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On Sublinear Quasi-Metrics and Neighborhoods in Locally Convex Cones

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Abstract. We consider the topological structure of the sublinear quasi-metrics in locally convex cones and define the notion of a locally convex quasi-metric cone. The presence of upper bounded neighborhoods, gives necessary and sufficient conditions for the quasi-metrizability of locally convex cones. In particular, we investigate the boundedness and separatedness of locally convex quasi-metric cones and characterize the metrizability of locally convex cones.

1. Introduction

The theory of locally convex cones carries a certain topological structure which generalizes the concept of (ordered) topological vector spaces. In the similar way that the topology of a locally convex space is defined by a family of seminorms, a locally convex topological structure on a cone can also be defined through a family of sublinear quasi-metrics [1, Ch I, 5.6]. In this paper, we define the unit neighborhood of a sublinear quasi-metric which leads to the notion of a locally convex quasi-metric cone topology. Then we investigate the sublinear quasi-metrics induced by neighborhoods and discuss the quasi-metrizable locally convex cones. Also, we study the boundedness and separatedness of locally convex quasi-metric cones and present necessary and sufficient conditions for (quasi) metrizability of locally convex cones.

An *ordered cone* is a set $\mathcal{P}$ endowed with an addition $(a, b) \longmapsto a+b$ and a scalar multiplication $(\lambda, a) \longmapsto \lambda a$ for real numbers $\lambda \geq 0$. The addition is supposed to be associative and commutative, there is a neutral element $0 \in \mathcal{P}$, and for the scalar multiplication the usual associative and distributive properties hold, that is, $\lambda(\mu a) = (\lambda\mu)a$, $(\lambda + \mu)a = \lambda a + \mu a$, $\lambda(a + b) = \lambda a + \lambda b$, $1a = a$, $0a = 0$ for all $a, b \in \mathcal{P}$ and $\lambda, \mu \geq 0$. In addition, the cone $\mathcal{P}$ carries a (partial) order, i.e., a reflexive transitive relation $\leq$ that is compatible with the algebraic operations, that is $a \leq b$ implies $a + c \leq b + c$ and $\lambda a \leq \lambda b$ for all $a, b, c \in \mathcal{P}$ and $\lambda \geq 0$. For example, the extended scalar field $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ of real numbers is a preordered cone. We consider the usual order and algebraic operations in $\overline{\mathbb{R}}$; in particular, $\lambda + \infty = +\infty$ for all $\lambda \in \overline{\mathbb{R}}$, $\lambda \cdot (+\infty) = +\infty$ for all $\lambda > 0$ and $0 \cdot (+\infty) = 0$. In any cone $\mathcal{P}$, equality is obviously such an order, hence all results about ordered cones apply to cones without order structures as well.

Let $(\mathcal{P}, \leq)$ be an ordered cone. An *abstract neighborhood system* in $\mathcal{P}$ is a subset $\mathcal{V}$ of positive elements that is directed downward, closed for addition and multiplication by (strictly) positive scalars. If the all elements of $\mathcal{P}$ are *bounded below*, i.e., for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \lambda v$ for some $\lambda > 0$, then $(\mathcal{P}, \mathcal{V})$

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is called a *full locally convex cone*. The elements v of $\mathcal{V}$ define *upper* (*lower*) *neighborhoods* for the elements of $\mathcal{P}$ by $v(a) = \{b \in \mathcal{P} : b \le a + v\}$ (respectively, $(a)v = \{b \in \mathcal{P} : a \le b + v\}$), creating the *upper*, respectively *lower topologies* on $\mathcal{P}$. Their common refinement is called the *symmetric topology.* Finally, a *locally convex cone* $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system $\mathcal{V}$. Endowed with the neighborhood system $\mathcal{V} = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone.

A collection U of convex sets of $U \subset \mathcal{P}^2$ is called a *convex quasi-uniform structure*, if the following conditions hold:

(U_1) $\triangle \subset U$ for all $U \in$ U, $\triangle = \{(a, a) : a \in \mathcal{P}\}$.
(U_2) For all $U, V \in$ U there is $W \in$ U such that $W \subseteq U \cap V$.
(U_3) $\lambda U \circ \mu U \subseteq (\lambda + \mu)U$ for all $\lambda, \mu > 0$ and $U \in$ U, where $\lambda U \circ \mu U = \{(a, b) \in \mathcal{P}^2 : \exists c \in \mathcal{P}$ with $(a, c) \in \lambda U$ and $(c, b) \in \mu U\}$.
(U_4) $\lambda U \in$ U for all $U \in$ U and $\lambda > 0$.

