f_{x_\lambda}(y_\mu) = \begin{cases} E, & y_\mu \ge x_\lambda, \\ \perp, & \text{otherwise.} \end{cases}$$

Then $f_{x_\lambda} \in \mathcal{U}_{\hat{\mathcal{R}}}$. In addition, for any $\eta \in \beta^*_\lambda(L)$, it holds that $A \le E = f_{x_\lambda}(x_\eta)$ and

$$A \le E \in \mathcal{C}_{\hat{\mathcal{R}}} \cap \psi_{x_\lambda}(L^X) \subseteq \mathcal{C}_{\hat{\mathcal{R}}} \cap \psi_{x_\eta}(L^X).$$

Thus $A \in \hat{\mathcal{R}}^{\mathcal{U}_{\hat{\mathcal{R}}}}_{x_\lambda}$. Therefore $\hat{\mathcal{R}}_{x_\lambda} \subseteq \hat{\mathcal{R}}^{\mathcal{U}_{\hat{\mathcal{R}}}}_{x_\lambda}$.

In conclusion, $\hat{\mathcal{R}}^{\mathcal{U}_{\hat{\mathcal{R}}}}_{x_\lambda} = \hat{\mathcal{R}}_{x_\lambda}$ for any $x_\lambda \in \beta^*(L^X)$. That is, $\hat{\mathcal{R}}_{\mathcal{U}_{\hat{\mathcal{R}}}} = \hat{\mathcal{R}}$. □

5. *L*-convex quasi-uniform preserving mappings

In this section, we introduce L-convex quasi-uniform preserving mappings, by which, we form the category of L-convex quasi-uniform spaces. We show that the category of L-convex spaces and the category of L-convex β^*-remotehood spaces can be embedded into the category of L-convex quasi-uniform spaces.

Lemma 5.1. *For a mapping $\varphi : X \to Y$, define $(\varphi^{\leftarrow}_L)^* : \Re(L^Y) \to (L^X)^{\beta^*(L^X)}$ by*

$$(\varphi^{\leftarrow}_L)^*(f)(x_\lambda) = (\varphi^{\leftarrow}_L \circ f \circ \varphi^{\rightarrow}_L)(x_\lambda)$$

for all $f \in \Re(L^Y)$ and $x_\lambda \in \beta^*(L^X)$. The following statements are valid.

(1) $(\varphi^{\leftarrow}_L)^*(f) \in \Re(L^X)$ for any $f \in \Re(L^Y)$;

(2) $(\varphi^{\leftarrow}_L)^*(g) \le (\varphi^{\leftarrow}_L)^*(h) \diamond (\varphi^{\leftarrow}_L)^*(h)$ for all $f, g \in \Re(L^Y)$ with $g \le h \diamond h$.

Proof. (1). It is direct.

(2). Let $x_\lambda \in \beta^*(L^X)$. For any $z_\eta \in \beta^*(L^X)$ with $z_\eta \not\le [(\varphi^{\leftarrow}_L)^*(h) \diamond (\varphi^{\leftarrow}_L)^*(h)](x_\lambda)$, there is a $y_\mu \in \beta^*(L^X)$ such that $(\varphi^{\leftarrow}_L)^*(h)(x_\lambda) \in \psi_{y_\mu}(L^X)$ and $z_\eta \not\le (\varphi^{\leftarrow}_L)^*(h)(y_\mu)$. By $(\varphi^{\leftarrow}_L)^*(h)(x_\lambda) \in \psi_{y_\mu}(L^X)$, it is clear that $y_\theta \not\le (\varphi^{\leftarrow}_L)^*(h)(x_\lambda)$ for any $\theta \in \beta^*_\lambda(L)$. Thus $\varphi^{\rightarrow}_L(y_\theta) \not\le h(\varphi^{\rightarrow}_L(x_\lambda))$. Hence $h(\varphi^{\rightarrow}_L(x_\lambda)) \in \psi_{\varphi^{\rightarrow}_L(y_\mu)}(L^Y)$. Also, by $z_\eta \not\le (\varphi^{\leftarrow}_L)^*(h)(y_\mu)$, it follows that $\varphi^{\rightarrow}_L(z_\eta) \not\le h(\varphi^{\rightarrow}_L(y_\mu))$ and

$$\varphi^{\rightarrow}_L(z_\eta) \not\le (h \diamond h)(\varphi^{\rightarrow}_L(x_\lambda)) \ge g(\varphi^{\rightarrow}_L(x_\lambda)).$$

