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Perfect Fluid Spacetimes and Gradient Solitons

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Abstract. The main object of this paper is to characterize the perfect fluid spacetimes if its metrics are Ricci solitons, gradient Ricci solitons, gradient A-Einstein solitons and gradient Schouten solitons.

1. Introduction

Let M be an n-dimensional Lorentzian manifold equipped with the Lorentzian metric g of signature $(\underbrace{+,+,\ldots,+}_{(n-1)\text{times}},-)$. The notion of generalized Robertson-Walker (GRW) spacetimes was introduced by Alías, Romero and Sánchez [1, 2] in 1995. An n-dimensional Lorentzian manifold with $n \geq 3$ is termed as a GRW spacetime if it can be written as a warped product of an open interval I of $\Re$ (set of real numbers) and a Riemannian manifold $\mathcal{M}^*$ of dimension $(n-1)$, that is, $M = -I \times \mathfrak{f}^2\mathcal{M}^*$, where $\mathfrak{f} > 0$ is a smooth function, named as warping function or scale factor. If the dimension of $\mathcal{M}^*$ is three and possesses the constant curvature, then the spacetime reduces to Robertson-Walker (RW) spacetime. Thus, the GRW spacetime is a natural extension of RW spacetime on which the standard cosmology is modeled. It also includes the Einstein-de Sitter spacetime, the Friedman cosmological models, the static Einstein spacetime, the de Sitter spacetime, and have applications as inhomogeneous spacetimes admitting an isotropic radiation [36]. The geometrical and physical properties of GRW spacetimes have been exhaustively presented in ([12], [13], [14], [20], [26], [28], [34]-[36]). In this series, Chen [14] and Mantica and Molinari [28] have given the following deepest results on GRW spacetime. We write their results in the following theorems.

Theorem 1.1. *[14] A Lorentzian n-manifold with $n \geq 3$ is a generalized Robertson-Walker spacetime if and only if it admits a timelike concircular vector field.*

Theorem 1.2. *[28] A Lorentzian manifold of dimension $n \geq 3$ is a GRW spacetime if and only if it admits a unit timelike torseforming vector field: $\nabla_k u_j = \varphi(g_{ij} + u_j u_i)$, that is also an eigenvector of the Ricci tensor.*

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An n-dimensional Lorentzian manifold is said to be a perfect fluid spacetime if its non-vanishing Ricci tensor S satisfies

$$S = \alpha g + \beta A \otimes A, \tag{1}$$

where α, β are scalar fields (not simultaneously zero), ρ is a vector field metrically equivalent to the 1-form A, that is, $g(X, \rho) = A(X)$ for all X and $g(\rho, \rho) = -1$. Here ρ is the unit timelike vector field (also called the velocity vector field) of the perfect fluid spacetime. Every RW spacetime is a perfect fluid spacetime [31], although in case of $n = 4$, the GRW spacetime is a perfect fluid spacetime if and only if it is a RW spacetime. In differential geometry, the Ricci tensor satisfying equation (1) is known as a quasi Einstein manifold [11]. For more details, we refer ([6], [15], [27], [37], [41]) and the references therein.

The problem of finding a canonical metric on a smooth manifold motivates Hamilton [22] to introduce the notion of Ricci flow. A metric of a (semi-) Riemannian manifold M satisfying an evolution equation $\frac{\partial}{\partial t} g_{ij}(t) = -2S_{ij}$ is called Ricci flow [22]. The self-similar solutions to the Ricci flow are the Ricci solitons. A metric of M is said to be a Ricci soliton [21] if it satisfies

$$\mathfrak{L}_V g + 2S + 2\lambda g = 0 \tag{2}$$

for some vector field V and real scalar λ. Here $\mathfrak{L}_V$ denotes the Lie derivative operator along the soliton vector field V. We denote (g, V, λ) as a Ricci soliton on M. If λ is positive, negative, or zero, then the Ricci soliton to be expanding, shrinking, or steady, respectively. Particularly, if V is Killing or identically zero, then the Ricci soliton is trivial and the manifold is Einstein. Also, if the soliton vector V of the Ricci soliton is the gradient of some smooth function $-f$, that is, $V = -Df$, then equation (2) reduces to

$$Hess\, f - S - \lambda g = 0, \tag{3}$$

where $Hess$ denotes the Hessian and D is the gradient operator of g. The metric satisfying equation (3) is said to be a gradient Ricci soliton. The smooth function f is called the potential function of the gradient Ricci soliton.

The metric g of a (semi-)Riemannian manifold M satisfying the equation

$$\mathfrak{L}_V g + 2S + (2\lambda - r)g + 2\mu_1 \eta \otimes \eta = 0 \tag{4}$$

is called an η-Einstein soliton [5]. Here r is the scalar curvature of g, V is the Einstein soliton vector of η-Einstein soliton, η is a 1-form and $\mu_1 \in \mathfrak{R}$. If we replace the 1-form η with A in (4), then we call η-Einstein soliton as A-Einstein soliton. If $r =$ constant, then A-Einstein soliton reduces to A-Ricci soliton. Also, if $r =$ constant and $\mu_1 = 0$, then A-Einstein soliton converts into the Ricci soliton. If the Einstein soliton vector of A-Einstein soliton is the gradient of a smooth function, then A-Einstein soliton becomes the gradient A-Einstein soliton.

Let M be an n-dimensional (semi)-Riemannian manifold and ξ, $\rho \in \mathfrak{R}$, $\xi \neq 0$. Then the metric g of M is said to be ξ-Einstein soliton [10] if it satisfies the equation

$$2S + \mathfrak{L}_V g = 2\xi r g + 2\lambda g.$$

The ξ-Einstein soliton is said to be shrinking, steady, or expanding if λ is positive, zero, or negative. If the soliton vector field of the ξ-Einstein soliton is the gradient of some smooth function f, then ξ-Einstein soliton reduces to gradient ξ-Einstein soliton and above equation takes the form

$$S + \nabla^2 f = \xi r g + \lambda g, \tag{5}$$

where $\nabla^2 f = Hess\, f$. The gradient ξ-Einstein soliton is said to be a gradient Schouten soliton if $\xi = \frac{1}{2(n-1)}$ and λ is the real constant.

Although most of the works of solitons have been done in Riemannian setting, the Ricci solitons and gradient Ricci solitons have been considered in the Lorentzian category ([3], [4], [7]).

In [10], Catino and Mazzieri proved that every complete gradient steady Schouten soliton is trivial, hence Ricci flat and a complete three-dimensional gradient shrinking Schouten soliton is isometric to a finite quotient of either $\mathbb{S}^3$ or $\mathbb{R}^3$ or $\mathbb{R}\times\mathbb{S}^2$ (see, Theorem 1.5 and Theorem 1.6 of [10]). Pina and Menezes [32] showed that if a gradient Schouten soliton is both complete, conformal to a Euclidean metric, and rotationally symmetric, then it is isometric to $\mathbb{R}\times\mathbb{S}^{n-1}$.

