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Barreledness in Locally Convex Direct Sum Cones

M.R. Motallebi[a]

[a]*Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran*

Abstract. We investigate the direct sum of barrel subsets in locally convex cones and present necessary and sufficient conditions for the barrelness of subsets in direct sum cones. This leads us to prove a direct sum locally convex cone topology is barreled if and only if its components are barreled.

1. Introduction

In the theory of locally convex cones, linear functionals may take infinite values $+\infty$ which is one of the basic coordinates in this theory; for instance, this makes the study of barreledness be in an interesting manner [8]. The topic of direct sums, has been generalized for locally convex cones in sources [2-7], includes the duality discussion and indicating that every linear functional on direct sum cone is written as the product of functionals on its components. To achieve the goal of this paper, the latter is essential and allows the study of barreledness for direct sum cone topologies. We consider the barrel subsets of direct sum cones and discuss their connections with the direct sum of barrel sets in the components; in particular, we obtain necessary and sufficient conditions for the barreledness of locally convex direct sum cones.

An *ordered cone* is a set $\mathcal{P}$ endowed with an addition $(a, b) \longmapsto a+b$ and a scalar multiplication $(\alpha, a) \longmapsto \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, there is a neutral element $0 \in \mathcal{P}$, and for the scalar multiplication the usual associative and distributive properties hold, that is, $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha + \beta)a = \alpha a + \beta a$, $\alpha(a + b) = \alpha a + \alpha b$, $1a = a$, $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. In addition, the cone $\mathcal{P}$ carries a (partial) order, i.e., a reflexive transitive relation $\leq$ that is compatible with the algebraic operations, that is $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. For example, the extended scalar field $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ of real numbers is an ordered cone. We consider the usual order and algebraic operations in $\overline{\mathbb{R}}$; in particular, $\alpha + \infty = +\infty$ for all $\alpha \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. In any cone $\mathcal{P}$, equality is obviously such an order, hence all results about ordered cones apply to cones without order structures as well.

A *full locally convex cone* $(\mathcal{P}, \mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, i.e., a subset of positive elements that is directed downward, closed for addition and multiplication by (strictly) positive scalars. The elements v of $\mathcal{V}$ define *upper* (*lower*) *neighborhoods* for the elements of $\mathcal{P}$ by $v(a) = \{b \in \mathcal{P} : b \leq a + v\}$ (respectively, $(a)v = \{b \in \mathcal{P} : a \leq b + v\}$), creating the *upper*, respectively *lower topologies* on $\mathcal{P}$. Their common refinement is called the *symmetric topology.* We assume all elements of $\mathcal{P}$ to be *bounded below*, i.e., for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \rho v$ for some $\rho > 0$. Finally, a *locally convex*

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Email address: motallebi@uma.ac.ir (M.R. Motallebi)

cone $(\mathcal{P},\mathcal{V})$ is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system $\mathcal{V}$.

For a locally convex cone $(\mathcal{P},\mathcal{V})$ the collection of all sets $\widetilde{v}\subset\mathcal{P}^2$, where $\widetilde{v}=\{(a,b): a\le b+v\}$ for all $v\in\mathcal{V}$, defines a *convex quasi-uniform structure* on $\mathcal{P}$. On the other hand, every convex quasi-uniform structure leads to a full locally convex cone, including $\mathcal{P}$ as a subcone and induces the same convex quasi-uniform structure. For details see [1, Ch I, 5.2].

For cones $\mathcal{P}$ and $\mathcal{Q}$, a map $t:\mathcal{P}\to\mathcal{Q}$ is called a *linear operator*, if $t(a+b)=t(a)+t(b)$ and $t(\alpha a)=\alpha t(a)$ for all $a,b\in\mathcal{P}$ and $\alpha\ge 0$. If $\mathcal{V}$ and $\mathcal{W}$ are abstract neighborhood systems on $\mathcal{P}$ and $\mathcal{Q}$, a linear operator $t:\mathcal{P}\to\mathcal{Q}$ is called *uniformly continuous* if for every $w\in\mathcal{W}$ there is $v\in\mathcal{V}$ such that $t(a)\le t(b)+w$ whenever $a\le b+v$. Uniform continuity implies continuity with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$.