So $z_\eta \not\le (\varphi^{\leftarrow}_L)^*(g)(x_\lambda)$. Therefore

$$(\varphi^{\leftarrow}_L)^*(g)(x_\lambda) \le [(\varphi^{\leftarrow}_L)^*(h) \diamond (\varphi^{\leftarrow}_L)^*(h)](x_\lambda).$$

By the arbitrariness of $x_\lambda \in \beta^*(L^X)$, it concludes that $(\varphi^{\leftarrow}_L)^*(g) \le (\varphi^{\leftarrow}_L)^*(h) \diamond (\varphi^{\leftarrow}_L)^*(h)$. □

Theorem 5.2. *Let $(Y, \mathcal{U}_Y)$ be an L-convex quasi-uniform space and $\varphi : X \to Y$ be a mapping. The set*

$$\varphi^{\leftarrow}_L(\mathcal{U}_Y) = \{f \in \Re(L^X) : \exists g \in \mathcal{U}_Y, f \le (\varphi^{\leftarrow}_L)^*(g)\}$$

is an L-convex quasi-uniformity on L^X.

Proof. (LCQU1). Let $(f_0)_X$ be the smallest element in $\Re(L^X)$ and $(f_0)_Y$ be the smallest element in $\Re(L^Y)$. Since $(f_0)_Y \in \mathcal{U}_Y$ by (LCQU1) and $\varphi_L^{\leftarrow}((f_0)_Y) = (f_0)_X$, it is clear that $(f_0)_X \in \varphi_L^{\leftarrow}(\mathcal{U}_Y)$.

(LCQU2). If $f \in \varphi_L^{\leftarrow}(\mathcal{U}_Y)$, then there is a $g \in \mathcal{U}_Y$ such that $f \leq (\varphi_L^{\leftarrow})^*(g)$. By $g \in \mathcal{U}_Y$, there is an $h \in \mathcal{U}_Y$ such that $g \leq h \diamond h$. By (2) of Lemma 5.1, it is clear that

$$f \leq (\varphi_L^{\leftarrow})^*(g) \leq (\varphi_L^{\leftarrow})^*(h) \diamond (\varphi_L^{\leftarrow})^*(h)$$

Since $(\varphi_L^{\leftarrow})^*(h) \in (\varphi_L^{\leftarrow})^*(\mathcal{U}_Y)$, (LCQU2) holds for $(\varphi_L^{\leftarrow})^*(\mathcal{U}_Y)$.

(LCQU3). Let $\{f_i\}_{i\in I}^{dir} \subseteq \varphi_L^{\leftarrow}(\mathcal{U}_Y)$. For any $i \in I$, there is an $h_i \in \mathcal{U}_Y$ such that $f_i \leq (\varphi_L^{\leftarrow})^*(h_i)$. Let

$$\Psi_i = \{h_i \in \mathcal{U}_Y : f_i \leq (\varphi_L^{\leftarrow})^*(h_i)\}.$$

For any $x_\lambda \in \beta^*(L^X)$, we have

$$f_i(x_\lambda) \leq \bigwedge_{h_i\in\Psi_i} ((\varphi_L^{\leftarrow})^*(h_i))(x_\lambda) = \varphi_L^{\leftarrow}(\bigwedge_{h_i\in\Psi_i} h_i(\varphi_L^{\rightarrow}(x_\lambda))) = (\varphi_L^{\leftarrow})^*(\bigwedge_{h_i\in\Psi_i} h_i)(x_\lambda).$$