The above studies motivate us to study the properties of perfect fluid spacetimes if the Lorentzian metrics are Ricci, gradient Ricci, gradient A-Einstein and gradient Schouten solitons. We structure the present paper as:

Section 2 is devoted to the study of Ricci solitons in a perfect fluid spacetime. The properties of the gradient Ricci solitons and gradient A-Einstein solitons in a perfect fluid spacetime are presented in Section 3 and Section 4, respectively. In Section 5, we study the geometrical and physical properties of gradient Schouten soliton in a perfect fluid spacetime.

2. Ricci soliton in perfect fluid spacetimes

This section deals with the study of Ricci solitons in perfect fluid spacetimes M.

Since $g(\rho,\rho) = -1$ and therefore its covariant derivative gives $g(\nabla_X\rho,\rho) = 0$ for all $X \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the collection of all smooth vector fields of M. Now, $(\nabla_Y A)(X) = \nabla_Y A(X) - A(\nabla_Y X) = \nabla_Y g(X,\rho) - g(\nabla_Y X,\rho) = g(X,\nabla_Y\rho)$, $\forall\ X,\ Y \in \mathfrak{X}(M)$. From (1), we get

$$r = n\alpha - \beta \tag{6}$$

and

$$QX = \alpha X + \beta A(X)\rho,\ \ \forall\ X \in \mathfrak{X}(M). \tag{7}$$

The Einstein's field equations without cosmological constant assume the form

$$2S = 2\kappa T + r\,g, \tag{8}$$

where κ is the gravitational constant and T is the energy momentum tensor. In case of perfect fluid spacetime T is defined as

$$T = pg + (p+\mu)A\otimes A,\ \ p+\mu \neq 0,\ \ \mu > 0 \tag{9}$$

where p and μ represent the isotropic pressure and energy density of the perfect fluid spacetime, respectively (see p. 69, [23]).

A Lorentzian manifold, whose Ricci tensor is of the form (1) is often called a perfect fluid spacetime with

$$\beta = \kappa(p+\mu),\quad \alpha = \frac{\kappa(p-\mu)}{2-n}, \tag{10}$$

where equations (8) and (9) are used.

The notion of torse-forming vector field on a Riemannian manifold has been introduced by Yano [40]. A vector field v on a (semi-)Riemannian manifold M is said to be a torse-forming vector field if for $X \in \mathfrak{X}(M)$ it satisfies the equation

$$\nabla_X v = \phi\, X + l(X)v,$$

where ϕ is scalar and l is a 1-form such that $g(X,v) = l(X)$ for all X. For more details we refer [16], [30] and the references therein.

Now, we prove the following theorem.

Theorem 2.1. *A Lorentzian manifold of dimension $n \geq 4$ admitting a Ricci soliton, whose soliton field is a unit timelike torseforming vector field, is a perfect fluid spacetime.*

Proof. Let M be a Lorentzian manifold of dimension n and $\nu = V$ is a unit timelike torse-forming vector field. Then from the above equation we can easily deduce that $\nabla_X V = \phi\{X + A(X)V\}$, where ϕ is a scalar and A is a 1-form such that $g(X, V) = A(X)$ for all X. Since $(\mathfrak{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V)$, therefore the last equation with (2) assume the form

$$S(X, Y) = -(\phi + \lambda)g(X, Y) - \phi A(X)A(Y),$$

which implies that the Lorentzian manifold with Ricci soliton is a perfect fluid spacetime. Hence the statement of Theorem 2.1 is proved. □

The Theorem 1.2 together with Theorem 2.1 state the following:

Corollary 2.2. *A GRW spacetime admitting Ricci soliton is a perfect fluid spacetime.*

Remark 2.3. *It may be mentioned that Mantica, Molinari and De [29] proved that a GRW spacetime with* $div\, C = 0$ *is a perfect fluid spacetime, where C denotes the conformal curvature tensor. In Corollary 2.2, we replaced the curvature condition* $div\, C = 0$ *by Ricci soliton.*

Theorem 2.4. *A Ricci soliton* (g, ρ, λ) *in a perfect fluid spacetime is shrinking, expanding, or steady if* $\alpha >, <,$ *or* $= \beta$*, respectively. Also, the integral curves generated by* ρ *are geodesics.*

Proof. From equations (1), (2) and $V = \rho$, we have

$$g(\nabla_X \rho, Y) + g(X, \nabla_Y \rho) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{11}$$

Setting $Y = X = \rho$ in equation (11), we have

$$S(\rho, \rho) + \lambda g(\rho, \rho) = 0,$$

since $g(\nabla_X \rho, \rho) = 0$. Again, from (1) we infer $S(\rho, \rho) = -\alpha + \beta$, since $g(\rho, \rho) = A(\rho) = -1$. Thus, $\lambda = \beta - \alpha$. This shows that the Ricci soliton on a perfect fluid spacetime is expanding, shrinking, or steady if $\beta >, <,$ or $= \alpha$, respectively. Again, putting $Y = \rho$ in equation (11), we find

$$\nabla_\rho \rho = 2(\beta - \alpha - \lambda). \tag{12}$$

Since $\lambda = \beta - \alpha$, therefore the above equation becomes $\nabla_\rho \rho = 0$. This reflects that the integral curves generated by the unit timelike vector field are geodesics. □

Remark 2.5. *It is observed that* α *and* β *defined in (1) are smooth functions, but* $\alpha - \beta$ *is a constant.*

Next, we prove the following corollaries.

Corollary 2.6. *Let an n-dimensional perfect fluid spacetime with* $n \geq 4$ *admit a Ricci soliton* (g, ρ, λ)*. Then the soliton* (g, ρ, λ) *is shrinking, expanding, or steady if the equation of state of the perfect fluid spacetime is* $(n-1)p + (n-3)\mu < 0,\ > 0,$ *or* $= 0$*, respectively.*

Proof. Let an $n\,(\geq 4)$-dimensional perfect fluid spacetime M admit a Ricci soliton (g, ρ, λ). From Theorem 2.4 and equation (10), we conclude that the soliton (g, ρ, λ) to be steady, shrinking or expanding if the equation of state of the perfect fluid spacetime is governed by $(n-1)p + (n-3)\mu = 0$, < 0 or > 0 , respectively, where $\mu > 0$ and $p + \mu \neq 0$ (p. 69, [23]). □

