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Some Characterizations of Strongly Partial Isometry Elements in Rings with Involutions

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Abstract. In this paper, we study an element which is both group invertible and Moore Penrose invertible to be EP, partial isometry and strongly EP by discussing the existence of solutions in a definite set of some given constructive equations. Mainly, let $a \in R^{\#} \cap R^{+}$. Then we firstly show that an element $a \in R^{EP}$ if and only if and Equation : $axa^{+} + a^{+}ax = 2x$ has at least one solution in $\chi_a = \{a, a^{\#}, a^{+}, a^{*}, (a^{\#})^{*}, (a^{+})^{*}\}$. Next, $a \in R^{SEP}$ if and only if Equation: $axa^{*} + a^{+}ax = 2x$ has at least one solution in χ_a. Finally, $a \in R^{PI}$ if and only if Equation: $aya^{*}x = xy$ has at least one solution in ρ_a^2, where $\rho_a = \{a, a^{\#}, a^{+}, (a^{\#})^{*}, (a^{+})^{*}\}$.

1. Introduction

Throughout this paper, R will denote a unital ring with identity 1. An involution $* : a \longmapsto a^{*}$ in a ring R is an anti-isomorphism of degree 2, that is,

$$(a^{*})^{*} = a, (ab)^{*} = b^{*}a^{*}, (a + b)^{*} = a^{*} + b^{*}.$$

The notion of Moore-Penrose invertible (or MP-invertible) has been investigated by many authors (see, for example, [13, 15, 16]). We say that $b = a^{\dagger}$ is the Moore-Penrose invertible of $a \in R$, if the following conditions hold:

$$aba = a, bab = b, (ab)^{*} = ab, (ba)^{*} = ba.$$

There is at most one b such that the above conditions hold. We write $R^{\dagger}$ for the set of all MP-invertibles of R. $a \in R$ is said to be group invertible if there is $a^{\#} \in R$ such that $aa^{\#}a = a; a^{\#}aa^{\#} = a^{\#}; aa^{\#} = a^{\#}a$. $a^{\#}$ is called a group inverse of a and it is uniquely determined by these equations. Denote by $R^{\#}$ the set of all group invertible elements of R.

An element $a \in R$ is said to be an EP element if $a \in R^{\dagger} \cap R^{\#}$ and $a^{\dagger} = a^{\#}$ [10]. The set of all EP elements of R will be denoted by R^{EP}. Mosić et al. in [1, Theorem 2.1] gave several equivalent conditions under which an element in R is an EP element. Patrćio and Puystjens in [7, Proposition 2] proved that for an element $a \in R$, $a \in R^{EP}$ if and only if $aR = a^{*}R$ or $aa^{\dagger} = a^{\dagger}a$. It is known by [17, Theorem 7.3] that $a \in R$ is EP if

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and only if a is group invertible and $aa^{\#}$ is symmetric. More results on *EP* elements can also be found in [6, 9, 11, 12, 14, 19].

Motivated by these results, this paper is intended to provide, by using certain equations admitting solutions in a definite set, further equivalent conditions for an element in a ring with involution to be a partial isometry. Since there are close connections between partial isometries, EP elements and normal elements in rings with involution [2, 5], we present also several characterizations of the latter two kinds of elements.

2. Results

Lemma 2.1. *([2, Lemma 1.1 and Theorem 1.2])Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equavilent:*
1) $a \in R^{EP}$;
2) $a^{+}a = aa^{+}$;
3) $a^{+}a = a^{\#}a$;
4) $aa^{+} = aa^{\#}$.

Observing the conditions 2) and 4) of Lemma 2.1, we obtain the following lemma.

Lemma 2.2. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equavilent:*
1) $a \in R^{EP}$;
2) $a^{+}a^{m+1} = a^{m}$ for some $m \geq 1$;
3) $a^{m} = a^{m+1}a^{+}$ for some $m \geq 1$.

Change the condition 2) of Lemma 2.1, we have the following lemma.

Lemma 2.3. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equavilent:*
1) $a \in R^{EP}$;
2) $aa^{+}a^{+} = a^{+}$;
3) $a^{+}a^{+}a = a^{+}$.

Lemma 2.4. *([2, Theorem 1.1]; [4]; [18]) (1) If $a \in R^{+}$, then $a^{+}aa^{*} = a^{*} = a^{*}aa^{+}$.*
(2) If $a \in R^{\#} \cap R^{+}$, then $a^{\#}a^{+}a = a^{\#} = aa^{+}a^{\#}$.

Substituting a^{*} for $a^{\#}$ in the left of condition 1) of Lemma 2.4, one obtains the following lemma.