So $f_i \leq (\varphi_L^{\leftarrow})^*(\bigwedge_{h_i\in\Psi_i} h_i)$. Since $\{f_i\}_{i\in I}^{dir} \in \varphi_L^{\leftarrow}(\mathcal{U}_Y)$, the set $\{\bigwedge_{h_i\in\Psi_i} h_i\} \subseteq \mathcal{U}_Y$ is also directed. Hence $\bigvee_{i\in I}^{dir} \bigwedge_{h_i\in\Psi_i} h_i \in \mathcal{U}_Y$ and

$$\bigvee_{i\in I}^{dir} f_i \leq \bigvee_{i\in I}^{dir} (\varphi_L^{\leftarrow})^*(\bigwedge_{h_i\in\Psi_i} h_i) = \varphi_L^{\leftarrow} \circ (\bigvee_{i\in I}^{dir} \bigwedge_{h_i\in\Psi_i} h_i) \circ \varphi_L^{\rightarrow} = (\varphi_L^{\leftarrow})^*(\bigvee_{i\in I}^{dir} \bigwedge_{h_i\in\Psi_i} h_i).$$

Therefore $\bigvee_{i\in I}^{dir} f_i \in \varphi_L^{\leftarrow}(\mathcal{U}_Y)$.

So $\varphi_L^{\leftarrow}(\mathcal{U}_Y)$ is an L-convex quasi-uniformity. $\square$

Theorem 5.3. *$C_{\varphi_L^{\leftarrow}(\mathcal{U}_Y)} = \varphi_L^{\leftarrow}(C_{\mathcal{U}_Y})$ for any L-convex quasi-uniform space $(Y, \mathcal{U}_Y)$ and any $\varphi : X \to Y$.*

Proof. If $A \in C_{\varphi_L^{\leftarrow}(\mathcal{U}_Y)}$, then $A = A_{\varphi_L^{\leftarrow}(\mathcal{U}_Y)}$. To prove that $A \in \varphi_L^{\leftarrow}(C_{\mathcal{U}_Y})$, let $x_\lambda \in \beta^*(L^X)$ with $A \in \psi_{x_\lambda}(L^X)$. Then $\varphi_L^{\rightarrow}(A) \in \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^Y)$ and $A_{\varphi_L^{\leftarrow}(\mathcal{U}_Y)} \in \psi_{x_\lambda}(L^X)$. It follows from (2) of Lemma 4.3 that any $\eta \in \beta_\lambda^*(L)$ implies an $f \in \varphi_L^{\leftarrow}(\mathcal{U}_Y)$ such that $A \leq f(x_\eta)$. Further, since $f \in \varphi_L^{\leftarrow}(\mathcal{U}_Y)$, there is a $g \in \mathcal{U}_Y$ such that $f \leq (\varphi_L^{\leftarrow})^*(g)$. Thus $A \leq (\varphi_L^{\leftarrow})^*(g)(x_\eta)$ followed by $\varphi_L^{\rightarrow}(A) \leq g(\varphi_L^{\rightarrow}(x_\eta))$. Hence $\varphi_L^{\rightarrow}(A)_{\mathcal{U}_Y} \in \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^Y)$. So $\varphi_L^{\rightarrow}(A)_{\mathcal{U}_Y} \leq \varphi_L^{\rightarrow}(A)$ by (4) of Lemma 3.3. Therefore $\varphi_L^{\rightarrow}(A) \in C_{\mathcal{U}_Y}$ and $\varphi_L^{\leftarrow}(\varphi_L^{\rightarrow}(A)) \in \varphi_L^{\leftarrow}(C_{\mathcal{U}_Y})$.