If $(\mathcal{P}, \mathcal{V})$ is a locally convex cone, then the collection of all sets $\widetilde{v} \subseteq \mathcal{P}^2$, where $\widetilde{v} = \{(a, b) : a \le b + v\}$ for all $v \in \mathcal{V}$, defines a convex quasi-uniform structure on $\mathcal{P}$. On the other hand, if a convex quasi-uniform structure on a cone $\mathcal{P}$ has the extra property

(U_5) for all $a \in \mathcal{P}$ and $U \in$ U, there is some $\lambda > 0$ such that $(0, a) \in \lambda U$,

then it leads to a full locally convex cone, including $\mathcal{P}$ as a subcone and induces the same convex quasi-uniform structure [1, Ch I, 5.2].

2. Sublinear quasi-metrics, neighborhoods and quasi-metrizablity

Let $\mathcal{P}$ be a cone, $\mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ be the product cone with the pointwise addition and scalar multiplication with non-negative scalars $\lambda \ge 0$ and $\overline{\mathbb{R}}_+ := [0, +\infty]$. According to [1, Ch I, 5.6], the function $d : \mathcal{P}^2 \to \overline{\mathbb{R}}_+$ is called a *sublinear quasi-metric*, if it satisfies:

(M_1) $d(a, a) = 0$ for all $a \in \mathcal{P}$.
(M_2) $d(a, b) \le d(a, c) + d(c, b)$ for all $a, b, c \in \mathcal{P}$.
(M_3) $d((a, b) + (a', b')) \le d(a, b) + d(a', b')$ for all $a, b, a', b' \in \mathcal{P}$.
(M_4) $d(\lambda(a, b)) = \lambda d(a, b)$ for all $a, b \in \mathcal{P}$ and $\lambda \ge 0$.

A family of sublinear quasi-metrics $(d_i)_{i \in \mathcal{I}}$ on $\mathcal{P}$ is called *directed*, if for every $i, j \in \mathcal{I}$, there are $k \in \mathcal{I}$ and $\lambda > 0$ such that $\max\{d_i(a, b), d_j(a, b)\} \le \lambda d_k(a, b)$ for all $a, b \in \mathcal{P}$.

Definition 2.1. Let $\mathcal{P}$ be a cone and $(d_i)_{i \in \mathcal{I}}$ a directed family of sublinear quasi-metrics on $\mathcal{P}$ satisfying:

(M_5) $d_i(0, a) < +\infty$ for all $a \in \mathcal{P}$ and $i \in \mathcal{I}$.

If for every finite subset F of $\mathcal{I}$ and $\lambda > 0$, we put

$$\mathrm{U}_{\lambda F} = \{(a, b) \in \mathcal{P}^2 : d_i(a, b) \le 1/\lambda \quad \text{for all } i \in F\}$$

and $\mathtt{U}_{\mathcal{I}} = \{\mathrm{U}_{\lambda F} : \lambda > 0$ and $F \subset \mathcal{I}$ is finite$\}$, then $\mathtt{U}_{\mathcal{I}}$ forms a convex quasi-uniform structure on $\mathcal{P}$ with condition (U_5) (cf. [1, Ch I, Proposition 5.7]).