Lemma 2.5. *Let $a \in R^{\#} \cap R^{+}$. If $a^{*} = a^{+}aa^{\#}$, then $a \in R^{EP}$ and $a^{+} = a^{*}$.*

Proof. Since $a^{*} = a^{+}aa^{\#}$, we have $a^{*}a = a^{+}aa^{\#}a = a^{+}a$. Hence $a^{*} = a^{+}$ by [5, Theorem 2.1]. Consequently, $a^{+} = a^{*} = a^{+}aa^{\#}$, one obtains $a \in R^{EP}$ by [1, Theorem 2.1(xxii)]. □

Let $a \in R^{\#} \cap R^{+}$. Then $a^{*} = a^{+}aa^{\#}$ if and only if $aa^{*} = aa^{\#}$. Hence Lemma 2.5 leads to the following corollary.

Corollary 2.6. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equavilent:*
1) $a \in R^{EP}$ and $a^{+} = a^{*}$;
2) $aa^{*} = aa^{\#}$;
3) $a^{*}a = a^{\#}a$;
4) $a^{*} = a^{\#}aa^{+}$;
5) $a^{*} = a^{+}aa^{\#}$.

Let $a \in R^{\#} \cap R^{+}$. If $a^{\#} = a^{+} = a^{*}$, then a is called a strongly *EP* element of R. We write by R^{SEP} to denote the set of all strongly *EP* elements of R.

Let $a \in R^{EP}$. Then we have $a^{2}a^{+} + a^{+}a^{2} = 2a$. Hence we can construct the following equation:

$$axa^{+} + a^{+}ax = 2x. \tag{1}$$

Using the equation (1), we can characterize strongly *EP* elements as follows.

Theorem 2.7. *Suppose $a \in R^{\#} \cap R^{+}$, then $a \in R^{EP}$ if and only if Equation (1) has at least one solution in $\chi_a = \{a, a^{\#}, a^{+}, a^{*}, (a^{\#})^{*}, (a^{+})^{*}\}$.*

Proof. $'' \Rightarrow ''$ Assume $a \in R^{EP}$, then $a^2a^+ + a^+a^2 = 2a$ by [1, Theorem 2.1(xxx)]. Hence $x = a$ is a solution to the equation.
$'' \Leftarrow ''$ 1) If $x = a$ is a solution, then $a^2a^+ + a^+a^2 = 2a$, this gives $a \in R^{EP}$ by [1, Theorem 2.1(XXX)];

2) If $x = a^{\#}$ is a solution, then one has $aa^{\#}a^+ + a^+aa^{\#} = 2a^{\#}$. Post-multiply it by a, we have $a^{\#}a + a^+a = 2a^{\#}a$ by Lemma 2.4(2), thus $a^+a = a^{\#}a$. We can deduce that $a \in R^{EP}$ by Lemma 2.1;

3) If $x = a^+$ is a solution, then $aa^+a^+ + a^+aa^+ = 2a^+$, that is, $a^+ = aa^+a^+$. By Lemma 2.3, $a \in R^{EP}$;

4) If $x = a^*$ is a solution, then $aa^*a^+ + a^+aa^* = 2a^*$, which implies that $a^* = aa^*a^+$. Pre-multiplying it by $1 - aa^+$, we get $(1 - aa^+)a^* = (1 - aa^+)aa^*a^+ = 0$. Applying the involution on the last equality, it turns out to be $a(1 - aa^+) = 0$, so $a = a^2a^+$. This means $a \in R^{EP}$ by Lemma 2.2;

5) If $x = (a^{\#})^*$ is a solution, one deduces that

$$a(a^{\#})^*a^+ + a^+a(a^{\#})^* = 2(a^{\#})^*. \tag{2}$$

Note that $a^+a(a^{\#})^* = (a^{\#}a^+a)^* = (a^{\#})^*$. Accordingly, (2) turns into $(a^{\#})^* = a(a^{\#})^*a^+$. Pre-multiply this equality by $1 - aa^+$, then we obtain $(1 - aa^+)(a^{\#})^* = (1 - aa^+)a(a^{\#})^*a^+ = 0$. Applying the involution on the equality, we get $a^{\#}(1 - aa^+) = 0$, Morever, pre-multiplying it by a^2, we get $a = a^2a^+$, which implies that $a \in R^{EP}$ by Lemma 2.2;

6) If $x = (a^+)^*$ is a solution, then

$$a(a^+)^*a^+ + a^+a(a^+)^* = 2(a^+)^*. \tag{3}$$

Since $aa^+(a^+)^* = (a^+aa^+)^* = (a^+)^*$, we can pre-multiply (3) by $1-aa^+$, and get $(1-aa^+)a^+a(a^+)^* = 0$. Multiplying it on the right by a^*, we arrive at $(1 - aa^+)a^+aaa^+ = 0$. In addition, post-multiplying this equality by $aa^{\#}$, we can see that $(1 - aa^+)a^+a = 0$, so $a^+a = aa^+a^+a$. According to the proof of Lemma 2.2, we know that $a \in R^{EP}$, as required. □

Modify Equation (1) to

$$axa^* + a^+ax = 2x. \tag{4}$$

Theorem 2.8. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if Equation (4) has at least one solution in χ_a.*

Proof. $'' \Rightarrow ''$ Obviously $x = a^+ = a^{\#} = a^*$ is a solution.
$'' \Leftarrow ''$ 1) If $x = a$ is a solution, then $a^2a^* + a^+a^2 = 2a$. Multiplying the equality on the left by a, we have $a^3a^* = a^2$. Hence $a \in R^{SEP}$ by [2, Theorem 2.2(xvii)];