Remark 1.1. In the extended real numbers $\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ we consider the usual order and algebraic operations, in particular $a+\infty=+\infty$ for all $a\in\overline{\mathbb{R}}$, $\alpha\cdot(+\infty)=+\infty$ for all $\alpha>0$ and $0\cdot(+\infty)=0$. Endowed with the neighborhood system $\mathcal{V}=\{\varepsilon\in\mathbb{R}:\varepsilon>0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. For every $v:=\varepsilon\in\mathcal{V}$ and $a\in\overline{\mathbb{R}}$, we have $v(a)\cap(a)v=[a-\varepsilon,a+\varepsilon]$; in particular, $v(+\infty)\cap(+\infty)v=\{+\infty\}$. If $\mathcal{P}$ is a locally convex cone, then the set of all uniformly continuous linear functionals $\mu:\mathcal{P}\to\overline{\mathbb{R}}$ is a cone called the *dual cone* of $\mathcal{P}$ and denoted by $\mathcal{P}^*$. In particular, $\overline{\mathbb{R}}^*=\{\lambda\in\mathbb{R}:\lambda\ge 0\}\cup\{\bar{0}\}$, where $\bar{0}(a)=0$ for all $a\in\mathbb{R}$ and $\bar{0}(+\infty)=+\infty$ [9, 2.2].

If $(\mathcal{P},\mathcal{V})$ is a locally convex cone, then a convex subset U of $\mathcal{P}^2$ is called a *barrel*, if it satisfies:

(U_1) For every $b\in\mathcal{P}$ there is a neighborhood $v\in\mathcal{V}$ such that for every $a\in v(b)\cap(b)v$ there is $\lambda>0$ with $(a,b)\in\lambda\mathrm{U}$.

(U_2) If $(a,b)\notin\mathrm{U}$, then there is $\mu\in\mathcal{P}^*$ such that $\mu(c)\le\mu(d)+1$ for all $(c,d)\in\mathrm{U}$ and $\mu(a)>\mu(b)+1$.

A locally convex cone $(\mathcal{P},\mathcal{V})$ is called *barreled* if for every barrel $\mathrm{U}\subset\mathcal{P}^2$ and every element $b\in\mathcal{P}$ there are a neighborhood $v\in\mathcal{V}$ and a $\lambda>0$ such that $(a,b)\in\lambda\mathrm{U}$ for all $a\in v(b)\cap(b)v$ [8].

2. Barreledness and direct sum cones

Let $\mathcal{P}_\gamma$, $\gamma\in\Gamma$ be cones and put $\mathcal{P}=\times_{\gamma\in\Gamma}\mathcal{P}_\gamma$. For elements $a,b\in\mathcal{P}$, $a=\times_{\gamma\in\Gamma}a_\gamma$, $b=\times_{\gamma\in\Gamma}b_\gamma$ and $\alpha\ge0$ we set $a+b=\times_{\gamma\in\Gamma}(a_\gamma+b_\gamma)$ and $\alpha a=\times_{\gamma\in\Gamma}(\alpha a_\gamma)$. With these operations $\mathcal{P}$ is a cone which is called the *product cone* of $\mathcal{P}_\gamma$. The subcone of the product cone $\mathcal{P}$ spanned by $\cup\mathcal{P}_\gamma$ (more precisely, by $\cup j_\gamma(\mathcal{P}_\gamma)$, where $j_\gamma:\mathcal{P}_\gamma\to\mathcal{P}$ is the injection mapping) is said to be the *direct sum cone* of $\mathcal{P}_\gamma$ and denoted by $\mathcal{Q}=\sum_{\gamma\in\Gamma}\mathcal{P}_\gamma$. If $(\mathcal{P}_\gamma,\mathcal{V}_\gamma)$, $\gamma\in\Gamma$ be a family of locally convex cones, then $\mathcal{W}=\times_{\gamma\in\Gamma}\mathcal{V}_\gamma$ leads to the *finest locally convex cone topology* on $\mathcal{Q}$ such that the all injection mappings j_γ are uniformly continuous:

Definition 2.1. For elements $a,b\in\mathcal{Q}$, $a=\sum_{\gamma\in\Delta}a_\gamma$, $b=\sum_{\gamma\in\Theta}b_\gamma$ and $w\in\mathcal{W}$, $w=\times_{\gamma\in\Gamma}v_\gamma$, we set

$$a\le_\Gamma b+w$$

if $a_\gamma\le_\gamma b_\gamma+\alpha_\gamma v_\gamma$ *for all* $\gamma\in\Delta\cup\Theta$, *where* $\sum_{\gamma\in\Delta\cup\Theta}\alpha_\gamma\le1$.

The subsets $\{(a,b)\in\mathcal{Q}^2: a\le_\Gamma b+w\}$ for all $w\in\mathcal{W}$ describe the finest convex quasi-uniform structure on $\mathcal{Q}$ which makes every injection mapping uniformly continuous. According to [1, Ch I, 5.4], there exists a full cone $\mathcal{Q}\oplus\mathcal{W}_0$ with abstract neighborhood system $\mathrm{W}=\{0\}\oplus\mathcal{W}$, whose neighborhoods yield the same quasi-uniform structure on $\mathcal{Q}$. The elements $w\in\mathcal{W}$, $w=\times_{\gamma\in\Gamma}v_\gamma$ form a basis for W in the following sense: For every $\mathrm{w}\in\mathrm{W}$ there is $w\in\mathcal{W}$ such that $a\le_\Gamma b+w$ for $a,b\in\mathcal{Q}$ implies that $a\le_\Gamma b\oplus\mathrm{w}$. The locally convex cone topology on $\mathcal{Q}$ induced by $\mathcal{W}$ is called the *locally convex direct sum cone of* $(\mathcal{P}_\gamma,\mathcal{V}_\gamma)$ and denoted by $(\mathcal{Q},\mathcal{W})$. For details see [3].

Theorem 2.2. *If* $\mathcal{Q}=\sum_{\gamma\in\Gamma}\mathcal{P}_\gamma$ *is a locally convex direct sum cone, then* $\mu\in\mathcal{Q}^*$ *if and only if* $\mu=\times_{\gamma\in\Gamma}\mu_\gamma$, *where* $\mu_\gamma=\mu\circ j_\gamma\in\mathcal{P}_\gamma^*$ *for all* $\gamma\in\Gamma$ *and* $\mathcal{Q}^*$, $\mathcal{P}_\gamma^*$ *are the dual cones of* $\mathcal{Q}$ *and* $\mathcal{P}_\gamma$, *respectively.*

Proof. See [7, Theorem 3.10]. $\square$

Lemma 2.3. *If for each $\gamma \in \Gamma$, U_γ is a convex subset of $\mathcal{P}_\gamma^2$ then $\sum_{\gamma\in\Gamma} \mathrm{U}_\gamma$ is a barrel in $(\mathcal{Q}, \mathcal{W})$ if and only if U_γ is a barrel in $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ for all $\gamma \in \Gamma$.*