For every finite set $F \subset \mathcal{I}$ and $\lambda > 0$, we set $a \le b + v_{\lambda F}$ for elements $a, b \in \mathcal{P}$ if and only if $(a, b) \in \mathrm{U}_{\lambda F}$ and put $\mathcal{V}_{\mathcal{I}} := \{v_{\lambda F} : \lambda > 0$ and $F \subset \mathcal{I}$ is finite$\}$. Then according to [1, Ch I, 5.4], there exists a full cone $\mathcal{P} \oplus \mathcal{V}_{\mathcal{I}_0}$ with abstract neighborhood system $\mathrm{V}_{\mathcal{I}} = \{0\} \oplus \mathcal{V}_{\mathcal{I}}$, whose neighborhoods yield the same quasi-uniform structure on $\mathcal{P}$. The elements of $\mathcal{V}_{\mathcal{I}}$ form a basis for $\mathrm{V}_{\mathcal{I}}$ in the following sense: For every $a, b \in \mathcal{P}$ and $\lambda > 0$, $a \le b + v_{\lambda F}$ implies that $a \le b \oplus \mathrm{v}_{\lambda F}$. The locally convex cone topology on $\mathcal{P}$ induced by $\mathcal{V}_{\mathcal{I}}$ is called the *locally convex cone generated by* $(d_i)_{i \in \mathcal{I}}$ and denoted by $(\mathcal{P}, \mathcal{V}_{\mathcal{I}})$. In particular, let d be a sublinear quasi-metric on $\mathcal{P}$ satisfying (M_5). If we define the *unit neighborhood* v_d for all $a, b \in \mathcal{P}$ on $\mathcal{P}$ by

$$a \le b + v_d \quad \text{if and only if} \quad d(a, b) \le 1$$

and put $\mathcal{V}_d = \{v_{\lambda d} : \lambda > 0\}$, then $\mathcal{V}_d$ induces a locally convex cone topology on $\mathcal{P}$ which is called the *locally convex quasi-metric cone* generated by d and denoted by $(\mathcal{P}, \mathcal{V}_d)$ (cf. [4, Definition 2.1]).

We say that a locally convex cone $(\mathcal{P}, \mathcal{V})$ is *quasi-metrizable* if there is a sublinear quasi-metric on $\mathcal{P}$ satisfying (M_5) such that $(\mathcal{P}, \mathcal{V})$ is equivalent to the locally convex quasi-metric cone $(\mathcal{P}, \mathcal{V}_d)$.

Remark 2.2. Suppose (E, p) is a semi-normed space with unit ball $\mathbb{B}_E$ and let $Conv(E)$ be the cone of all non-empty convex subsets of E with the usual addition and scalar multiplication of sets. If we define the function $D : Conv(E)^2 \to \overline{\mathbb{R}}_+$ for all $A, B \in Conv(E)$ by

$$D(A, B) = \inf\{\lambda > 0 : A \subset B + \lambda\mathbb{B}_E\},$$

then clearly D satisfies (M_1)-(M_4). We note that $D(A, B) := +\infty$, whenever $\{\lambda > 0 : A \subset B + \lambda\mathbb{B}_E\} = \emptyset$.

For every $A \in Conv(E)$, there is $\lambda > 0$ such that $\lambda\mathbb{B}_E \cap A \neq \emptyset$ so $\{0\} \subset A + \lambda\mathbb{B}_E$, i.e., D satisfies (M_5). Thus $(Conv(E), \mathcal{V}_D)$ is a locally convex quasi-metric cone, where $\mathcal{V}_D = \{v_{\lambda D} : \lambda > 0\}$. Via the embedding $x \to \{x\} : E \to Conv(E)$, we may consider E as a subcone of $Conv(E)$ hence $(E, \mathcal{V}_D)$ is also a locally convex quasi-metric cone. We note that $D(\{a\}, \{b\}) = p(a - b)$ for all $a, b \in E$, consequently $a \leq b + v_D$ if and only if $p(a - b) \leq 1$ so the lower, upper and symmetric topologies of $(E, \mathcal{V}_D)$ are identical to the given topology of (E, p) (cf. [9, 2.1 (c)]).

Example 2.3. Let $Conv(\overline{\mathbb{R}})$ be the cone of all non-empty convex subsets of $\overline{\mathbb{R}}$ with the usual addition and scalar multiplication of sets by non-negative scalars $\lambda \geq 0$. We define the function $D : Conv(\overline{\mathbb{R}})^2 \to \overline{\mathbb{R}}_+$ for all $A, B \in Conv(\overline{\mathbb{R}})$ by

$$D(A, B) = \inf\{\lambda > 0 : A \subset\downarrow B + \lambda\mathbb{B}_{\mathbb{R}}\},$$