Remark 2.7. *For* $n = 4$*, if the soliton* (g, ρ, λ) *is*
(*i*) *shrinking, then the equation of state* $w = \frac{p}{\mu} < -\frac{1}{3}$ *is required for acceleration of cosmic inflation. More generally, the expansion of the universe is accelerating for any equation of state* $w < -\frac{1}{3}$ *([9], [24]).*
(*ii*) *expanding, then the equation of state* $w = \frac{p}{\mu} > -\frac{1}{3} > -1$ *describes the scienrio of non-phantom dark energy.*
(*iii*) *steady, then the equation of state* $w = \frac{p}{\mu} = -\frac{1}{3}$*, which shows the limiting case of dark energy and the limiting case of violating the strong energy condition in the context of perfect fluid spacetimeun.*

Corollary 2.8. *Let a perfect fluid spacetime M admit a steady Ricci soliton (g, ρ, λ). Then the velocity vector field ρ of M is Ricci inheritance vector if and only if it is Killing. Additionally, if α is a non-zero constant then ρ is Ricci collineation if and only if ρ is Killing.*

Proof. Let the perfect fluid spacetimes confess a steady Ricci soliton. Then from Theorem 2.4, we find $\lambda = 0 \implies \alpha = \beta \neq 0$. Using this relation in (1) and then taking the Lie derivative of the obtained equation along the velocity vector field ρ, we have

$$\begin{aligned}(\mathfrak{L}_\rho S)(X,Y) &= \rho(\alpha)\{g(X,Y) + A(X)A(Y)\} \\ &\quad +\alpha\{(\mathfrak{L}_\rho g)(X,Y) + (\mathfrak{L}_\rho A)(X)A(Y) + (\mathfrak{L}_\rho A)(Y)A(X)\}\end{aligned} \tag{13}$$

for all $X, Y \in \mathfrak{X}(M)$. Again, the Lie derivative of $A(X) = g(X, \rho)$ along ρ gives

$$(\mathfrak{L}_\rho A)(X) = (\mathfrak{L}_\rho g)(X, \rho) + g(X, \mathfrak{L}_\rho \rho). \tag{14}$$

In consequence of equations (1), (12), (14), $(\mathfrak{L}_\rho g)(X, \rho) = g(\nabla_X \rho, \rho) + g(X, \nabla_\rho \rho)$ and $g(\nabla_X \rho, \rho) = 0$, equation (13) assumes the form

$$\mathfrak{L}_\rho S = \frac{\rho(\alpha)}{\alpha} S + \alpha\, \mathfrak{L}_\rho g.$$

A vector field U on a perfect fluid spacetime is said to be a Ricci inheritance vector (briefly, *RIV*) ([19], p. 161) if it satisfies the relation $\mathfrak{L}_U S = 2\mathfrak{a}S$, where $\mathfrak{a}$ is some smooth function on M. A *RIV* reduces to Ricci collineation (briefly, *RC*) if $\mathfrak{a} = 0$, otherwise it is called a proper *RIV*. Hence the proof. □

3. Gradient Ricci soliton in a perfect fluid spacetime

Theorem 3.1. *Let the perfect fluid spacetimes admit a gradient Ricci soliton and its velocity vector field ρ is Killing. Then the state equation of the perfect fluid spacetime is governed by either*
(i) $p = \frac{3-n}{n-1}\mu$ and the soliton is steady, or
(ii) $p = -\mu +$ constant or the gradient Ricci soliton is trivial.

Proof. Let us suppose that the soliton vector field V of the Ricci soliton (g, V, λ) in an n-dimensional perfect fluid spacetime M is a gradient of some smooth function $-f$. Then equation (2) takes the form

$$\nabla_X Df = QX + \lambda X \tag{15}$$

for all $X \in \mathfrak{X}(M)$. The equation (15) along with the relation

$$R(X,Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df \tag{16}$$

give

$$R(X,Y)Df = (\nabla_X Q)(Y) - (\nabla_Y Q)(X). \tag{17}$$

The covariant derivative of (7) gives

$$(\nabla_X Q)(Y) = X(\alpha)Y + X(\beta)A(Y)\rho + \beta(\nabla_X A)(Y)\rho + \beta A(Y)\nabla_X \rho. \tag{18}$$

In view of equations (17) and (18), we lead

$$\begin{aligned}R(X,Y)Df &= X(\alpha)Y - Y(\alpha)X + \{X(\beta)A(Y) - Y(\beta)A(X) \\ &\quad +\beta(\nabla_X A)(Y) - \beta(\nabla_Y A)(X)\}\rho + \beta\{A(Y)\nabla_X \rho - A(X)\nabla_Y \rho\}.\end{aligned} \tag{19}$$

Taking a set of orthonormal frame field and contracting equation (19) along the vector field X, we have

$$S(Y, Df) = (1-n)Y(\alpha) + Y(\beta) + \rho(\beta)A(Y) + \beta[(\nabla_\rho A)(Y) - (\nabla_Y A)(\rho) + A(Y)\, div\rho]. \tag{20}$$

Again, from (1) we have

$$S(Y, Df) = \alpha Y(f) + \beta A(Y)\rho(f). \tag{21}$$

Setting $Y = \rho$ in equations (20) and (21) and then equating the values of $S(\rho, Df)$, we find

$$(\alpha - \beta)\rho(f) = (1 - n)\rho(\alpha) - \beta\, div\rho. \tag{22}$$

Suppose that the velocity vector field ρ of the perfect fluid spacetime is Killing, that is, $\mathfrak{L}_\rho g = 0$. Then we get $div\,\rho = 0$, $\mathfrak{L}_\rho p = 0$ and $\mathfrak{L}_\rho \mu = 0$ (see [19], p. 89). Thus, from equations (10) and (22) we get

$$(\alpha - \beta)\rho(f) = 0, \tag{23}$$

which shows that either $\alpha = \beta$ or $\rho(f) = 0$ on a perfect fluid spacetime with the gradient Ricci soliton. Now, we divide our study into two cases as:
Case I. We suppose that $\alpha = \beta$ and $\rho(f) \neq 0$ and therefore from equation (10), we conclude that

$$p = \frac{3-n}{n-1}\mu.$$

This gives the equation of state in a perfect fluid spacetime. If $n = 4$, then the equation of state is $\mu + 3p = 0$, which implies that it is the limiting case of dark energy and limiting case of violating the strong energy condition in the context of perfect fluid spacetimes. Also, $\lambda = \beta - \alpha = 0$ and hence the gradient Ricci soliton is steady.
Case II. We consider that $\rho(f) = 0$ and $\alpha \neq \beta$. The covariant derivative of $g(\rho, Df) = 0$ along the vector field X gives

$$g(\nabla_X \rho, Df) = -[\lambda + (\alpha - \beta)]A(X), \tag{24}$$

where equations (7) and (15) are used. Since the velocity vector field ρ is Killing in a perfect fluid spacetime, that is, $g(\nabla_X \rho, Y) + g(X, \nabla_Y \rho) = 0$. Setting $Y = \rho$ in this equation, we find that $g(X, \nabla_\rho \rho) = 0$ because $g(\nabla_X \rho, \rho) = 0$. Thus, we conclude that $\nabla_\rho \rho = 0$. Changing X with ρ in equation (24) and using the last equation, we infer that

$$\lambda = \beta - \alpha. \tag{25}$$

This reflects that the gradient Ricci soliton in a perfect fluid spacetime is shrinking or expanding if $\beta < \alpha$ or $\beta > \alpha$, respectively. Next, the equations (20) and (21) together with the hypothesis take the form

$$S(Y, Df) = \alpha Y(f) \tag{26}$$

and

$$S(Y, Df) = (1 - n)Y(\alpha) + Y(\beta). \tag{27}$$

In view of (25)-(27), we conclude

$$\alpha Y(f) + (n-2)Y(\alpha) = 0 \Longleftrightarrow \alpha Df + (n-2)D\alpha = 0. \tag{28}$$