Since $\varphi_L^{\rightarrow}(A)_{\mathcal{U}_Y} \in \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^Y)$, it follows that $\varphi_L^{\leftarrow}(\varphi_L^{\rightarrow}(A)_{\mathcal{U}_Y}) \in \psi_{x_\lambda}(L^Y)$. Thus $\varphi_L^{\leftarrow}(\varphi_L^{\rightarrow}(A)_{\mathcal{U}_Y}) \leq A$ by (4) of Lemma 3.3. This implies that $A = \varphi_L^{\leftarrow}(\varphi_L^{\rightarrow}(A)_{\mathcal{U}_Y}) \in \varphi_L^{\leftarrow}(C_{\mathcal{U}_Y})$. Therefore $C_{\varphi_L^{\leftarrow}(\mathcal{U}_Y)} \subseteq \varphi_L^{\leftarrow}(C_{\mathcal{U}_Y})$.

Conversely, if $A \in \varphi_L^{\leftarrow}(C_{\mathcal{U}_Y})$, then $\varphi_L^{\rightarrow}(A) \in C_{\mathcal{U}_Y}$. To prove that $A \in C_{\varphi_L^{\leftarrow}(\mathcal{U}_Y)}$, let $x_\lambda \in \beta^*(L^X)$ with $A \in \psi_{x_\lambda}(L^X)$. Then $\varphi_L^{\rightarrow}(A) \in \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^Y)$ and

$$\varphi_L^{\rightarrow}(A)_{\mathcal{U}_Y} = co_{C_{\mathcal{U}_Y}}(\varphi_L^{\rightarrow}(A)) = \varphi_L^{\rightarrow}(A) \in \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^Y).$$

It follows from (2) of Lemma 4.3 that any $\eta \in \beta_\lambda^*(L)$ implies some $g \in \mathcal{U}_Y$ such that $\varphi_L^{\rightarrow}(A) \leq g(\varphi_L^{\rightarrow}(x_\eta))$. Thus $A \leq (\varphi_L^{\leftarrow})^*(g)(x_\eta)$. Notice that $(\varphi_L^{\leftarrow})^*(g) \in \varphi_L^{\leftarrow}(\mathcal{U}_Y)$. It follows from (2) of Lemma 4.3 that

$$co_{\varphi_L^{\leftarrow}(\mathcal{U}_Y)}(A) = A_{\varphi_L^{\leftarrow}(\mathcal{U}_Y)} \in \psi_{x_\lambda}(L^X).$$

Further, it follows from (4) of Lemma 3.3 that $A_{\varphi_L^{\leftarrow}(\mathcal{U}_Y)} \leq A$. Hence $A = A_{\varphi_L^{\leftarrow}(\mathcal{U}_Y)} \in C_{\varphi_L^{\leftarrow}(\mathcal{U}_Y)}$. As a result, $\varphi_L^{\leftarrow}(C_{\mathcal{U}_Y}) \subseteq C_{\varphi_L^{\leftarrow}(\mathcal{U}_Y)}$. Therefore $C_{\varphi_L^{\leftarrow}(\mathcal{U}_Y)} = \varphi_L^{\leftarrow}(C_{\mathcal{U}_Y})$. $\square$

Next, we introduce the notion of L-convex quasi-uniform preserving mapping.

Definition 5.4. Let $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$ be L-convex quasi-uniform spaces. A mapping $\varphi : X \to Y$ is called an L-convex quasi-uniformity preserving mapping, if any $g \in \mathcal{U}_Y$ implies some $f \in \mathcal{U}_X$ such that $g(\varphi_L^{\rightarrow}(x_\lambda)) \in \psi_{\varphi_L^{\rightarrow}(y_\mu)}(L^Y)$ for all $x_\lambda, y_\mu \in \beta^*(L^X)$ with $f(x_\lambda) \in \psi_{y_\mu}(L^X)$.

The category of L-convex quasi-uniform spaces and L-convex quasi-uniformitity preserving mappings is denoted by L-**CQUS**.

Theorem 5.5. *Let $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$ be L-convex quasi-uniform spaces. For a mapping $\varphi : X \to Y$, the following results are equivalent.*

(1) φ is an L-convex quasi-uniform preserving mapping.