where $\downarrow B = \{a \in \overline{\mathbb{R}} : a \leq b \text{ for some } b \in B\}$. Since $A \subset\downarrow A + \lambda\mathbb{B}_{\mathbb{R}}$ for all $\lambda > 0$ and $A \in Conv(\overline{\mathbb{R}})$, we have $D(A, A) = 0$, i.e., (M_1) holds. Let $A, B, C \in Conv(\overline{\mathbb{R}})$. If $D(A, B) = +\infty$ or $D(B, C) = +\infty$, then clearly (M_2) holds. If $D(A, B) = \lambda$ and $D(B, C) = \mu$ for some $\lambda, \mu > 0$, then $A \subset\downarrow B + \lambda\mathbb{B}_{\mathbb{R}}$ and $B \subset\downarrow C + \mu\mathbb{B}_{\mathbb{R}}$ which yields $A \subset\downarrow C + (\lambda + \mu)\mathbb{B}_{\mathbb{R}}$, i.e., $D(A, C) \leq D(A, B) + D(B, C)$. The condition (M_3) is similar to (M_2) and (M_4) is trivial. For every $A \in Conv(\overline{\mathbb{R}})$, there is $\lambda > 0$ such that $0 \in\downarrow A + \lambda\mathbb{B}_{\mathbb{R}}$ so $D(\{0\}, A) < +\infty$, i.e., (M_5) is also satisfied for D. Thus $(Conv(\overline{\mathbb{R}}), \mathcal{V}_D)$ is a locally convex quasi-metric cone. In particular, $(Conv(\mathbb{R}), \mathcal{V}_D)$ and $(Conv(\mathbb{R}_+), \mathcal{V}_D)$ are locally convex quasi metric cones.

We may consider $\overline{\mathbb{R}}$ as a subcone of $Conv(\overline{\mathbb{R}})$ so $(\overline{\mathbb{R}}, \mathcal{V}_d)$ is also a locally convex quasi-metric cone, where the sublinear quasi-metric $d : \overline{\mathbb{R}}^2 \to \overline{\mathbb{R}}_+$ for all $(x, y) \in \overline{\mathbb{R}}^2$ is given by $d(x, y) = D(\{x\}, \{y\})$, i.e.,

$$d(x, y) = \begin{cases} \max\{x - y, 0\}, & \text{if} \quad y \neq +\infty, \\ 0, & \text{if} \quad y = +\infty. \end{cases}$$

In particular, $(\mathbb{R}, \mathcal{V}_d)$ and $(\mathbb{R}_+, \mathcal{V}_d)$ are locally convex quasi-metric cones.

If $(\mathcal{P}, \mathcal{V})$ is a locally convex cone then $\mathcal{V}_v = \{\lambda v : \lambda > 0\}$ is a neighborhood system on $\mathcal{P}$ for all $v \in \mathcal{V}$, and $(\mathcal{P}, \mathcal{V}_v)$ is again a locally convex cone [8, p. 13].

Proposition 2.4. *If $(\mathcal{P}, \mathcal{V})$ is a locally convex cone and $v \in \mathcal{V}$, then*

(a) *the function $d_v : \mathcal{P}^2 \to \overline{\mathbb{R}}_+$ defined by*

$$d_v(a, b) = \inf\{\lambda > 0 : a \leq b + \lambda v\} \quad \textit{for all} \quad (a, b) \in \mathcal{P}^2$$

is a sublinear quasi-metric on $\mathcal{P}$ satisfying (M_5),

(b) *$(\mathcal{P}, \mathcal{V}_v)$ is quasi-metrizable.*

Proof. (a) Since $a \leq a + \lambda v$ for all $a \in \mathcal{P}$ and $\lambda > 0$, we have $d_v(a, a) = 0$, i.e., (M_1) holds. For (M_2), let $a, b, c \in \mathcal{P}$. If $d_v(a, c) = +\infty$ or $d_v(c, b) = +\infty$ then clearly (M_2) holds. If $d_v(a, c) = \lambda$ and $d_v(c, b) = \mu$ for some $\lambda, \mu \geq 0$, then $a \leq c + \lambda v, c \leq b + \mu v$, so $a \leq b + (\lambda + \mu)v$, hence $d_v(a, b) \leq d_v(a, c) + d_v(c, b)$. In a similar way, we can verify (M_3) and the condition (M_4) is clear. Thus d_v is a sublinear quasi-metric. Since every element $a \in \mathcal{P}$ is bounded below, $0 \leq a + \lambda v$ for some $\lambda > 0$, hence $d_v(0, a) \leq \lambda < +\infty$ i.e., d_v satisfies (M_5). For (b), we have $a \leq b + v_{d_v}$ for elements $a, b \in \mathcal{P}$ if and only if $a \leq b + v$, that is, $\mathcal{V}_v$ and $\mathcal{V}_{d_v}$ are equivalent to each other. □