Considering a set of orthonormal frame field and contracting equation (17) along the vector field X and using the fact that trace$\{Y \longrightarrow (\nabla_Y Q)X\} = \frac{1}{2}\nabla_X r$, we lead

$$S(Y, Df) = -\frac{1}{2}Y(r) = \alpha Y(f), \tag{29}$$

where equation (26) has been used. Again from equations (6) and (25), we infer that

$$Y(r) = (n-1)Y(\alpha). \tag{30}$$

In consequence of equations (28)-(30), we conclude that

$$(n-3)Y(\alpha) = 0. \tag{31}$$

Since the dimension of the perfect fluid spacetime is greater than or equal to four, therefore equation (31) shows that α = constant. Consequently, β = constant and the scalar curvature of the perfect fluid spacetime is constant. Now, using equation (31) in (28) we have

$$\alpha Y(f) = 0,$$

which implies that either $\alpha = 0$ or $Y(f) = 0 \implies f = \text{constant}$. If $\alpha = 0$ and f is a non-zero non-constant function on a perfect fluid spacetime, then from equation (1) we have

$$S = -rA \otimes A, \tag{32}$$

where $r = -\beta = \text{constant} \neq 0$ (see equation (6)). From equations (10) and (32), we observe that the state equation of the perfect fluid spacetime is given by $p = -\mu + constant$. Next, we consider that $\alpha \neq 0$ and $Df = 0$ and therefore $f = constant$. Thus, the gradient Ricci soliton on a perfect fluid spacetime is trivial. □

Corollary 3.2. *Suppose the perfect fluid spacetime M admits a gradient Ricci soliton with $\alpha \neq \beta$. If the velocity vector field of M is Killing, then M possesses the constant scalar curvature.*

Proof. Follows from the above theorem. □

Corollary 3.3. *Let the metric of a perfect fluid spacetime M is a non-trivial gradient Ricci soliton. If the velocity vector field ρ of M is Killing and $\alpha \neq \beta$, then M possesses the cyclic parallel Ricci tensor.*

Proof. Let the perfect fluid spacetimes admit a non-trivial gradient Ricci soliton. Also, we suppose that the smooth functions α and β defined in (1) are not equal, that is, $\alpha \neq \beta$. Then from equations (1) and (23), we can easily derive equation (32). Taking covariant derivative of equation (32), we have

$$(\nabla_X S)(Y,Z) = \beta[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z)], \tag{33}$$

since β is constant. The cyclic sum of (33) for vector fields X, Y and Z gives

$$\begin{aligned}(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = \beta\{&A(X)[(\nabla_Y A)(Z)\\ &+(\nabla_Z A)(Y)] + A(Y)[(\nabla_X A)(Z) + (\nabla_Z A)(X)]\\ &+A(Z)[(\nabla_X A)(Y) + (\nabla_Y A)(X)]\}.\end{aligned} \tag{34}$$

Since the velocity vector field ρ of the perfect fluid spacetime is Killing. Then we have

$$g(\nabla_X \rho, Y) + g(X, \nabla_Y \rho) = 0$$

for arbitrary vector fields X and Y. Since $(\nabla_X A)(Y) = g(Y, \nabla_X \rho)$, therefore the above equation takes the form

$$(\nabla_X A)(Y) + (\nabla_Y A)(X) = 0. \tag{35}$$

By considering equations (34) and (35), we lead to

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0.$$

This shows that the Ricci tensor of the perfect fluid spacetime is cyclic parallel. Thus, we get the corollary. □

Corollary 3.4. *If a perfect fluid spacetime admitting a non-trivial gradient Ricci soliton and a Killing vector field ρ, then it possesses the cyclic parallel energy momentum tensor.*

Proof. It is well-known that a cyclic parallel Ricci tensor possesses the constant scalar curvature and therefore the above equation together with equation (8) reflect that

$$(\nabla_X T)(Y, Z) + (\nabla_Y T)(Z, X) + (\nabla_Z T)(X, Y) = 0.$$

This shows that the energy momentum tensor of the perfect fluid spacetime is cyclic parallel. For more details about the energy momentum tensor, we cite [25] and the references therein. Thus, we get the result. □

Theorem 3.5. *If the velocity vector field of a perfect fluid spacetime is Killing, then it is parallel, provided* $\alpha \geq \beta$.

Proof. In [39], Watanabe proved that for a vector field X in M

$$\int_M [S(X, X) - |\nabla X|^2 - (divX)^2] dV \leq 0.$$

The equality holds if and only if X is a Killing vector field. For a Killing vector field X we have $divX = 0$ and therefore the above equation becomes

$$\int_M [S(X, X) - |\nabla X|^2] dV = 0.$$

Let $X = \rho$ be a Killing vector field, then from the above equation we have

$$\int_M [S(\rho, \rho) - |\nabla \rho|^2] dV = 0,$$

which gives

$$\int_M [\beta - \alpha - |\nabla \rho|^2] dV = 0, \tag{36}$$

where equation (1) is used. It is well known that if the Ricci tensor $S(X, X) \leq 0$, then a Killing vector field X in M has vanishing covariant derivative [8]. This result together with equation (36) completes the proof. □

Since the velocity vector field ρ of the perfect fluid spacetime is Killing, therefore from equation (1) we have

$$\begin{aligned}(\nabla_Z S)(X, Y) = {} & Z(\alpha)g(X, Y) + Z(\beta)A(X)A(Y) \\ & + \beta\{(\nabla_Z A)(X)A(Y) + (\nabla_Z A)(Y)A(X)\},\end{aligned}$$

which gives

$$\begin{aligned}(\nabla_Z S)(X, Y) + (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) = {} & Z(\alpha)g(X, Y) + X(\alpha)g(Y, Z) \\ & + Y(\alpha)g(Z, X) + Z(\beta)A(X)A(Y) + X(\beta)A(Z)A(Y) + Y(\beta)A(X)A(Z) \\ & + \beta[A(Z)\{(\nabla_X A)(Y) + (\nabla_Y A)(X)\} + A(Y)\{(\nabla_X A)(Z) \\ & + (\nabla_Z A)(X)\} + A(X)\{(\nabla_Z A)(Y) + (\nabla_Y A)(Z)\}].\end{aligned}$$