(2) For any $g \in \mathcal{U}_Y$, there is an $f \in \mathcal{U}_X$ such that $(\varphi_L^{\leftarrow})^*(g) \le f$.

(3) $(\varphi_L^{\leftarrow})^*(\Phi) \subseteq \mathcal{U}_X$ for any L-convex quasi-uniform subbase Φ of $\mathcal{U}_Y$.

(4) $(\varphi_L^{\leftarrow})^*(\mathcal{B}) \subseteq \mathcal{U}_X$ for any L-convex quasi-uniform base $\mathcal{B}$ of $\mathcal{U}_Y$.

(5) $(\varphi_L^{\leftarrow})^*(\mathcal{U}_Y) \subseteq \mathcal{U}_X$.

Proof. (1) $\Rightarrow$ (2). Let $g \in \mathcal{U}_Y$. Since φ is an L-convex quasi-uniformity preserving mapping, there is an $f \in \mathcal{U}_X$ such that $g(\varphi_L^{\rightarrow}(x_\lambda)) \in \psi_{\varphi_L^{\rightarrow}(y_\mu)}(L^Y)$ for all $x_\lambda, y_\mu \in \beta^*(L^X)$ with $f(x_\lambda) \in \psi_{y_\mu}(L^X)$. To prove that $(\varphi_L^{\leftarrow})^*(g) \le f$, let $x_\lambda, y_\eta \in \beta^*(L^X)$ with $y_\eta \not\le f(x_\lambda)$, we next verify that $y_\eta \not\le (\varphi_L^{\leftarrow})^*(g)(x_\lambda)$.

Since $y_\eta \not\le f(x_\lambda)$, there is a $\mu \in \beta^*(\eta)$ such that $y_\mu \not\le f(x_\lambda)$. Thus $f(x_\lambda) \in \psi_{y_\mu}(L^X)$ and $g(\varphi_L^{\rightarrow}(x_\lambda)) \in \psi_{\varphi_L^{\rightarrow}(y_\mu)}(L^Y)$. Hence $\varphi_L^{\rightarrow}(y_\eta) \not\le g(\varphi_L^{\rightarrow}(x_\lambda))$ which implies that $y_\eta \not\le (\varphi_L^{\leftarrow})^*(g)(x_\lambda)$. So $(\varphi_L^{\leftarrow})^*(g)(x_\lambda) \le f(x_\lambda)$. Therefore $(\varphi_L^{\leftarrow})^*(g) \le f$.

(2) $\Rightarrow$ (3). It directly follows from (LCQU2).

(3) $\Rightarrow$ (4). It is clear since an L-convex quasi-uniform base is an L-convex quasi-uniform subbase.

(4) $\Rightarrow$ (5). For any $g \in \mathcal{U}_Y$, there is an $h \in \mathcal{B}$ such that $g \le h$. Then $(\varphi_L^{\leftarrow})^*(g) \le (\varphi_L^{\leftarrow})^*(h) \in \mathcal{U}_X$ followed by $(\varphi_L^{\leftarrow})^*(g) \in \mathcal{U}_X$. Thus $(\varphi_L^{\leftarrow})^*(\mathcal{U}_Y) \subseteq \mathcal{U}_X$.

(5) $\Rightarrow$ (1). If $g \in \mathcal{U}_Y$, then $(\varphi_L^{\leftarrow})^*(g) \in (\varphi_L^{\leftarrow})^*(\mathcal{U}_Y) \subseteq \mathcal{U}_X$. By (LCQU2), there is an $f \in \mathcal{U}_X$ such that $(\varphi_L^{\leftarrow})^*(g) \le f \diamond f \le f$.