We say that a sublinear quasi-metric is a *sublinear metric*, if it also satisfies:

(M_6) $d^{-1}(a,b) = d(a,b)$ for all $a,b \in \mathcal{P}$, where $d^{-1}(a,b) = d(b,a)$.
(M_7) $d(a,b) \neq 0$ for $a,b \in \mathcal{P}$ if $a \neq b$.

If a sublinear metric d on $\mathcal{P}$ satisfies (M_5) then $(\mathcal{P}, \mathcal{V}_d)$ is called the *locally convex metric cone* generated by d on $\mathcal{P}$.

Proposition 2.5. *If $(\mathcal{P}, \mathcal{V}_d)$ is a locally convex quasi-metric cone and $d(a,0) < +\infty$ for all $a \in \mathcal{P}$, then the function $d^s : \mathcal{P}^2 \to \overline{\mathbb{R}}_+$ defined by*

$$d^s(a,b) = \max\{d(a,b), d^{-1}(a,b)\} \quad \textit{for all} \quad (a,b) \in \mathcal{P}^2$$

is a sublinear quasi-metric on $\mathcal{P}$ satisfying (M_5) *and $(\mathcal{P}, \mathcal{V}_{d^s})$ is a locally convex quasi-metric cone.*

Proof. Clearly, d^s satisfies (M_1)-(M_4), so it is a sublinear quasi-metric. By the assumption, for every $a \in \mathcal{P}$, there is $\lambda > 0$ such that $d(a,0) \leq \lambda$. On the other hand, by the condition (M_5) for d, we have $d(0,a) \leq \lambda'$ for some $\lambda' > 0$. Thus $d^s(0,a) \leq \max\{\lambda, \lambda'\} < +\infty$, so d^s satisfies (M_5) and $(\mathcal{P}, \mathcal{V}_{d^s})$ is a locally convex quasi-metric cone. □

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *separated* if $\bar{a} = \bar{b}$ for $a,b \in \mathcal{P}$ implies $a = b$, where $\bar{a}$ is the closure of $\{a\}$ with respect to the lower topology [1, Ch I, 3.8]. We recall that according to the Proposition 3.9 in [1], $\mathcal{P}$ is separated if and only if the symmetric topology is Hausdorff (equivalently the upper topology is T_0), i.e., $\overline{a}^s = \{a\}$ for all $a \in \mathcal{P}$, where $\overline{a}^s$ is the closure of $\{a\}$ in the symmetric topology. The separating families of linear mappings have been studied for polar (or weak) topologies in [3]-[7]. Here, we consider the separating families of sublinear quasi-metrics and discuss the metrizability of locally convex cones. We say that a family of sublinear quasi-metrics $(d_i)_{i \in \mathcal{I}}$ on a cone $\mathcal{P}$ is *separating* if for all $a,b \in \mathcal{P}$ with $a \neq b$ there is $i \in \mathcal{I}$ such that $d_i^s(a,b) \neq 0$.

Proposition 2.6. *If $\mathcal{P}$ is a cone and $(d_i)_{i \in \mathcal{I}}$ is a directed family of sublinear quasi-metrics on $\mathcal{P}$ satisfying* (M_5), *then $(d_i)_{i \in \mathcal{I}}$ is separating if and only if $(\mathcal{P}, \mathcal{V}_{\mathcal{I}})$ is separated.*