In view of (35), the above equation assumes the following form

$$\begin{aligned}(\nabla_Z S)(X, Y) + (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) = {} & Z(\alpha)g(X, Y) + X(\alpha)g(Y, Z) \\ & + Y(\alpha)g(Z, X) + Z(\beta)A(X)A(Y) + X(\beta)A(Z)A(Y) + Y(\beta)A(X)A(Z),\end{aligned}$$

which implies that the Ricci tensor is conformal Killing [33], provided β is constant. In [28], Mantica and Molinari proved that a perfect fluid spacetime with conformal Killing Ricci tensor is governed by the following equation of state,

$$p = -\mu \frac{n+1}{n-1} + constant.$$

The equation of state with constant = 0 violates the weak energy condition $|\frac{p}{\mu}| \leq 1$. In $n = 4$, the equation of state is $p = -\frac{5}{3}\mu$. The matter with $\frac{p}{\mu} \leq -1$ is named as "photon energy".

Corollary 3.6. *If the velocity vector field ρ of a perfect fluid spacetime M is Killing, then the equation of state is governed by $p = -\mu\frac{n+1}{n-1}$ + constant, provided β is constant. Also, if the constant is zero and $\dim M = 4$, then matter is named as photon energy.*

4. Gradient A-Einstein soliton in perfect fluid spacetimes

Proposition 4.1. *Let a perfect fluid spacetime M admit a gradient A-Einstein soliton and the velocity vector field of M is Killing. Then M possesses the constant isotropic pressure.*

Proof. This section is concerned with the study of perfect fluid spacetimes with a gradient A-Einstein metric. Throughout this section, we suppose that the Einstein potential function f satisfies $\rho(f) \neq 0$. Let us put $V = Df$ in equation (4), we have

$$Hess f + S + (\lambda - \frac{r}{2})g + \mu_1 A \otimes A = 0,$$

which gives

$$\nabla_X Df = -QX - \left(\lambda - \frac{r}{2}\right)X - \mu_1 A(X)\rho \tag{37}$$

for all $X \in \mathfrak{X}(M)$. The covariant derivative of (37) together with (16) give

$$\begin{aligned} R(X,Y)Df &= (\nabla_Y Q)(X) - (\nabla_X Q)(Y) + \frac{1}{2}\{X(r)Y - Y(r)X\} \\ &\quad -\mu_1\{(\nabla_X A)(Y)\rho - (\nabla_Y A)(X)\rho + A(Y)\nabla_X\rho - A(X)\nabla_Y\rho\}. \end{aligned} \tag{38}$$

Taking an orthonormal frame field on a perfect fluid and contracting equation (38) for X, we infer

$$S(Y, Df) = -\frac{n-2}{2}Y(r) - \mu_1[(\nabla_\rho A)(Y) + A(Y) div\rho], \tag{39}$$

where $\frac{Y(r)}{2} = (div\, Q)(Y) = \text{trace}\{X \longrightarrow (\nabla_X Q)(Y)\}$. Again, in view of equation (18), equation (38) assumes the form

$$\begin{aligned} R(X,Y)Df &= -X(\alpha)Y - X(\beta)A(Y)\rho - \beta(\nabla_X A)(Y)\rho \\ &\quad -\beta A(Y)\nabla_X\rho + Y(\alpha)X + Y(\beta)A(X)\rho + \beta(\nabla_Y A)(X)\rho \\ &\quad +\beta A(X)\nabla_Y\rho + \frac{X(r)}{2}Y - \frac{Y(r)}{2}X \\ &\quad -\mu_1[(\nabla_X A)(Y)\rho - (\nabla_Y A)(X)\rho + A(Y)\nabla_X\rho - A(X)\nabla_Y\rho]. \end{aligned} \tag{40}$$

Considering an orthonormal frame field and contracting equation (40) for X, we lead

$$\begin{aligned} S(Y, Df) &= (n-1)Y(\alpha) - Y(\beta) - \rho(\beta)A(Y) - \frac{(n-1)}{2}Y(r) \\ &\quad -(\beta + \mu_1)[(\nabla_\rho A)(Y) + A(Y) div\rho]. \end{aligned} \tag{41}$$

In consequence of equations (39) and (41), we find

$$\frac{1}{2}Y(r) = (n-1)Y(\alpha) - Y(\beta) - \rho(\beta)A(Y) - \beta[(\nabla_\rho A)(Y) + A(Y) div\rho]. \tag{42}$$

If possible, we suppose that the velocity vector field ρ is Killing on a perfect fluid spacetime. Then, we have $\rho(p) = 0$, $\rho(\mu) = 0$ and $div\,\rho = 0$, which implies that $\nabla_\rho \rho = 0$, and $\rho(\alpha) = \rho(\beta) = 0$. Thus, the equation (42) becomes

$$2(n-1)Y(\alpha) - 2Y(\beta) = Y(r). \tag{43}$$

From equation (18), we have

$$(\nabla_X Q)(Y) - (\nabla_Y Q)(X) = X(\alpha)Y + X(\beta)A(Y)\rho + \beta(\nabla_X A)(Y)\rho$$
$$+\beta A(Y)\nabla_X\rho - Y(\alpha)X - Y(\beta)A(X)\rho - \beta(\nabla_Y A)(X)\rho - \beta A(X)\nabla_Y\rho.$$

Considering an orthonormal frame field and contracting the above equation for X, we lead

$$Y(r) = 2Y(\alpha), \tag{44}$$

where equation (6) and $\frac{Y(r)}{2} = (div\,Q)(Y) = \text{trace}\{X \longrightarrow (\nabla_X Q)(Y)\}$ have been used. By considering equations (43) and (44), we conclude that

$$(n-2)Y(\alpha) - Y(\beta) = 0 \implies (n-2)\alpha - \beta = \text{constant}, \tag{45}$$

which gives

$$p = \text{contant},$$

where equation (10) has been used. This shows that the isotropic pressure of the perfect fluid spacetime under assumption is constant. □

Proposition 4.2. *Every gradient A-Einstein soliton on a perfect fluid spacetime is a gradient A-Ricci soliton, provided the velocity vector field of the perfect fluid spacetime is Killing and $\rho(f) \neq 0$.*

Proof. Again, from equation (1) we get

$$S(Y, Df) = \alpha g(Y, Df) + \beta A(Y)\rho(f). \tag{46}$$

In view of equations (43) and (44), equation (41) takes the form

$$S(Y, Df) = -Y(\beta)$$

and therefore the above equation together with equation (46) infer that

$$\alpha Y(f) + \beta A(Y)\rho(f) + Y(\beta) = 0 \implies (\alpha - \beta)\rho(f) = 0. \tag{47}$$