Proof. Suppose $\sum_{\gamma\in\Gamma} \mathrm{U}_\gamma$ is a barrel and for $\gamma \in \Gamma$, let $b_\gamma \in \mathcal{P}_\gamma$. From (U_1), there exists $w \in \mathcal{W}$, $w = \times_{\gamma\in\Gamma} v_\gamma$ such that for every $a \in w(b_\gamma) \cap (b_\gamma)w$ there is $\lambda > 0$ with $(a, b_\gamma) \in \lambda\left(\sum_{\gamma\in\Gamma} \mathrm{U}_\gamma\right)$. If $a_\gamma \in v_\gamma(b_\gamma) \cap (b_\gamma)v_\gamma$, then $a_\gamma \in w(b_\gamma) \cap (b_\gamma)w$; whence $(a_\gamma, b_\gamma) \in \lambda \mathrm{U}_\gamma$, i.e., U_γ satisfies in (U_1). If $(a_\gamma, b_\gamma) \notin \mathrm{U}_\gamma$ then $(a_\gamma, b_\gamma) \notin \mathrm{U}$ so there is $\mu \in \mathcal{Q}^*$ such that $\mu \circ j_\gamma(a_\gamma) > \mu \circ j_\gamma(b_\gamma) + 1$ and $\mu \circ j_\gamma(c_\gamma) \le \mu \circ j_\gamma(d_\gamma) + 1$ for all $(c_\gamma, d_\gamma) \in \mathrm{U}_\gamma$. Because $\mu_\gamma = \mu \circ j_\gamma \in \mathcal{P}_\gamma^*$ for all $\gamma \in \Gamma$ by Theorem 2.2, (U_2) holds.

Conversely, for each $\gamma \in \Gamma$, let U_γ be a barrel and $b \in \mathcal{Q}$, $b = \sum_{\gamma\in\Theta} b_\gamma$. For every $\gamma \in \Gamma$, there is $v_\gamma \in \mathcal{V}_\gamma$ such that for every $a_\gamma \in v_\gamma(b_\gamma) \cap (b_\gamma)v_\gamma$ there is $\lambda_\gamma > 0$ with $(a_\gamma, b_\gamma) \in \lambda_\gamma \mathrm{U}_\gamma$. If we put $w \in \mathcal{W}$, $w = \times_{\xi\in\Gamma} v_\xi$ where $v_\xi = v_\gamma$ for $\gamma \in \Theta$ and $v_\xi \in \mathcal{V}_\gamma$ otherwise; then for every $a \in w(b) \cap (b)w$, $a = \sum_{\gamma\in\Delta} a_\gamma$, we have

$$(a, b) = \left(\sum_{\gamma\in\Delta} a_\gamma, \sum_{\gamma\in\Theta} b_\gamma\right) \in \sum_{\gamma\in\Delta\cup\Theta} \lambda_\gamma \mathrm{U}_\gamma \subset \left(\sum_{\gamma\in\Delta\cup\Theta} \lambda_\gamma\right) \sum_{\gamma\in\Delta\cup\Theta} \mathrm{U}_\gamma,$$

i.e., $\sum_{\gamma\in\Gamma} \mathrm{U}_\gamma$ satisfies in (U_1). If $(a, b) \notin \sum_{\gamma\in\Gamma} \mathrm{U}_\gamma$, $a = \sum_{\gamma\in\Delta} a_\gamma$, $b = \sum_{\gamma\in\Theta} b_\gamma$, then there is a finite set $\Lambda \subset \Delta \cup \Theta$ such that $(a_\gamma, b_\gamma) \notin \mathrm{U}_\gamma$ for all $\gamma \in \Lambda$ so there is a $\mu_\gamma \in \mathcal{P}_\gamma^*$ such that

$$\mu_\gamma(a_\gamma) > \mu_\gamma(b_\gamma) + 1 \text{ and } \mu_\gamma(c_\gamma) \le \mu_\gamma(d_\gamma) + 1 \text{ for all } (c_\gamma, d_\gamma) \in \mathrm{U}_\gamma \tag{1}$$

Let n_Λ be the number of elements in Λ. If we put $\mu = \times_{\xi\in\Gamma} \mu_\xi$ where $\mu_\xi = (1/n_\Lambda)\mu_\gamma \in \mathcal{P}_\gamma^*$ for all $\xi \in \Lambda$ and $\mu_\xi = 0$ otherwise, then from (1), $\mu(a) > \mu(b) + 1$ and $\mu(c) \le \mu(d) + 1$ for all $(c, d) \in \sum_{\gamma\in\Gamma} \mathrm{U}_\gamma$. By Theorem 2.2, $\mu \in \mathcal{Q}^*$ hence (U_2) holds. $\square$

Proposition 2.4. *If U is a convex subset of $\mathcal{Q}^2$, then the following are equivalent:*

(a) U *is a barrel in* $(\mathcal{Q}, \mathcal{W})$.
(b) $\sum_{\gamma\in\Gamma} \mathrm{U} \cap \mathcal{P}_\gamma^2$ *is a barrel in* $(\mathcal{Q}, \mathcal{W})$.
(c) *For every* $\gamma \in \Gamma$, $\mathrm{U} \cap \mathcal{P}_\gamma^2$ *is a barrel in* $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$.