Proof. Let $(\mathcal{P}, \mathcal{V}_{\mathcal{I}})$ be separated and let $a \neq b$. The symmetric topology of $\mathcal{P}$ is Hausdorff by [1, Ch I, Proposition 3.9], so there is a finite set $F \subset \mathcal{I}$ and $\lambda > 0$ such that $a \leq b + v_{\lambda F}$ but $b \not\leq a + v_{\lambda F}$, hence $d_i(b,a) > \lambda$ for some $i \in F$. For the converse, let $a,b \in \mathcal{P}$ with $a \neq b$. By the assumption, there is $i \in \mathcal{I}$ such that $d_i^s(a,b) \neq 0$, which implies that $d_i(a,b) \neq 0$ or $d_i(b,a) \neq 0$, i.e., $a \not\leq b + v_{d_i}$ or $b \not\leq a + v_{d_i}$. That is, the upper topology of $(\mathcal{P}, \mathcal{V}_{\mathcal{I}})$ is T_0, so $(\mathcal{P}, \mathcal{V}_{\mathcal{I}})$ is separated by [1, Ch I, Proposition 3.9]. □

In particular, a sublinear quasi-metric d on $\mathcal{P}$ is *separating*, if for all $a,b \in \mathcal{P}$ with $a \neq b$ we have $d^s(a,b) \neq 0$, i.e., if and only if d^s satisfies in (M_7). Hence:

Corollary 2.7. *A locally convex quasi-metric cone $(\mathcal{P}, \mathcal{V}_d)$ is separated if and only if $(\mathcal{P}, \mathcal{V}_{d^s})$ is a locally convex metric cone.*

An element $a \in \mathcal{P}$ is called *v-bounded* if $a \leq \lambda v$ for some $\lambda > 0$, and a is called *bounded* if it is v-bounded for all $v \in \mathcal{V}$ [1, Ch I, 2.3]. If all elements of $\mathcal{P}$ are bounded, then they are bounded below with respect to the symmetric topology. Thus the symmetric convex quasi-uniform structure defines a locally convex cone topology as well. Let us denote this by $(\mathcal{P}, \mathcal{V}^s)$, i.e., for $a,b \in \mathcal{P}$ and $v \in \mathcal{V}$, we have $a \leq b + v^s$ if and only if $a \leq b + v$ and $b \leq a + v$. By a simple verification, we notice that the upper, lower and symmetric topologies of $(\mathcal{P}, \mathcal{V}^s)$ coincide to the original symmetric topology [1, P. 35].

Remark 2.8. If $(\mathcal{P}, \mathcal{V})$ is a locally convex cone, then

(i) If $0 \in \mathcal{V}$, then $(\mathcal{P}, \mathcal{V})$ is equivalent to $(\mathcal{P}, \mathcal{V}_0)$, where $\mathcal{V}_0 = \{0\}$. Indeed, for $v_0 = 0$, if $a \leq b + v_0$ for $a,b \in \mathcal{P}$ then $a \leq b + v$ for all $v \in \mathcal{V}$, i.e., $\mathcal{V}_0$ is finer than $\mathcal{V}$, but $\mathcal{V}_0 \subset \mathcal{V}$ so $\mathcal{V}$ is equivalent to $\mathcal{V}_0$.

(ii) If the elements of $\mathcal{P}$ are bounded and $0 \in \mathcal{V}$, then $(\mathcal{P}, \mathcal{V})$ is separated if and only if $\mathcal{P} = \{0\}$; for, we have $0 \leq a + v_0$ and $a \leq v_0$ for all $a \in \mathcal{P}$, i.e., $a \in \overline{0}^s = \{0\}$.

Proposition 2.9. *If $(\mathcal{P}, \mathcal{V})$ is a locally convex cone and $v \in \mathcal{V}$ is upper bounded, then*

(a) *$(\mathcal{P}, \mathcal{V})$ is equivalent to $(\mathcal{P}, \mathcal{V}_v)$,*
(b) *if $(\mathcal{P}, \mathcal{V})$ is separated, then $(\mathcal{P}, \mathcal{V}_v)$ is also separated.*

Proof. (a) If $0 \in \mathcal{V}$, the assertion holds by Remark 2.8 (i). Let $0 \notin \mathcal{V}$. For every $u \in \mathcal{V}$, there is a $\lambda > 0$ such that $\frac{1}{\lambda} v \leq u$, so the neighborhood system $\mathcal{V}_v$ is equivalent to $\mathcal{V}$. Part (b) is clear by (a). □

Theorem 2.10. *A locally convex cone $(\mathcal{P}, \mathcal{V})$ is quasi-metrizable if and only if $\mathcal{V}$ contains an upper bounded neighborhood.*