For all $x_\lambda, y_\mu \in \beta^*(L^X)$ with $f(x_\lambda) \in \psi_{y_\mu}(L^X)$, it is clear that $(\varphi_L^{\leftarrow})^*(g) \in \psi_{y_\mu}(L^X)$. Thus $g(\varphi_L^{\rightarrow}(x_\lambda)) \in \psi_{\varphi_L^{\rightarrow}(y_\mu)}(L^Y)$. Therefore φ is an L-convex quasi-uniformity preserving mapping. □

Theorem 5.6. *Let $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$ be L-convex quasi-uniform spaces. If $\varphi : X \to Y$ is an L-convex quasi-uniformity preserving mapping, then $\varphi : (X, \mathcal{C}_{\mathcal{U}_X}) \to (Y, \mathcal{C}_{\mathcal{U}_Y})$ is an L-convexity preserving mapping.*

Proof. Let $B \in \mathcal{C}_{\mathcal{U}_Y}$. To prove that $\varphi_L^{\leftarrow}(B) \in \mathcal{C}_{\mathcal{U}_X}$, let $x_\lambda \in \beta^*(L^X)$ with $\varphi_L^{\leftarrow}(B) \in \psi_{x_\lambda}(L^X)$. Then $co_{\mathcal{C}_{\mathcal{U}_Y}}(B) = B \in \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^Y)$. It follows from (2) of Lemma 4.3 that any $\eta \in \beta_\lambda^*(L)$ implies some $g \in \mathcal{U}_Y$ such that $B \le g(\varphi_L^{\rightarrow}(x_\eta))$. Thus $\varphi_L^{\leftarrow}(B) \le (\varphi_L^{\leftarrow})^*(g)(x_\eta)$. By (2) of Theorem 5.5, there is an $f \in \mathcal{U}_X$ such that $\varphi_L^{\leftarrow}(B) \le (\varphi_L^{\leftarrow})^*(g)(x_\eta) \le f(x_\eta)$.

We say that $co_{\mathcal{C}_{\mathcal{U}_X}}(\varphi_L^{\leftarrow}(B)) \in \psi_{x_\lambda}(L^X)$. Otherwise, $co_{\mathcal{C}_{\mathcal{U}_X}}(\varphi_L^{\leftarrow}(B)) \notin \psi_{x_\lambda}(L^X)$. It follows from (2) of Lemma 4.3 that there is an $\eta \in \beta_\lambda^*(L)$ such that $\varphi_L^{\leftarrow}(B) \not\le h(x_\eta)$ for any $h \in \mathcal{U}_X$. In particular, $\varphi_L^{\leftarrow}(B) \not\le f(x_\eta)$. It is a contradiction. Thus $co_{\mathcal{C}_{\mathcal{U}_X}}(\varphi_L^{\leftarrow}(B)) \in \psi_{x_\lambda}(L^X)$. Hence $\varphi_L^{\leftarrow}(B) = co_{\mathcal{C}_{\mathcal{U}_X}}(\varphi_L^{\leftarrow}(B)) \in \mathcal{C}_{\mathcal{U}_X}$. Therefore φ is an L-convexity preserving mapping. □

Lemma 5.7. *Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L-convex spaces. If $\varphi : X \to Y$ is an L-convexity preserving mapping, then $(\varphi_L^{\leftarrow})^*(f_A) = f_{\varphi_L^{\leftarrow}(A)}$ for any $A \in \mathcal{C}_Y$.*

Proof. Let $x_\lambda \in \beta^*(L^X)$. For any $D \in \mathcal{C}_Y \cap \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^Y)$ with $D \le A$, we have $\varphi_L^{\leftarrow}(D) \in \mathcal{C}_X \cap \psi_{x_\lambda}(L^X)$ and $\varphi_L^{\leftarrow}(D) \le \varphi_L^{\leftarrow}(A) \in \mathcal{C}_X$. Thus

$$\begin{aligned}
(\varphi_L^{\leftarrow})^*(f_A)(x_\lambda) &= \varphi_L^{\leftarrow}(f_A(\varphi_L^{\rightarrow}(x_\lambda))) \\
&= \bigvee\{\varphi_L^{\leftarrow}(D) : D \in \mathcal{C}_Y \cap \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^Y), D \le A\} \\
&\le \bigvee\{G : G \in \mathcal{C}_X \cap \psi_{x_\lambda}(L^X), G \le \varphi_L^{\leftarrow}(A)\} \\
&= f_{\varphi_L^{\leftarrow}(A)}(x_\lambda).
\end{aligned}$$