2) If $x = a^{\#}$ is a solution, one has that $aa^{\#}a^* + a^+aa^{\#} = 2a^{\#}$. Hence $a \in R^{SEP}$ by [2, Theorem 2.2(iv)];

3) If $x = a^+$ is a solution, then $aa^+a^* + a^+aa^+ = 2a^+$. It can be concluded that $aa^+a^* = a^+$. Then post-multiply the equality by a, and we have $aa^+a^*a = a^+a$. Applying the involution on the last equality, one has $a^+a = a^*a^2a^+$. Multiplying the equality on the right by $aa^{\#}$, we arrive at $a^+a = a^*a$. Hence $a^* = a^+$, it follows that $a^* = a^+ = aa^+a^*$. So, $a = a^2a^+$, one obtains $a \in R^{EP}$. Thus $a \in R^{SEP}$;

4) If $x = a^*$ is a solution, one concludes that $aa^*a^* + a^+aa^* = 2a^*$, which forces that $aa^*a^* = a^*$. Taking the involution on the equality, we get $a = a^2a^*$. Hence $a \in R^{SEP}$ by [2, Theorem 2.2(xvii)];

5) If $x = (a^{\#})^*$ is a solution, then $a(a^{\#})^*a^* + a^+a(a^{\#})^* = 2(a^{\#})^*$. This leads to $a(aa^{\#})^* = (a^{\#})^*$. Consequently, $a^{\#} = aa^{\#}a^*$. Furthermore, pre-multiply it by a^3, and we obtain $a^2 = a^3a^*$. In the light of the proof of 1), $a \in R^{SEP}$;

6) If $x = (a^+)^*$ is a solution, we have $a(a^+)^*a^* + a^+a(a^+)^* = 2(a^+)^*$. Taking the involution on the equality, one has that $aa^+a^* + a^+a^+a = 2a^+$. Pre-multiply the equality by $1 - a^+a$, it turns out to be $(1 - a^+a)aa^+a^* = 0$. Again applying the involution on the last equality, we get $a^2a^+(1 - a^+a) = 0$. Furthermore, multiplying it on the left by $a^+a^{\#}$, we obtain $a^+(1 - a^+a) = 0$, giving that $a^+ = a^+a^+a$. By Lemma 2.3, we have $a \in R^{EP}$, this gives $2a^+ = aa^+a^* + a^+a^+a = a^+aa^* + a^+aa^+ = a^* + a^+$, it follows that $a^* = a^+$. Hence $a \in R^{SEP}$, as required. □

We modify the equation (1) to

$$axa^{+} + a^{*}ax = 2x. \tag{5}$$

Theorem 2.9. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Equation* (5) *has at least one solution in $\{a, a^{\#}, a^{+}\}$.*

Proof. " $\Rightarrow$ " Obviously $x = a^{+} = a^{\#} = a^{*}$ is a solution.

" $\Leftarrow$ " 1) If $x = a$ is a solution, then $a^2a^{+} + a^{*}a^2 = 2a$. Multiplying the equality on the right by a, we have $a^{*}a^3 = a^2$. Hence $a \in R^{SEP}$ by [2, Theorem 2.2(xvi)];

2) If $x = a^{\#}$ is a solution, one has that $aa^{\#}a^{+} + a^{*}aa^{\#} = 2a^{\#}$. Multiplying the equality on the right by a, one has $aa^{\#} = a^{*}a$. Hence $a \in R^{SEP}$ by [2, Theorem 2.2(v)];

3) If $x = a^{+}$ is a solution, then $aa^{+}a^{+} + a^{*}aa^{+} = 2a^{+}$, that is, $aa^{+}a^{+} + a^{*} = 2a^{+}$. Pre-multiply the equality by $1 - a^{+}a$, and we have $(1 - a^{+}a)aa^{+}a^{+}a = 0$. Applying the involution on the last equality, one obtains that $a^{+}a^2a^{+}(1 - a^{+}a) = 0$. Multiplying it on the left by $a^{+}a^{\#}a$, one has $a^{+}(1 - a^{+}a) = 0$. Hence $a \in R^{EP}$ by Lemma 2.3, this gives $a^{\#} = a^{+}$, it follows that $2a^{+} = aa^{+}a^{+} + a^{*} = aa^{+}a^{\#} + a^{*} = a^{\#} + a^{*} = a^{+} + a^{*}$. Thus $a^{+} = a^{*}$, this implies $a \in R^{SEP}$. □

If we use $a^{\#}$ in place of a^{+} in Equation (1), one has the following equation.

$$axa^{\#} + a^{+}ax = 2x. \tag{6}$$

Theorem 2.10. *Suppose $a \in R^{\#} \cap R^{+}$, then $a \in R^{EP}$ if and only if Equation* (6) *has at least one solution in χ_a.*

Proof. " $\Rightarrow$ " Assume $a \in R^{EP}$, then $x = a$ is a solution to the equation.