Since $\rho(f) \neq 0$ (by hypothesis), therefore from equation (47) we conclude that $\alpha = \beta$. This relation along with equations (6) and (45) show that r, α and β are constants. Since r = constant, therefore the gradient A-Einstein soliton becomes a gradient A-Ricci soliton. Hence,the theorem □

Theorem 4.3. *Every perfect fluid spacetime admitting a gradient A-Einstein soliton is a GRW spacetime, provided the velocity vector field ρ of the perfect fluid spacetime is Killing and $\rho(f) \neq 0$.*

Proof. Thus, equation (47) together with Proposition 4.2 assume the form

$$Df = -\rho(f)\rho.$$

Taking covariant derivative of the above equation and then making use of (37), we lead

$$QX + \left(\lambda - \frac{r}{2}\right)X + \mu_1 A(X)\rho = X(\rho(f))\rho + \rho(f)\nabla_X\rho. \tag{48}$$

Taking the inner product of equation (48) with the velocity vector field of the perfect fluid spacetime, we have

$$\left(\lambda - \frac{r}{2}\right)A(X) - \mu_1 A(X) = -X(\rho(f)), \tag{49}$$

where equation (7) and hypothesis that $\alpha = \beta$ are used. Using the last equation and (7) in equation (48), we find

$$\nabla_X\rho = \varphi[X + A(X)\rho],$$

where $\varphi = \frac{1}{\rho(f)}\left(\alpha + \lambda - \frac{r}{2}\right)$. The last equation together with Theorem 1.2 state that the perfect fluid spacetime under consideration is a GRW spacetime. □

Suppose that the perfect fluid spacetime admits a gradient A-Einstein soliton. Also, the velocity vector field ρ is Killing and $\rho(f) \neq 0$, then equation (49) is satisfied. Hence we can state the following:

Corollary 4.4. *Let the perfect fluid spacetimes admit a gradient A-Einstein soliton. If the velocity vector field ρ of the perfect fluid spacetime is Killing and $\rho(f) \neq 0$, then the gradient of the Einstein potential function is pointwise collinear with the velocity vector field ρ.*

Now replacing X with ρ in equation (49), we find

$$\lambda - \frac{r}{2} - \mu_1 = \rho(\rho(f)).$$

Let us take $\rho = \frac{\partial}{\partial t}$, then the above equation becomes

$$\frac{\partial^2 f}{\partial t^2} = \psi, \tag{50}$$

where $\psi = \lambda - \frac{r}{2} - \mu_1 =$ constant. Thus we have

Corollary 4.5. *Let the metric of a perfect fluid spacetime M be a gradient A-Einstein soliton. If the velocity vector field ρ of M is Killing and $\rho(f) \neq 0$, then M satisfies the partial differential equation (50).*

5. Gradient Schouten soliton on perfect fluid spacetimes

Theorem 5.1. *If the Lorentzian metric of a perfect fluid spacetime M is a gradient Schouten soliton, then either*
(*i*) *the gradient of the Schouten potential function is pointwise collinear with the velocity vector field of M, or*
(*ii*) *the soliton is trivial or M is Ricci simple.*

Proof. We discuss the properties of the gradient Schouten soliton on a perfect fluid spacetime M in this section.

From equation (5) we have

$$\nabla_X Df = -QX + \left(\frac{r}{2(n-1)} + \lambda\right)X \tag{51}$$

for all vector field X on M. In consequence of equations (51) and (16), we find

$$R(X,Y)Df = (\nabla_Y Q)(X) - (\nabla_X Q)(Y) + \frac{1}{2(n-1)}[X(r)Y - Y(r)X]. \tag{52}$$

Taking an orthonormal frame field and contracting equation (52) along the vector field X, we lead

$$S(Y, Df) = 0. \tag{53}$$

With the help of equation (18), equation (52) takes the form

$$\begin{aligned} R(X,Y)Df &= Y(\alpha)X + Y(\beta)A(X)\rho + \beta(\nabla_Y A)(X)\rho + \beta A(X)\nabla_Y \rho \\ &-X(\alpha)Y - X(\beta)A(Y)\rho - \beta(\nabla_X A)(Y)\rho - \beta A(Y)\nabla_X \rho \\ &+\frac{1}{2(n-1)}[X(r)Y - Y(r)X]. \end{aligned} \tag{54}$$

Considering an orthonormal frame field and contracting equation (54) along the vector field X, we infer

$$S(Y, Df) = (n-1)Y(\alpha) - Y(\beta) - \rho(\beta)A(Y) - \beta(\nabla_\rho A)(Y) - \frac{1}{2}Y(r) - \beta A(Y) div\rho.$$

The above equation along with equations (6) and (53) assumes the form

$$-\frac{1}{2}Y(r) = -Y(\alpha) - \rho(\beta)A(Y) - \beta(\nabla_\rho A)(Y) - \beta A(Y) div\rho. \tag{55}$$

Setting $Y = \rho$ in equation (55) we lead

$$(n-2)\rho(\alpha) + \rho(\beta) + 2\beta div\rho = 0, \tag{56}$$

where equation (6) is used. Again, from equation (1) we have

$$S(Y, Df) = \alpha Y(f) + \beta A(Y)\rho(f). \tag{57}$$

In view of equations (53) and (57), we infer

$$\alpha Y(f) = -\beta A(Y)\rho(f). \tag{58}$$

Replacing Y with ρ in equation (58) we get $(\alpha - \beta)\rho(f) = 0$, which shows that either $\alpha = \beta$ or $\rho(f) = 0$. If possible, we suppose that $\alpha = \beta \neq 0$ and $\rho(f) \neq 0$, then equation (58) gives

$$Y(f) = -A(Y)\rho(f) \Longleftrightarrow Df = -\rho(f)\rho. \tag{59}$$

This confesses that the gradient of Schouten potential function is pointwise collinear with the velocity vector field of the perfect fluid spacetime. Next, we consider that $\alpha \neq \beta$ and $\rho(f) = 0$ and therefore equation (58) infers that either $\alpha = 0 \implies S = \beta A \otimes A$ or $Y(f) = 0 \implies f =$ constant. A semi-Riemannian manifold M is said to be Ricci simple [18] if its non-vanishing Ricci tensor S satisfies the equation $S = \beta A \otimes A$. From the expression $S = \beta A \otimes A$, it is obvious that $\beta = -r$. □

Remark 5.2. *The geometrical and physical interpretations of the expression $S = -rA \otimes A$ are given in [26]. Actually, the relation $S = -rA \otimes A$, $g(\rho, \rho) = -1$, $A(X) = g(X, \rho)$ for any vector field X represents the stiff matter fluid ([38], p. 601) and the mass-less scalar field spacetimes with timelike gradient $\nabla_j \psi$. It is also proved that if $S = -rA \otimes A$ and quasi-conformal curvature tensor is harmonic on a Lorentzian manifold M of dimension $n > 3$, then M is a GRW spacetime [17].*