Proof. By applying Lemma 2.3 for $\mathrm{U}_\gamma := \mathrm{U} \cap \mathcal{P}_\gamma^2$ for all $\gamma \in \Gamma$, parts (b) and (c) are equivalent. Let U be a barrel and $b \in \mathcal{Q}$, $b = \sum_{\gamma\in\Theta} b_\gamma$. There is $w \in \mathcal{W}$, $w = \times_{\gamma\in\Gamma} v_\gamma$ such that for every $a \in w(b) \cap (b)w$ there is $\lambda > 0$ with $(a, b) \in \lambda \mathrm{U}$ which yields $(a_\gamma, b_\gamma) \in \lambda(\mathrm{U} \cap \mathcal{P}_\gamma^2)$ for all $\gamma \in \Delta \cup \Theta$, hence

$$(a, b) = \sum_{\gamma\in\Delta\cup\Theta} (a_\gamma, b_\gamma) \in \lambda\left(\sum_{\gamma\in\Delta\cup\Theta} \mathrm{U} \cap \mathcal{P}_\gamma^2\right) \subset \lambda\left(\sum_{\gamma\in\Gamma} \mathrm{U} \cap \mathcal{P}_\gamma^2\right),$$

i.e., (U_1) holds for $\sum_{\gamma\in\Gamma} \mathrm{U} \cap \mathcal{P}_\gamma^2$. For (U2), if $(a, b) \notin \sum_{\gamma\in\Gamma} \mathrm{U}_\gamma$, $a = \sum_{\gamma\in\Delta} a_\gamma$, $b = \sum_{\gamma\in\Theta} b_\gamma$ then, by the convexity of U, $(a, b) \notin n_{(\Delta\cup\Theta)}\mathrm{U}$, so there is $\mu \in \mathcal{Q}^*$ such that

$$\mu(a) > \mu(b) + n_{(\Delta\cup\Theta)} \text{ and } \mu(c) \le \mu(d) + n_{(\Delta\cup\Theta)} \quad \text{for all} \quad (c, d) \in \mathrm{U}. \tag{2}$$

If we put $\mu = \times_{\xi\in\Gamma} \mu_\xi$, where $\mu_\xi = \left(1/n_{(\Delta\cup\Theta)}\right) \mu_\gamma$ for $\xi \in \Delta\cup\Theta$ and $\mu_\xi = 0$ otherwise; then by (2), $\mu(a) > \mu(b)+1$ and $\mu(c) \le \mu(d) + 1$ for all $(c, d) \in \sum_{\gamma\in\Gamma} \mathrm{U}_\gamma$. Thus (a) implies (b). Assume (b) and let $b \in \mathcal{Q}$, $b = \sum_{\gamma\in\Theta} b_\gamma$. To show (a), by (U_1), there is $w \in \mathcal{W}$, $w = \times_{\gamma\in\Gamma} v_\gamma$ such that for every $a \in w(b) \cap (b)w$ there is $\lambda > 0$ with $(a, b) \in \lambda \sum_{\gamma\in\Gamma} \mathrm{U}_\gamma$, which yields $(a_\gamma, b_\gamma) \in \lambda \mathrm{U}$ for all $\gamma \in \Delta \cup \Theta$, so

$$(a, b) = \sum_{\gamma\in\Delta\cup\Theta} \left(a_\gamma, b_\gamma\right) \in \lambda\left(\sum_{\gamma\in\Delta\cup\Theta} \mathrm{U}\right) \subset n_{(\Delta\cup\Theta)}\lambda \mathrm{U},$$