" $\Leftarrow$ " 1) If $x = a$ is a solution, then $a^2a^{\#} + a^{+}a^2 = 2a$, this gives $a = a^{+}a^2$. Hence $a \in R^{EP}$ by Lemma 2.2;

2) If $x = a^{\#}$ is a solution, then one has $aa^{\#}a^{\#} + a^{+}aa^{\#} = 2a^{\#}$, that is $a^{+}aa^{\#} = a^{\#}$. Post-multiply it by a, we have $a^{+}a = a^{\#}a$. Hence $a \in R^{EP}$ by Lemma 2.1;

3) If $x = a^{+}$ is a solution, then $aa^{+}a^{\#} + a^{+}aa^{+} = 2a^{+}$, that is, $a^{+} = aa^{+}a^{\#} = a^{\#}$ by Lemma 2.4. Hence $a \in R^{EP}$;

4) If $x = a^{*}$ is a solution, then $aa^{*}a^{\#} + a^{+}aa^{*} = 2a^{*}$, which implies that $a^{*} = aa^{*}a^{\#}$. Post-multiplying it by $1 - a^{+}a$, we get $a^{*}(1 - a^{+}a) = aa^{*}a^{\#}(1 - a^{+}a) = 0$. Applying the involution on the last equality, it turns out to be $(1 - a^{+}a)a = 0$, so $a = a^{+}a^2$. This means $a \in R^{EP}$ by Lemma 2.2;

5) If $x = (a^{\#})^{*}$ is a solution, one deduces that

$$a(a^{\#})^{*}a^{\#} + a^{+}a(a^{\#})^{*} = 2(a^{\#})^{*}. \tag{7}$$

This implies $(a^{\#})^{*} = a(a^{\#})^{*}a^{\#}$. Post-multiply this equality by $1 - a^{+}a$, then we obtain $(a^{\#})^{*}(1 - a^{+}a) = 0$. Applying the involution on the equality, we get $(1 - a^{+}a)a^{\#} = 0$. According to the proof of (2), we get $a \in R^{EP}$;

6) If $x = (a^{+})^{*}$ is a solution, then

$$a(a^{+})^{*}a^{\#} + a^{+}a(a^{+})^{*} = 2(a^{+})^{*}. \tag{8}$$

Similar to the proof of 6) in Theorem 2.1, we have $a \in R^{EP}$, as required. □

Pre-multiplying Equation (6) by a, we have the following equation.

$$a^2xa^{\#} = ax. \tag{9}$$

Change the left sided of Equation (9) as follows.

$$xa^2a^{+} = ax. \tag{10}$$

Theorem 2.11. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{EP}$ if and only if the equation* (10) *has at least one solution in χ_a.*

Proof. $'' \Rightarrow ''$ Assume that $a \in R^{EP}$, then $x = a$ is a solution to the equation (10).

$'' \Leftarrow ''$ 1) If $x = a$ is a solution, then $a^3a^+ = a^2$. Hence $a \in R^{EP}$ by Lemma 2.2;

2) If $x = a^{\#}$ is a solution, then one has $a^{\#}a^2a^+ = aa^{\#}$, that is $aa^+ = aa^{\#}$. Hence $a \in R^{EP}$;

3) If $x = a^+$ is a solution, then $a^+a^2a^+ = aa^+$, this infers that $a = a^+a^2$ by multiplying the equality on the right by a. Hence $a \in R^{EP}$ by Lemma 2.2;

4) If $x = a^*$ is a solution, then $a^*a^2a^+ = aa^*$, which implies that $aR = a^*R$ by [3, Lemma 2.3, Lemma 2.4]. This means $a \in R^{EP}$;

5) If $x = (a^{\#})^*$ is a solution, one deduces that $(a^{\#})^*a^2a^+ = a(a^{\#})^*$. Then, by [3, Lemma 2.2, Lemma 2.3], we have $aR \subseteq a^*R$, this implies $(1 - a^+a)aR \subseteq (1 - a^+a)a^*R = 0$. Hence we get $a \in R^{EP}$ by Lemma 2.2;

6) If $x = (a^+)^*$ is a solution, then $(a^+)^*a^2a^+ = a(a^+)^*$, by [3, Lemma 2.1, Lemma 2.4], one has $Ra^+ = R(a^+)^*a^2a^+ = Ra(a^+)^* \subseteq R(a^+)^* = Ra$, it infers that $Ra^+(1 - a^+a) = Ra(1 - a^+a) = 0$. Hence $a \in R^{EP}$ by Lemma 2.3. □

Applying the involution on the equation (10), one obtains the following equation.

$$aa^+a^*x = xa^*. \tag{11}$$

Since $a \in R^{EP}$ if and only if $a^* \in R^{EP}$ and $\chi_a = \chi_{a^*}$, Theorem 2.5 implies the following corollary.