Conversely, let $E \in \mathcal{C}_X \cap \psi_{x_\lambda}(L^X)$ with $E \le \varphi_L^{\leftarrow}(A)$. Then $\varphi_L^{\rightarrow}(E) \le A$ and $\varphi_L^{\rightarrow}(E) \in \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^Y)$. Also, $co_{\mathcal{U}_{\mathcal{C}_Y}}(\varphi_L^{\rightarrow}(E)) = co_{\mathcal{C}_Y}(\varphi_L^{\rightarrow}(E)) \le A$ by Theorem 4.8.

Let $z_\theta \in \beta^*(E)$. Then $\varphi_L^{\rightarrow}(z_\theta) \le \varphi_L^{\rightarrow}(E) \in \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^X)$ which implies $\varphi_L^{\rightarrow}(z_\theta) \in \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^X)$. Define a mapping $f_{\varphi_L^{\rightarrow}(z_\theta)} : \beta^*(L^Y) \to L^Y$ by for any $y_\mu \in \beta^*(L^Y)$,

$$f_{\varphi_L^{\rightarrow}(z_\theta)}(y_\mu) = \begin{cases} \varphi_L^{\rightarrow}(z_\theta), & y_\mu \ge \varphi_L^{\rightarrow}(x_\lambda), \\ \underline{\perp}, & \text{otherwise.} \end{cases}$$

It is clear that $f_{\varphi_L^{\rightarrow}(z_\theta)} \in \Re(L^Y)$ and $f_{\varphi_L^{\rightarrow}(z_\theta)}(\varphi_L^{\rightarrow}(x_\lambda)) \in \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^Y)$. Further, it follows from Theorem 4.8 and (3) of Lemma 4.3 that

$$co_{C_Y}(\varphi_L^{\rightarrow}(z_\theta)) = co_{C_Y}(f_{\varphi_L^{\rightarrow}(z_\theta)}(\varphi_L^{\rightarrow}(x_\lambda)) = (f_{\varphi_L^{\rightarrow}(z_\theta)}(\varphi_L^{\rightarrow}(x_\lambda)))_{\mathcal{U}_{C_Y}} \in C_Y \cap \psi_{\varphi_L^{\rightarrow}(x_\lambda)}(L^Y).$$

In addition, $co_{C_Y}(\varphi_L^{\rightarrow}(z_\theta)) \leq co_{C_Y}(\varphi_L^{\rightarrow}(E)) \leq A$. Thus $co_{C_Y}(\varphi_L^{\rightarrow}(z_\theta)) \leq f_A(\varphi_L^{\rightarrow}(x_\lambda))$ which implies that

$$z_\theta \leq \varphi_L^{\leftarrow}(co_{C_Y}(\varphi_L^{\rightarrow}(z_\theta))) \leq \varphi_L^{\leftarrow}(f_A(\varphi_L^{\rightarrow}(x_\lambda))) = (\varphi_L^{\leftarrow})^*(f_A)(x_\lambda).$$

Hence $E = \bigvee_{z_\theta \in \beta^*(E)} z_\theta \leq (\varphi_L^{\leftarrow})^*(f_A)(x_\lambda)$. Therefore

$$f_{\varphi_L^{\leftarrow}(A)}(x_\lambda) = \bigvee \{E \in C_X \cap \psi_{x_\lambda}(L^X) : E \leq \varphi_L^{\leftarrow}(A)\} \leq (\varphi_L^{\leftarrow})^*(f_A)(x_\lambda).$$

In conclusion, $f_{\varphi_L^{\leftarrow}(A)}(x_\lambda) = (\varphi_L^{\leftarrow})^*(f_A)(x_\lambda)$ for any $x_\lambda \in \beta^*(L^X)$. That is, $f_{\varphi_L^{\leftarrow}(A)} = (\varphi_L^{\leftarrow})^*(f_A)$, as desired. □