i.e., (U_1) holds for U. For (U_2), if $(a, b) \notin \mathrm{U}$, $a = \sum_{\gamma\in\Delta} a_\gamma$, $b = \sum_{\gamma\in\Theta} b_\gamma$, then there is a finite set $\Lambda \subset \Delta \cup \Theta$ such that

$$\left(\sum_{\gamma\in\Lambda} a_\gamma, \sum_{\gamma\in\Lambda} b_\gamma\right) \notin \left(n_{(\Delta\cup\Theta)} - n_\Lambda\right)\left(\sum_{\gamma\in\Gamma} \mathrm{U} \cap \mathcal{P}_\gamma^2\right),$$

so there is $\mu \in \mathcal{Q}^*$ such that

$$\sum_{\gamma\in\Lambda} \mu_\gamma(a_\gamma) > \sum_{\gamma\in\Lambda} \mu_\gamma(b_\gamma) + n_{(\Delta\cup\Theta)} - n_\Lambda$$

and

$$\sum_{\gamma\in\Lambda} \mu_\gamma(c_\gamma) \leq \sum_{\gamma\in\Lambda} \mu_\gamma(d_\gamma) + n_{(\Delta\cup\Theta)} - n_\Lambda \quad \text{for all} \quad (c, d) \in \sum_{\gamma\in\Gamma} \mathrm{U}_\gamma.$$

If we put $\mu = \times_{\xi\in\Gamma}\mu_\xi$ where $\mu_\xi = (1/(n_{(\Delta\cup\Theta)} - n_\Lambda))\, \mu_\gamma \in \mathcal{P}_\gamma^*$ for all $\xi \in \Lambda$ and $\mu_\xi = 0$ otherwise; then $\mu \in \mathcal{Q}^*$ by Theorem 2.2 and we have

$$\begin{aligned} \mu(a) &= \left(1/(n_{(\Delta\cup\Theta)} - n_\Lambda)\right)\sum_{\gamma\in\Lambda} \mu_\gamma(a_\gamma) \\ &> (1/(n_{(\Delta\cup\Theta)} - n_\Lambda))\sum_{\gamma\in\Lambda} \mu_\gamma(b_\gamma) + 1 \\ &= \mu(b) + 1 \end{aligned}$$

and

$$\begin{aligned} \mu(c) &= \left(1/(n_{(\Delta\cup\Theta)} - n_\Lambda)\right)\sum_{\gamma\in\Lambda} \mu_\gamma(c_\gamma) \\ &\leq \left(1/(n_{(\Delta\cup\Theta)} - n_\Lambda)\right)\sum_{\gamma\in\Lambda} \mu_\gamma(d_\gamma) + 1 \\ &= \mu(d) + 1 \qquad (\text{for all} \quad (c, d) \in \mathrm{U}) \end{aligned}$$

□

Theorem 2.5. *If for each $\gamma \in \Gamma$, $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is a locally convex cone, then the following are equivalent:*

(a) *The direct sum cone topology $(\mathcal{Q}, \mathcal{W}) = \left(\sum_{\gamma\in\Gamma} \mathcal{P}_\gamma, \times_{\gamma\in\Gamma}\mathcal{V}_\gamma\right)$ is barreled.*
(b) *For every barrel subset* U *of $\mathcal{Q}^2$, there is a $w \in \mathcal{W}$ and a $\lambda > 0$ such that*

$$(a, b) \in \lambda\left(\sum_{\gamma\in\Gamma} \mathrm{U} \cap \mathcal{P}_\gamma^2\right) \quad \textit{for all} \quad a \in w(b) \cap (b)w.$$

(c) *For every $\gamma \in \Gamma$, $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is barreled.*