Corollary 2.12. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{EP}$ if and only if the equation* (11) *has at least one solution in χ_a.*

Using a^* in place of a in the equation (11), one has the following equation.

$$a^+a^2x = xa. \tag{12}$$

Hence Corollary 2.2 implies the following corollary.

Corollary 2.13. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{EP}$ if and only if the equation* (12) *has at least one solution in χ_a.*

Using a^+ in place of a^* in the right of Equation (11), one has the following equation.

$$aa^+a^*x = xa^+. \tag{13}$$

Theorem 2.14. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Equation* (13) *has at least one solution in χ_a.*

Proof. $'' \Rightarrow ''$ Obviously $x = a^+ = a^{\#} = a^*$ is a solution.

$'' \Leftarrow ''$ 1) If $x = a$ is a solution, then $aa^+a^*a = aa^+$. Applying the involution on the equality, one has $aa^+ = a^*a^2a^+$. Multiplying the last equality on the right by $aa^{\#}$, we have $aa^{\#} = a^*a$, this implies $a \in R^{SEP}$ by Corollary 2.1;

2) If $x = a^{\#}$ is a solution, one has that $aa^+a^*a^{\#} = a^{\#}a^+$. Multiplying the equality on the right by a^+a, we have $a^{\#}a^+ = a^{\#}a^+a^+a$ by Lemma 2.4, this gives $aa^+ = aa^+a^+a$ by pre-multiplying a^2. Hence $a \in R^{EP}$ by Lemma 2.3, this gives $aa^+ = a^{\#}a^+a^2 = aa^+a^*a^{\#}a^2 = aa^+a^*a$. Hence $a \in R^{SEP}$ by 1);

3) If $x = a^+$ is a solution, then $aa^+a^*a^+ = a^+a^+$. Hence $aR = aa^+R = (a^+)^*a^*R = (a^+)^*a^*a^*R = (a^2a^+)^*R = aa^+a^*R = aa^+a^*a^*R = aa^+a^*a^+R = a^+a^+R \subseteq a^+R$, this implies $a \in R^{EP}$. Thus $a^{\#}a^+ = a^+a^+ = aa^+a^*a^+ = aa^+a^*a^{\#}$, one obtains $a \in R^{SEP}$ by 2);

4) If $x = a^*$ is a solution, one concludes that $aa^+a^*a^* = a^*a^+$, which forces that $a^3a^+ = (a^+)^*a$ by applying the involution on the equality. Noting that $Ra^3 = Ra$ and $R(a^+)^* = Ra$. Then $Ra^+ = Raa^+ = Ra^3a^+ = R(a^+)^*a = Ra^2 = Ra$, which implies $a \in R^{EP}$. It follows that $a^2 = a^3a^+ = (a^+)^*a$. Pre-multiplying the last equality by a^*, we get $a^*a^2 = a$. Hence $a \in R^{SEP}$ by [2, Theorem 2.2(xvii)];

5) If $x = (a^\#)^*$ is a solution, then $aa^+a^*(a^\#)^* = (a^\#)^*a^+$. Taking the involution on the equality, one has $aa^+ = (a^+)^*a^\#$, which implies $a^* = a^*aa^+ = a^*(a^+)^*a^\# = a^+aa^\#$. Hence $a \in R^{SEP}$ by Lemma 2.5;

6) If $x = (a^+)^*$ is a solution, we have $aa^+a^*(a^+)^* = (a^+)^*a^+$, that is, $aa^+a^+a = (a^+)^*a^+$. Per-multiplying the equality by a^*, one has $a^*a^+a = a^+$, it follows that $Ra^+ = Ra^*a^+a = Ra^*(a^+a)^* = Ra^*a^*(a^+)^* = Ra^*(a^+)^* = Ra^+a = Ra$. Hence $a \in R^{EP}$. It follows that $aa^+ = aa^+a^+a = (a^+)^*a^+$ and $a^* = a^*aa^+ = a^*(a^+)^*a^+ = a^+$. Therefore $a \in R^{SEP}$. □

If we modify the equation (13) as follows.

$$aa^*a^+x = xa^+. \tag{14}$$

Then we have the following problem.

Problem 2.15. *Let $a \in R^\# \cap R^+$. If Equation* (14) *has at least one solution in χ_a, is $a \in R^{SEP}$?*

For this problem, we have studied the conclusions of three cases, and other cases need to be further reached. The details are as follows:

(1) If $x = a$ is a solution, then $aa^*a^+a = aa^+$, this gives $a^+ = a^*a^+a$. By [5], $a \in R^{SEP}$.

(2) If $x = a^\#$ is a solution, then $aa^*a^+a^\# = a^\#a^+$. Post-multiply this equality by a^2, one yields $aa^*a^+a = a^\#a$, this gives $a^\#a^+ = (aa^*a^+a)a^\#a^\# = a^\#aa^\#a^\# = a^\#a^\#$. Hence $a \in R^{EP}$ by [2, Theorem 2.1]. Thus $aa^+ = a^+a = a^\#a = a^\#a^\#a^2 = a^\#a = a^\#a^+a^2 = aa^*a^+a^\#a^2 = aa^*a^+a$. By (1), we have $a \in R^{SEP}$.