Theorem 5.3. *Let the perfect fluid spacetimes admit a gradient Schouten soliton with $\rho(f) \neq 0$, then it satisfies the differential equations (63) and (64). Also, the soliton is shrinking, steady, or expanding as $\alpha \substack{\geq \\ = \\ <} \frac{\rho(\alpha)\rho(f)}{\alpha}$, respectively.*

Proof. Let us assume that the potential Schouten function on a perfect fluid spacetime satisfies $\rho(f) \neq 0$, then from Theorem 5.1 it is clear that the equation (59) is satisfied. Taking covariant derivative of (59) along the vector field X, we find

$$QX - \left(\frac{r}{2(n-1)} + \lambda\right)X = X(\rho(f))\rho + \rho(f)\nabla_X \rho, \tag{60}$$

where equation (51) is used. The inner product of equation (60) with ρ reflects that

$$\left(\frac{r}{2(n-1)} + \lambda\right)A(X) = X(\rho(f)), \tag{61}$$

which gives

$$\rho(\rho(f)) = -\left(\lambda + \frac{\alpha}{2}\right), \tag{62}$$

where equations (1), (6) and our hypothesis ($\alpha = \beta$) have been used. Since $\rho(f) \neq 0$, therefore equation (59) shows $|Df| \neq 0$. Let $\rho = \frac{\partial}{\partial t}$, then equation (62) becomes

$$\frac{\partial^2 f}{\partial t^2} = -\left(\lambda + \frac{\alpha}{2}\right). \tag{63}$$

This reflects that the non-trivial gradient Schouten soliton on a perfect fluid spacetime satisfies the second order partial differential equation (63). From equation (56) we have

$$(n-1)\rho(\alpha) + 2\alpha\, div\rho = 0. \tag{64}$$

Again, considering an orthonormal frame field and contracting equation (60) along the vector field X, we conclude that

$$\lambda = \frac{1}{2}\left[\alpha - \frac{\rho(\alpha)\rho(f)}{\alpha}\right],$$

where equation (6) is used. Thus, the gradient Schouten soliton on a perfect fluid spacetime is shrinking, steady, or expanding accordingly as $\alpha \substack{\geq \\ = \\ <} \frac{\rho(\alpha)\rho(f)}{\alpha}$, respectively. □

Theorem 5.4. *Let the metric of a perfect fluid spacetime M is a gradient Schouten soliton. Then M is a GRW spacetime, provided $\lambda \neq \frac{\alpha}{2}$ and $\rho(f) \neq 0$. Also, if $2\lambda - \alpha = 0$ then the velocity vector field of M is parallel and the gradient Schouten soliton is shrinking.*

Proof. Equations (1), (6), (60) and (61) together with our hypothesis that $\rho(f) \neq 0$ give

$$\nabla_X \rho = \frac{1}{\rho(f)}\left[\lambda - \frac{\alpha}{2}\right]\{X + A(X)\rho\}, \tag{65}$$

provided $2\lambda - \alpha \neq 0$. Thus, the above equation together with Theorem 1.2 tell us that the perfect fluid spacetime under consideration is a *GRW*-spacetime. Next, we suppose $2\lambda - \alpha = 0$, then from (65) it is obvious that the velocity vector field of the perfect fluid is parallel and $\lambda = \frac{\kappa}{(n-1)(n-2)}\mu > 0$, where equation (10) is used. This infers that the gradient Schouten soliton on a perfect fluid spacetime is shrinking. □

Let us suppose that the perfect fluid spacetimes admit a gradient Schouten soliton with $\rho(f) \neq 0$, and $2\lambda - \alpha = 0$. Thus, from equation (65), we have $\nabla\rho = 0 \implies (\mathfrak{L}_\rho g)(X,Y) = g(X, \nabla_Y \rho) + g(X, \nabla_Y \rho) = 0$. This shows that the perfect fluid spacetime, under consideration, is Killing. Duggal and Sharma listed in their book that if the velocity vector field ρ of the perfect fluid spacetime is Killing, then $div\,\rho = 0$, $\mathfrak{L}_\rho p = 0$ and $\mathfrak{L}_\rho \mu = 0$ (see [19], p. 89). This shows that p and μ are invariant along ρ. Now, we conclude our results in the following:

Corollary 5.5. *If a perfect fluid spacetime M admits a gradient Schouten soliton with $\rho(f) \neq 0$ and $2\lambda - \alpha = 0$, then the isotropic pressure and energy density of M are invariant along the velocity vector field of M.*

Let us consider that the perfect fluid spacetimes admit a gradient Schouten soliton and $\rho(f) \neq 0$, then equations (58) and (65) infer that $\alpha = \beta \neq 0$ and

$$\nabla_X \rho = \sigma\{X + A(X)\rho\}, \tag{66}$$

where $\sigma = \frac{1}{\rho(f)}\left[\lambda - \frac{\alpha}{2}\right] \neq 0$. The Lie derivative of (1) along the velocity vector field ρ together with hypothesis $\alpha = \beta$ give

$$\begin{aligned}(\mathfrak{L}_\rho S)(X,Y) = \rho(\alpha)[g(X,Y) + A(X)A(Y)] + \alpha\{(\mathfrak{L}_\rho g)(X,Y) \\ +(\mathfrak{L}_\rho A)(X)A(Y) + (\mathfrak{L}_\rho A)(Y)A(X)\}\end{aligned} \tag{67}$$

for all vector fields X and Y. Again, the Lie derivative of $A(X) = g(X,\rho)$ along ρ gives $(\mathfrak{L}_\rho A)(X) = (\mathfrak{L}_\rho g)(X,\rho) + g(X, \mathfrak{L}_\rho \rho)$. Making use of this fact, $(\mathfrak{L}_Z g)(X,Y) = g(\nabla_X Z, Y) + g(X, \nabla_Y Z)$ and equation (66) in equation (67), we obtain

$$\mathfrak{L}_\rho S = 2\sigma_g S,$$

where $\sigma_g = \frac{\rho(\alpha)+2\sigma\alpha}{2\alpha}$. Thus we can state the following:

Corollary 5.6. *Let a perfect fluid spacetime admit a gradient Schouten soliton with $\rho(f) \neq 0$, then the velocity vector field of the perfect fluid spacetime is RIV. In particular, the velocity vector field of the perfect fluid spacetime is RC if $\rho(\alpha) + 2\sigma\alpha = 0$.*

References

[1] Alías, L. , Romero, A. and Sánchez, M., *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes*, Gen. Relat. Gravit. **27** (1995), 71.

[2] Alías, L. , Romero, A. and Sánchez, M.,*Compact spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes; in Geometry and Topology of Submanifolds VII*, River Edge NJ, USA, World Scientific, **67**, 1995.