Proof. By Proposition 2.4, (a) implies (b). Assume (b), fix $\gamma \in \Gamma$ and let $\mathrm{U}_\gamma \subset \mathcal{P}_\gamma^2$ be a barrel. If we set $\mathrm{U} = \sum_{\xi\in\Gamma} \mathrm{U}_\xi$, where $\mathrm{U}_\xi = \mathrm{U}_\gamma$ for $\xi = \gamma$ and $\mathrm{U}_\xi = \mathcal{P}_\gamma^2$ otherwise; then U is a barrel in $(\mathcal{Q}, \mathcal{W})$; so for $b_\gamma \in \mathcal{P}_\gamma$, there is a $w \in \mathcal{W}$ and a $\lambda > 0$ such that $(a, b) \in \lambda\mathrm{U}$ for all $a \in w(b_\gamma) \cap (b_\gamma)w$, i.e., $(a_\gamma, b_\gamma) \in \lambda\mathrm{U}_\gamma$ for all $a_\gamma \in v_\gamma(b_\gamma) \cap (b_\gamma)v_\gamma$. That is, (c) is obtained from (b). Finally, let (c) holds, U be a barrel in $(\mathcal{Q}, \mathcal{W})$ and $b \in \mathcal{Q}$, $b = \sum_{\gamma\in\Theta} b_\gamma$. By Proposition 2.4 (c), for every $\gamma \in \Gamma$, there is a neighborhood $v_\gamma \in \mathcal{V}_\gamma$ and a $\lambda_\gamma > 0$ such that $(a_\gamma, b_\gamma) \in \lambda_\gamma\, \mathrm{U}_\gamma$ for all $a_\gamma \in v_\gamma(b_\gamma) \cap (b_\gamma)v_\gamma$. Put $w \in \mathcal{W}$, $w = \times_{\xi\in\Gamma}v_\xi$ where

$$v_\xi = \begin{cases} v_\gamma & \text{for} \quad \xi \in \Theta, \\ \left(1/\lambda_\gamma\right) v_\gamma & \text{for} \quad \xi \in \Gamma \setminus \Theta. \end{cases}$$

If $a \in w(b) \cap (b)w$, $a = \sum_{\gamma\in\Delta} a_\gamma$, then

$$\begin{cases} a_\gamma \in \left(\alpha_\gamma v_\gamma\right)\left(b_\gamma\right) \bigcap \left(b_\gamma\right)\left(\alpha_\gamma v_\gamma\right) & \text{for} \quad \gamma \in \Theta, \\ \lambda_\gamma a_\gamma \in \left(\alpha_\gamma v_\gamma\right)\left(0_{\mathcal{P}_\gamma}\right) \bigcap \left(0_{\mathcal{P}_\gamma}\right)\left(\alpha_\gamma v_\gamma\right) & \text{for} \quad \gamma \in \Delta \setminus \Theta, \end{cases}$$

which yields

$$\begin{cases} \left(a_\gamma, b_\gamma\right) \in \lambda_\gamma \mathrm{U}_\gamma & \text{for} \quad \gamma \in \Theta, \\ \left(a_\gamma, 0_{\mathcal{P}_\gamma}\right) \in \alpha_\gamma \mathrm{U}_\gamma & \text{for} \quad \gamma \in \Delta \setminus \Theta. \end{cases}$$

Therefore

$$\begin{aligned} (a, b) &= \sum_{\gamma\in\Theta} \left(a_\gamma, b_\gamma\right) + \sum_{\gamma\in\Delta\setminus\Theta} \left(a_\gamma, 0_{\mathcal{P}_\gamma}\right) \\ &\in \sum_{\gamma\in\Theta} \lambda_\gamma \mathrm{U}_\gamma + \sum_{\gamma\in\Delta\setminus\Theta} \alpha_\gamma \mathrm{U}_\gamma \\ &\subset \left(\sum_{\gamma\in\Theta} \lambda_\gamma + \sum_{\gamma\in\Delta\setminus\Theta} \alpha_\gamma\right) \mathrm{U}_\gamma \\ &\subset \left(\sum_{\gamma\in\Theta} \lambda_\gamma + 1\right) \mathrm{U}, \end{aligned}$$

since U is convex and $\sum_{\gamma\in\Gamma\setminus\Theta} \alpha_\gamma \leq 1$. That is, (a) holds. □