(3) If $x = a^+$ is a solution, then $aa^*a^+a^+ = a^+a^+$. By [19, Lemma 2.11],we have $aa^*a^+ = a^+$. Hence $a \in R^{SEP}$ by [5].

Unfortunately, we haven't yet reached whether $a \in R^{SEP}$ when $x = a^*, (a^+)^*$ or $(a^\#)^*$.

Also, Equation (13) can be changed as follows.

$$aa^+xa^* = xa^+. \tag{15}$$

Let $a \in R$. a is said to be partial isometry if $a^* = a^+$. We denote the set of all partial isometry elements of R by R^{PI}.

Theorem 2.16. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if Equation* (15) *has at least one solution in χ_a.*

Proof. " $\Rightarrow$ " Obviously $x = a$ is a solution.

" $\Leftarrow$ " 1) If $x = a$ is a solution, then $aa^+aa^* = aa^+$, this gives $aa^* = aa^+$. Hence $a \in R^{PI}$ by [2, Theorem 2.1(i)];

2) If $x = a^\#$ is a solution, one has that $aa^+a^\#a^* = a^\#a^+$. It follows that $a^\#a^* = a^\#a^+$ from Lemma 2.4, this gives $aa^* = aa^+$ by pre-multiplying a^2. Hence $a \in R^{PI}$ by 1);

3) If $x = a^+$ is a solution, then $aa^+a^+a^* = a^+a^+$. Pre-multiplying the equality by $1 - aa^+$, one has $(1 - aa^+)a^+a^+ = 0$, we arrive at $(1 - aa^+)a^+a^* = 0$ because $a^* = a^+aa^*$. Applying the involution on the equality, we have $a(a^+)^*(1 - aa^+) = 0$. Since $Ra(a^+)^* = Ra(a^+aa^+)^* = Ra^2a^+(a^+)^* = Raa^+(a^+)^* = R(a^+)^* = R(a^+aa^+)^* = R(a^+)^*a^+a \subseteq Ra^+a = Ra^*(a^+)^* \subseteq R(a^+)^*$, $Ra(a^+)^* = Ra^+a = Ra$, it follows that $Ra(1 - aa^+) = Ra(a^+)^*(1 - aa^+) = 0$, which implies $a \in R^{EP}$. Hence $a^\#a^+ = a^+a^+ = aa^+a^+a^* = aa^+a^\#a^*$, this infers that $a \in R^{PI}$ by 2);

4) If $x = a^*$ is a solution, one concludes that $aa^+a^*a^* = a^*a^+$. Hence $a \in R^{PI}$ by the proof of 4) of Theorem 2.6;

5) If $x = (a^\#)^*$ is a solution, then $aa^+(a^\#)^*a^* = (a^\#)^*a^+$. Taking the involution on the equality, one has $aa^\#aa^+ = (a^+)^*a^\#$, which implies $aa^+ = (a^+)^*a^\#$. Post-multiplying a^2, we have $a^2 = (a^+)^*a$, pre-multiplying a^*, one has $a^*a^2 = a^+a^2$. Hence $a \in R^{PI}$ by [2, Theorem 2.1(ii)];

6) If $x = (a^+)^*$ is a solution, we have $aa^+(a^+)^*a^* = (a^+)^*a^+$, that is, $aa^+ = (a^+)^*a^+$. Post-multiplying the equality by a, one has $a = (a^+)^*a^+a = (a^+)^*$, Therefore $a \in R^{PI}$. □

Pre-multiplying the equation (15) by a^+, we have the following equation.

$$a^+xa^* = a^+xa^+. \tag{16}$$

Theorem 2.17. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if Equation* (16) *has at least one solution in* $\{a, a^\#, a^*, (a^\#)^*, (a^+)^*\}$.

Proof. " $\Rightarrow$ " Obviously $x = a$ is a solution.

" $\Leftarrow$ " 1) If $x = a$ is a solution, then $a^+aa^* = a^+aa^+ = a^+$, this gives $a^* = a^+$. Hence $a \in R^{PI}$;

2) If $x = a^\#$ is a solution, one has $a^+a^\#a^* = a^+a^\#a^+$. It follows that $a^\#a^* = a^\#a^+$ by pre-multiplying a. Hence $a \in R^{PI}$ by [2, Theorem 2.1(iv)];

3) If $x = a^*$ is a solution, one concludes that $a^+a^*a^* = a^+a^*a^+$. Pre-multiplying the equality by a and applying the involution, we have $a^3a^+ = (a^+)^*a^2a^+$. Post-multiplying the last equality by $a^\#a$, one obtains $a^2 = (a^+)^*a$. Hence $a \in R^{PI}$ by the proof of 5) of Theorem 2.7;

4) If $x = (a^\#)^*$ is a solution, then $a^+(a^\#)^*a^* = a^+(a^\#)^*a^+$. Pre-multiply the equality by a and then taking the involution, one has $aa^+ = (a^+)^*a^\#aa^+$, Post-multiplying the last equality by a^2, one has $a^2 = (a^+)^*a$, which implies $a \in R^{PI}$ by 3);