Theorem 5.8. *Let (X, C_X) and (Y, C_Y) be L-convex spaces. If $\varphi : X \to Y$ is an L-convexity preserving mapping, then $\varphi : (X, \mathcal{U}_{C_X}) \to (Y, \mathcal{U}_{C_Y})$ is an L-convex quasi-uniformity preserving mapping.*

Proof. Let $g \in \mathcal{U}_{C_Y}$. Then there is a set $A \in C_Y$ such that $g(z_\eta) \leq f_A(y_\mu) \in C_Y$ for any $y_\mu \in \beta^*(L^Y)$. Thus $\varphi_L^{\leftarrow}(A) \in C_X$ and $g(\varphi_L^{\rightarrow}(x_\lambda)) \leq f_A(\varphi_L^{\rightarrow}(x_\lambda)) \in C_Y$ for any $x_\lambda \in \beta^*(L^X)$. This implies that

$$f_{\varphi_L^{\leftarrow}(A)}(x_\lambda) = (\varphi_L^{\leftarrow})^*(g)(x_\lambda) \leq \varphi_L^{\leftarrow}(f_A(\varphi_L^{\rightarrow}(x_\lambda))) = (\varphi_L^{\leftarrow})^*(f_A)(x_\lambda) \in C_X.$$

By this result and Lemma 5.7, it follows that $(\varphi_L^{\leftarrow})^*(f_A) = f_{\varphi_L^{\leftarrow}(A)} \in \mathcal{U}_{C_X}$.

For any $y_\mu \in \beta^*(L^X)$ with $(\varphi_L^{\leftarrow})^*(g_A)(x_\lambda) \in \psi_{y_\mu}(L^X)$, it follows that

$$g(\varphi_L^{\rightarrow}(x_\lambda)) = \varphi_L^{\rightarrow}(\varphi_L^{\leftarrow}(g(\varphi_L^{\rightarrow}(x_\lambda)))) = \varphi_L^{\rightarrow}((\varphi_L^{\leftarrow})^*(g_A)(x_\lambda)) \in \psi_{\varphi_L^{\rightarrow}(y_\mu)}(L^Y).$$

So $\varphi : (X, \mathcal{U}_{C_X}) \to (Y, \mathcal{U}_{C_Y})$ is an L-convex quasi-uniformity preserving mapping. □

Based on Theorems 4.6 and 5.8, we define a functor $\mathbb{U} : L\text{-}\mathbf{CS} \to L\text{-}\mathbf{CQUS}$ by

$$\mathbb{U}((X, C)) = (X, \mathcal{U}_C), \quad \mathbb{U}(f) = f.$$

Based on Theorem 4.8, $\mathbb{U}$ is an injective functor. Thus the category L-**CS** can be embedded as a subcategory into the category of L-**CQUS**. Further, based on Theorem 3.10, the category L-**CBRS** can be embedded as a subcategory into the category of L-**CQUS**.

6. Conclusions

We define a new remotehood space, namely L-convex β^*-remotehood space, which can be used to characterize L-convex space and L-convex remotehood space. Further, we present the notion of β^*-remotehood mappings, based on whose properties, we further introduce L-convex quasi-uniform space. We find that L-convexities and L-convex quasi-uniformities are mutually induced. In addition, we prove that the category of L-convex spaces and the category of L-convex β^*-remotehood spaces can be embedded into the category L-convex quasi-uniform spaces as subcategories.

In [11], Shi defined pointwise quasi-uniformities by fuzzy remote mappings in fuzzy set theory. As we can see, fuzzy remote mappings are different with L-fuzzy β^*-remote mappings. But they possess similar properties. Thus, it may be worth to discuss relations between Shi's quasi-uniformity and L-convex quasi-uniformity. In addition, it may also be worth to consider how to characterize L-convex quasi-uniformities by fuzzy proximities or fuzzy metrics.

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