Proof. If $0 \in \mathcal{V}$, then by Remark 2.8 (i), $\mathcal{V}$ is equivalent to $\mathcal{V}_0 = \{0\}$, but $\mathcal{V}_0$ is equivalent to $\mathcal{V}_{d_0}$ by Proposition 2.4 (b), where $d_0 : \mathcal{P}^2 \to \overline{\mathbb{R}}_+$ for all $(a, b) \in \mathcal{P}^2$ is given by

$$d_0(a,b) = \begin{cases} 0, & \text{if} \quad a \leq b, \\ +\infty, & \text{if} \quad a \nleq b, \end{cases}$$

i.e., $(\mathcal{P}, \mathcal{V})$ is quasi-metrizable. Suppose $0 \notin \mathcal{V}$ and let $v \in \mathcal{V}$ be upper bounded. By Proposition 2.9 (a), the neighborhood system $\mathcal{V}$ is equivalent to $\mathcal{V}_v$, so $(\mathcal{P}, \mathcal{V})$ is quasi-metrizable by Proposition 2.4 (b). Conversely, let $(\mathcal{P}, \mathcal{V})$ be quasi-metrizable and let d be a sublinear quasi-metric on $\mathcal{P}$ with condition (M_5) such that $(\mathcal{P}, \mathcal{V})$ is equivalent to $(\mathcal{P}, \mathcal{V}_d)$. Fix $v \in \mathcal{V}$. Then for every $u \in \mathcal{V}$ there exist $\lambda, \mu > 0$ such that $v \leq \mu v_d \leq \lambda u$, i.e., v is upper bounded. □

Proposition 2.11. *If $(\mathcal{P}, \mathcal{V})$ is a locally convex cone and $v \in \mathcal{V}$, then*

(a) *the function $d_v^s : \mathcal{P}^2 \to \overline{\mathbb{R}}_+$ defined by*

$$d_v^s(a,b) = \max\{d_v(a,b), d_v^{-1}(a,b)\} \quad \textit{for all} \quad (a,b) \in \mathcal{P}^2$$

is a sublinear quasi-metric on $\mathcal{P}$ satisfying (M_6),
(b) *d_v^s satisfies* (M_5) *if and only if the elements of $\mathcal{P}$ are bounded,*
(c) *$(\mathcal{P}, \mathcal{V}_{d_v^s})$ is a locally convex metric cone if and only if $(\mathcal{P}, \mathcal{V}_v)$ is separated and the elements of $\mathcal{P}$ are bounded,*
(d) *$(\mathcal{P}, \mathcal{V}_v)$ is metrizable if and only if it is separated and $d_v = d_v^{-1}$.*

Proof. The proof of (a) is similar to Proposition 2.4 (a). For (b), if $b \in \mathcal{P}$ is bounded, then there is $\lambda > 0$ such that $0 \leq b + \lambda v$, $b \leq \lambda v$ which yields $d_v^s(0, b) \leq \lambda$, i.e., d_v^s satisfies (M_5). Conversely, if d_v^s satisfies (M_5) then by a similar verification the elements of $\mathcal{P}$ are bounded. By Proposition 2.4 (b), $(\mathcal{P}, \mathcal{V}_v)$ is equivalent to $(\mathcal{P}, \mathcal{V}_{d_v})$, so part (c) follows from (b) and Corollary 2.7. For (d), if $(\mathcal{P}, \mathcal{V}_v)$ is metrizable, then obviously it is separated and $d_v = d_v^{-1}$. For the converse, if $v = 0$ then $(\mathcal{P}, \mathcal{V}_0)$ is equivalent to $(\mathcal{P}, \mathcal{V}_{d_0^s})$ by Proposition 2.4 (b), where the sublinear metric $d_0^s : \mathcal{P}^2 \to \overline{\mathbb{R}}_+$ for all $(a, b) \in \mathcal{P}^2$ is given by

$$d_0^s(a,b) = \begin{cases} 0, & \text{if} \quad a = b, \\ +\infty, & \text{if} \quad a \neq b, \end{cases}$$