5) If $x = (a^+)^*$ is a solution, we have $a^+(a^+)^*a^* = a^+(a^+)^*a^+$, that is, $a^+ = a^+(a^+)^*a^+$, this gives $a = aa^+a = aa^+(a^+)^*a^+a = (a^+)^*$. Therefore $a \in R^{PI}$. □

Proposition 2.18. *Let $a \in R^\# \cap R^+$, if $a^+a^+a^* = a^+a^+a^+$, then $a \in R^{PI}$.*

Proof. Since $a^+a^+a^* = a^+a^+a^+$, $a^+a^* = a^+a^+$ by [19, Lemma2.11]. Hence $a \in R^{PI}$ by [19, Corollary 2.10]. □

Theorem 2.19. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if Equation*

$$aya^*x = xy. \tag{17}$$

has at least one solution in ρ_a^2, where $\rho_a = \{a, a^\#, a^+, (a^\#)^*, (a^+)^*\}$.

Proof. " $\Rightarrow$ " If $a \in R^{PI}$, then $a^* = a^+$, it follows that

$$\begin{cases} x = a \\ y = a \end{cases}$$

is a solution.

" $\Leftarrow$ " 1) If $y = a$, then Equation (17) changes as follows

$$a^2a^*x = xa. \tag{18}$$

(1) If $x = a$ is a solution, then $a^2a^*a = a^2$. Pre-multiplying the equality by $a^+a^\#$, one has $a^*a = a^+a$, this implies $a \in R^{PI}$;

(2) If $x = a^\#$ is a solution, then $a^2a^*a^\# = a^\#a$. Post-multiplying it by a^2, we have $a^2a^*a = a^2$, by (1), we can get $a \in R^{PI}$;

(3) If $x = a^+$ is a solution, then $a^2a^*a^+ = a^+a$. Post-multiplying the equality by aa^+, one obtains $a^+a = a^+a^2a^+$, this gives $a = a^2a^+$, $a \in R^{EP}$, so $a^\# = a^+$, it follows $a^2a^*a^\# = a^2a^*a^+ = a^+a = a^\#a$, by (2), $a \in R^{PI}$;

(4) If $x = (a^+)^*$ is a solution, then $a^2a^*(a^+)^* = (a^+)^*a$, that is, $a^2 = (a^+)^*a$. Pre-multiplying it by a^*, one has $a^*a^2 = a^+a^2$, it infers $a \in R^{PI}$;

(5) If $x = (a^\#)^*$ is a solution, then $a^2a^*(a^\#)^* = (a^\#)^*a$. Post-multiplying aa^+, we have $(a^\#)^*a = (a^\#)^*a^2a^+$. Pre-multiplying $(a^+)^*a^*a^*$, we have $a = a^2a^+$, it follows that $a \in R^{EP}$, then $a^\# = a^+$, and $a^2a^*(a^+)^* = (a^+)^*a$, by (4), $a \in R^{PI}$.

2) If $y = a^{\#}$, then

$$aa^{\#}a^{*}x = xa^{\#}. \tag{19}$$

i) If $x = a$ is a solution, then $aa^{\#}a^{*}a = aa^{\#}$. Pre-multiplying the equality by a^2, we have $a^2a^*a = a^2$, by (1), $a \in R^{PI}$.

ii) If $x = a^{\#}$ is a solution, then $aa^{\#}a^*a^{\#} = a^{\#}a^{\#}$. Post-multiplying a^2, we have $aa^{\#}a^*a = aa^{\#}$, by i), we can get $a \in R^{PI}$;

iii) If $x = a^+$ is a solution, then $aa^{\#}a^*a^+ = a^+a^{\#}$. Pre-multiplying a, we have $aa^*a^+ = a^{\#}$, post-multiplying $1 - aa^+$, one has $a^{\#} = a^{\#}aa^+, a \in R^{EP}$, this gives $aa^{\#}a^*a^{\#} = aa^{\#}a^*a^+ = a^+a^{\#} = a^{\#}a^{\#}$, by ii), $a \in R^{PI}$;

iv) If $x = (a^+)^*$ is a solution, then $aa^{\#}a^*(a^+)^* = (a^+)^*a^{\#}$, that is, $aa^{\#} = (a^+)^*a^{\#}$, post-multiplying a^2, we have $a^2 = (a^+)^*a$, by (4), we have $a \in R^{PI}$;

v) If $x = (a^{\#})^*$ is a solution, then $aa^{\#}a^*(a^{\#})^* = (a^{\#})^*a^{\#}$. Pre-multiplying $1 - aa^+$, we have $(1 - aa^+)(a^{\#})^*a^{\#} = 0$. Post-multiplying $a^2a^+(a^*)^2$, we have $(1 - aa^+)a^* = 0$, this gives $a \in R^{EP}$. Hence $aa^{\#}a^*(a^+)^* = aa^{\#}a^*(a^{\#})^* = (a^{\#})^*a^{\#} = (a^+)^*a^{\#}$, by iv), $a \in R^{PI}$.