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *weakly cone-complete* if for all $b \in \mathcal{P}$ and $v \in \mathcal{V}$, every sequence $(a_n)_{n\in\mathbb{N}}$ in $v(b) \cap (b)v$ that converges to b in the symmetric topology of $\mathcal{P}$ and $\alpha_n \geq 0$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$, there is $a \in v(b) \cap (b)v$ such that

$$\mu(a) = \sum_{n=1}^{\infty} \alpha_n \mu(a_n) \tag{3}$$

for all $\mu \in \mathcal{P}^*$ with $\mu(b) < +\infty$. A *neighborhood base* for a locally convex cone $(\mathcal{P}, \mathcal{V})$ is a subset $\mathcal{V}_0$ of $\mathcal{V}$ such that for every $v \in \mathcal{V}$ there exists some $v_0 \in \mathcal{V}_0$ with $v_0 \leq v$. According to [8, Theorem 2.3], every weakly cone-complete locally convex cone $(\mathcal{P}, \mathcal{V})$ with a countable neighborhood base is barreled.

Example 2.6. With the neighborhood system $\mathcal{V} = \{\varepsilon \in \overline{\mathbb{R}} : \varepsilon > 0\}$, the cone $\overline{\mathbb{R}}$ is weakly cone-complete with the countable neighborhood base $\mathcal{V}_0 = \{\frac{1}{n} : n \in \mathbb{N}\}$, so it is barreled. For, let $b \in \overline{\mathbb{R}}$, $v \in \mathcal{V}, v = \varepsilon$, $(a_n)_{n\in\mathbb{N}} \subseteq \varepsilon(b) \cap (b)\varepsilon$ converges to b in the symmetric topology of $\overline{\mathbb{R}}$ and let $\sum_{n=1}^{\infty} \alpha_n = 1$, $\alpha_n \geq 0$. If $b = +\infty$ then $\varepsilon(+\infty) \cap (+\infty)\varepsilon = \{+\infty\}$, so $a_n = +\infty$ for all $n \in \mathbb{N}$. From Remark 1.1, we have $\mu(+\infty) < +\infty$ for $\mu \in \overline{\mathbb{R}}^*$ if and only if $\mu = 0$, hence (3) holds for $a = +\infty$. If $b \neq +\infty$ then $\mu(b) < +\infty$ all $\mu \in \overline{\mathbb{R}}^*$ by Remark 1.1. By the assumption we have $b - \varepsilon \leq a_n \leq b + \varepsilon$ for all $n \in \mathbb{N}$, which yield $a_n \neq +\infty$ for all $n \in \mathbb{N}$ and $b - \varepsilon \leq \sum_{n=1}^{\infty} \alpha_n a_n \leq b + \varepsilon$, i.e., $\sum_{n=1}^{\infty} \alpha_n a_n$ is convergent and $+\infty \neq a = \sum_{n=1}^{\infty} \alpha_n a_n \in \varepsilon(b) \cap (b)\varepsilon$. Then $\mu(a) = \lambda(\sum_{n=1}^{\infty} \alpha_n a_n) = \sum_{n=1}^{\infty} \alpha_n \mu(a_n)$ for all $\mu \in \overline{\mathbb{R}}^*$ with $\mu = \lambda \geq 0$ and $\overline{0}(a) = \overline{0}(a_n) = 0$ for $\mu = \overline{0}$.

Now, if we set $\mathcal{Q} = \sum_{\gamma\in\Gamma} \mathcal{P}_\gamma$ and $\mathcal{W} = \times_{\gamma\in\Gamma} \mathcal{V}_\gamma$, where $\mathcal{P}_\gamma = \overline{\mathbb{R}}$ and $\mathcal{V}_\gamma = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$ for all $\gamma \in \Gamma$, then $(\mathcal{Q}, \mathcal{W})$ is barreled by Theorem 2.5, since $(\mathcal{P}_\gamma, \mathcal{V}_\gamma), \gamma \in \Gamma$ are barreled.

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