[3] Barbosa, E., Pina, R. and Tenenblat, K., *On gradient Ricci solitons conformal to a pseudo-Euclidean space*, Israel J. Math. **200** (2014), 213–224.

[4] Batat, W., Brozos-Vazquez, M., Garcia-Rio, E. and Gavino-Fernandez, S., *Ricci soliton on Lorentzian manifolds with large isometry groups*, Bull. Lond. Math. Soc. **43** (2011), 1219–1227.

[5] Blaga, A. M., *On gradient η-Einstein solitons*, Kragujev. J. Math. **42** (2018), 229–237.

[6] Blaga, A. M., *Solitons and geometrical structures in a perfect fluid spacetime*, Rocky Mountain J. Math. **50** (2020), 41–53.

[7] Brozos-Vazqnez, M., Calvaruso, G., Garcia-Rio, E. and Gavino-Fernandez, S., *Three-dimensional Lorentzian homogeneous Ricci solitons*, Israel J. Math. **188** (2012), 385–403.

[8] Bochner, S., *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc. **52** (1946), 776–797.

[9] Caldwell, R. R., Kamionkowski, M. and Weinberg, N. N., *Phantom Energy: Dark Energy with $w < -1$ causes a cosmic doomsday*, Phys. Rev. Lett. **91** (2013), 071301.

[10] Catino, G. and Mazzieri, L., *Gradient Einstein solitons*, Nonlinear Analysis **132** (2016), 66–94.

[11] Chaki and R. K. Maity, *On quasi Einstein manifolds*, Publ. Math. Debrecen **57** (2000), 257–306.

[12] S. K. Chaubey, *Generalized Robertson-Walker space-times with W_1-curvature tensor*, J. Phys. Math. **10** (2019), 1000303 (7 pages).

[13] Chaubey, S. K. . Suh Y. J., and De, U. C., *Characterizations of the Lorentzian manifolds admitting a type of semi-symmetric metric connection*, Anal. Math. Phy. **10** (2020), 61.

[14] Chen, B. Y., *A simple characterization of generalized Robertson–Walker spacetimes*, Gen. Relativ. Gravit. **46** (2014), 1833 (5 pages).

[15] Chen, B. Y., *Pseudo-Riemannian Geometry, δ-invariants and Applications*, World Scientific, 2011.

[16] Chen, B. Y., *Classification of torqued vector fields and its applications to Ricci solitons*, Kragujev. J. Math. **41** (2017), 239–250.

[17] De, U. C. and Suh, Y. J., *Some characterizations of Lorentzian manifolds*, Int. J. Geom. Methods Mod. Phys. **16** (2019), 1950016.

[18] Deszcz, R., Hotloś M. and Sentürk, Z., *On curvature properties of quasi-Einstein hypersurfaces in semi-Euclidean spaces*, Soochow J. Math. **27** (2001), 375–389.

[19] Duggal, K. L., Sharma,R. *Symmetries of spacetimes and Riemannian manifolds, Mathematics and its Applications* **487**, Kluwer Academic Press, Boston, London, 1999.

[20] Gutiérrez, M. and Olea, B., *Global decomposition of a Lorentzian manifold as a generalized Robertson-Walker space*, Differ. Geom. Appl. **27** (2009), 146–156.

[21] Hamilton, R. S., *Three-manifolds with positive Ricci curvature*, J. Differ. Geom. **17** (1982), 255–306.

[22] Hamilton, R. S., *The Ricci flow on surfaces*, Contemp. Math. No. 71, Amer. Math. Soc., 1988.

[23] Hawking, S. W. and Ellis, G. F. R., *The large scale structure of spacetime*, Cambridge University Press, Cambridge, 1973.

[24] Jenny, H., *Welcome to the Dark side*, Nature **448** (2007), 240–245.

[25] Mallick, S., De U. C. and Suh, Y. J., *Spacetimes with different forms of energy momentum tensor*, J. Geom. Phy. *151* (2020), 103622.

[26] Mantica, C. A., Suh Y. J. and De, U. C., *A note on generalized Robertson–Walker space-times*, Int. J. Geom. Methods Mod. Phys. **13** (2016), 1650079 (9 pages).

[27] Mantica, C. A., De, U. C., Suh, Y. J. and Molinari, L. G., *Perfect fluid spacetimes with harmonic generalized curvature tensor*, Osaka J. Math. **56** (2019), 173–182.

[28] Mantica, C. A. and Molinari, L. G., *Generalized Robertson-Walker spacetimes-A survey*, Int. J. Geom. Methods Mod. Phys. **14** (2017), 1730001 (27 pages).

[29] Mantica, C. A., Molinari, L. G., and De, U. C., *A condition for a perfect-fluid space-time to be a generalized Robertson-Walker space-time*, J. Math. Phys. **57** (2016), 022508.

[30] Mihai, A. and Mihai, I., *Torse forming vector fields and exterior concurrent vector fields on Riemannian manifolds and applications*, J. Geom. Phy. **73** (2013), 200–208.

[31] O'Neill, B., *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

[32] Pina, R. and Menezes, I., *On gradient Schouten solitons conformal to a pseudo-Euclidean space*, manuscripta math. (2019). https://doi.org/10.1007/s00229-019-01159-0.

[33] Rani, R., Edgar, S. B., Barnes, A., *Killing tensors and conformal Killing tensors from conformal Killing vectors*, Class. Quantum Grav. **20** (2003), 1929–1942.

[34] Romero, A., Rubio, R. N., and Salamanca, J. J., *Uniqueness of complete maximal hypersurfaces in spatially parabolic generalized Robertson-Walker spacetimes*, Class. Quantum Grav. **30**(2013), 115007.

[35] Romero, A., Rubio, R. N., and Salamanca, J. J., *Uniqueness of complete maximal hypersurfaces in spatially parabolic generalized Robertson-Walker spacetimes. Applications to uniqueness results*, Int. J. Geom. Meth. Mod. Phys. **10** (2013), 1360014.

[36] Sánchez, M., *On the geometry of generalized Robertson-Walker spacetimes: geodesics*, Gen. Relat. Gravit. **30** (1998), 915–932.

[37] Sharma, R. and Ghosh, A., *Perfect fluid space-times whose energy-momentum tensor is conformal Killing*, J. Math. Phys. **51** (2010), 022504.

[38] Stephani, H., Kramer, D., Mac-Callum,M., Hoenselaers, C. and Hertl, E., *Exact Solutions of Einstein's Field Equations*, 2nd edn. Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2009.

[39] Watanabe, Y., *Integral inequalities in compact orientable manifolds, Riemannian or Kählerian*, Kodai Math. Sem. Rep. **20** (1968), 264–271.

[40] Yano, K., *On torse forming direction in a Riemannian space*, Proc. Imp. Acad. Tokyo **20** (1944), 340–345.

[41] Ö. Zengin, F., *m-projectively flat spacetimes*, Math. Reports **14** (2012), 363–370.