3) If $y = a^+$, then

$$aa^+a^*x = xa^+. \tag{20}$$

By Theorem 2.6, $a \in R^{PI}$.

4) If $y = (a^+)^*$, then

$$a(a^+)^*a^*x = x(a^+)^*. \tag{21}$$

That is,

$$a^2a^+x = x(a^+)^*. \tag{22}$$

(a) If $x = a$ is a solution, then $a^2a^+a = a(a^+)^*$, that is $a^2 = a(a^+)^*$. Similar to the proof of (4), we have $a \in R^{PI}$;

(b) If $x = a^{\#}$ is a solution, then $a^2a^+a^{\#} = a^{\#}(a^+)^*$, that is $aa^{\#} = a^{\#}(a^+)^*$, pre-multiplying it by a^2, we have $a^2 = a(a^+)^*$, by (a), we can get $a \in R^{PI}$;

(c) If $x = a^+$ is a solution, then $a^2a^+a^+ = a^+(a^+)^*$. Pre-multiplying it by $1-aa^+$, we have $(1-aa^+)a^+(a^+)^* = 0$, post-multiplying a^*, we have $(1 - aa^+)a^+ = 0$, this implies $a \in R^{EP}$. Hence $x = a^{\#}$ is a solution of the equation (22), by (b), $a \in R^{PI}$;

(d) If $x = (a^+)^*$ is a solution, then $a^2a^+(a^+)^* = (a^+)^*(a^+)^*$. Applying the involution on the equality, we have $a^+a^* = a^+a^+$, pre-multiplying the equality by a and then, applying the involution, we have $a^2a^+ = (a^+)^*aa^+$, post-multiply a, one has $a^2 = (a^+)^*a$, by (4), $a \in R^{PI}$;

(e) If $x = (a^{\#})^*$ is a solution, then $a^2a^+(a^{\#})^* = (a^{\#})^*(a^+)^*$. Post-multiplying the equality by aa^+, we have $(a^{\#})^*(a^+)^* = (a^{\#})^*(a^+)^*aa^+$. Applying the involution on the last equality, we have $a^+a^{\#} = aa^+a^+a^{\#}$. Post-multiplying it by a^2, we have $a^+a = aa^+a^+a$, hence $a \in R^{EP}$, this implies $x = (a^+)^*$ is a solution of Equation (22), by (d), $a \in R^{PI}$.

5) If $y = (a^{\#})^*$, then

$$a(a^{\#})^*a^*x = x(a^{\#})^*. \tag{23}$$

a) If $x = a$ is a solution, then $a(a^{\#})^*a^*a = a(a^{\#})^*$, pre-multiplying a^+, we have $(a^{\#})^*a^*a = (a^{\#})^*$. Applying the involution, one obtains $a^*aa^{\#} = a^{\#}$, this implies $a \in R^{SEP}$. Hence $a \in R^{PI}$;

b) If $x = a^{\#}$ is a solution, then $a(a^{\#})^*a^*a^{\#} = a^{\#}(a^{\#})^*$. Post-multiplying it by a^+a, we have $a^{\#}(a^{\#})^*a^+a = a^{\#}(a^{\#})^*$. Pre-multiplying it by a^+a^2, we have $(a^{\#})^*a^+a = (a^{\#})^*$. Applying the involution on the equality, we have $a^{\#} = a^+aa^{\#}, a \in R^{EP}$. Thus $aa^+ = aa^{\#} = aaa^+a^{\#} = a(a^+)^*a^*a^{\#} = a(a^{\#})^*a^*a^{\#} = a^{\#}(a^{\#})^* = a^+(a^+)^*, a = a^2a^+ = aa^+(a^+)^* = (a^+)^*, a \in R^{PI}$;

c) If $x = a^+$ is a solution, then $a(a^{\#})^*a^*a^+ = a^+(a^{\#})^*$. Pre-multiplying it by aa^+, we have $a^+(a^{\#})^* = aa^+a^+(a^{\#})^*$, post-multiplying the last equality by $(a^*)^2$, we have $a^+a^* = aa^+a^+a^*$. Applying the involution, we have

$a(a^+)^*(1-aa^+) = 0$. Noting that $Ra(a^+)^* = Ra$. Then $a(1-aa^+) = 0$, $a \in R^{EP}$. So $x = a^\#$ is a solution, by b), $a \in R^{PI}$;

d) If $x = (a^+)^*$ is a solution, then $a(a^\#)^*a^*(a^+)^* = (a^+)^*(a^\#)^*$, so $a^+aa^\#a^* = a^\#a^+$, by applying the involution. Pre-multiplying it by a^2, we obtain $aa^* = aa^+$, $a \in R^{PI}$;

e) If $x = (a^\#)^*$ is a solution, then $a(a^\#)^*a^*(a^\#)^* = (a^\#)^*(a^\#)^*$. Applying the involution on the equality, one has $a^\#a^* = a^\#a^\#$. Thus $a \in R^{PI}$. □

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