i.e., $(\mathcal{P}, \mathcal{V}_0)$ is metrizable. Let $v \neq 0$ and $a, b \in \mathcal{P}$ with $a \neq b$. If $d_v^s(a, b) = 0$ then $d_v(a, b) = d_v(b, a) = 0$, i.e., $a \leq b + \lambda v$ and $b \leq a + \lambda v$ for all $\lambda > 0$ which yields $\overline{a}^v = \overline{b}^v$ where $\overline{a}^v$ is the closure of a in the lower topology induced by $\mathcal{V}_v$, hence $a = b$; since $(\mathcal{P}, \mathcal{V}_v)$ is separated, i.e., d_v^s satisfies (M_7). Thus $(\mathcal{P}, \mathcal{V}_{d_v})$ is a locally convex metric cone, since $d_v = d_v^{-1}$ so $(\mathcal{P}, \mathcal{V}_v)$ is metrizable. □

Theorem 2.12. *A locally convex cone $(\mathcal{P}, \mathcal{V})$ is metrizable if and only if $\mathcal{P}$ is separated and $\mathcal{V}$ contains an upper bounded neighborhood v with $d_v = d_v^{-1}$.*

Proof. Suppose $(\mathcal{P}, \mathcal{V})$ is separated and let $d_v = d_v^{-1}$. If $0 \in \mathcal{V}$ then $(\mathcal{P}, \mathcal{V})$ is metrizable by Remark 2.8 (i) and Proposition 2.11 (d). Suppose $0 \notin \mathcal{V}$ and let $v \in \mathcal{V}$ be upper bounded. Then $(\mathcal{P}, \mathcal{V})$ is equivalent to $(\mathcal{P}, \mathcal{V}_v)$ by Proposition 2.9 (a), so $(\mathcal{P}, \mathcal{V})$ is metrizable by Proposition 2.11 (d). The converse evidently holds by Theorem 2.10. □

As a consequence of Theorem 2.12 and Proposition 2.11, we have:

Corollary 2.13. *If $(\mathcal{P},\mathcal{V})$ is separated and the elements of $\mathcal{P}$ are bounded, then $(\mathcal{P},\mathcal{V}^s)$ is metrizable if and only if $\mathcal{V}$ contains an upper bounded neighborhood.*

Example 2.14. (i) With the sublinear quasi metric d introduced in Example 2.3, the locally convex cone $(\overline{\mathbb{R}},\mathcal{V})$ is quasi-metrizable, where $\mathcal{V}=\{\epsilon\in\mathbb{R}:\epsilon>0\}$; indeed, for every $\epsilon>0$, we have $a\leq b+\epsilon$ for $a,b\in\overline{\mathbb{R}}$ if and only if $d(a,b)\leq\epsilon$, i.e., $\mathcal{V}$ is equivalent to $\mathcal{V}_d$. In particular, $(\mathbb{R},\mathcal{V})$, $(\mathbb{R}_+,\mathcal{V})$ are equivalent to $(\mathbb{R},\mathcal{V}_d)$ and $(\mathbb{R}_+,\mathcal{V}_d)$, respectively hence they are quasi-metrizable. We note that d does not satisfies (M_6), (M_7); so these cones are not metrizable.

The sublinear function $d^s:\overline{\mathbb{R}}^2\to\overline{\mathbb{R}}_+$ is given by

$$d^s(x,y)=\begin{cases}|x-y|, & \text{if } x,y\neq+\infty,\\ 0, & \text{if } x,y=+\infty,\\ +\infty, & \text{if } x=+\infty \text{ or } y=+\infty,\end{cases}$$

which satisfies (M_7) so $(\overline{\mathbb{R}},\mathcal{V}_{d^s})$ is a locally convex metric cone. We note that $d^s(x,y)=|x-y|$ for all $x,y\in\mathbb{R}$ so $(\mathbb{R},\mathcal{V}_{d^s})$ and $(\mathbb{R}_+,\mathcal{V}_{d^s})$ coincide to the usual metric space on $\mathbb{R}$ and $\mathbb{R}_+$.

(ii) With the singleton neighborhood system $\mathcal{V}_0=\{0\}$, the subcone $\overline{\mathbb{R}}_+$ of $\overline{\mathbb{R}}$ is also a full locally convex cone and the symmetric topology of $(\overline{\mathbb{R}}_+,\mathcal{V}_0)$ is the discrete topology on $\overline{\mathbb{R}}_+$ [9, Example 2.1 (b)]. With the sublinear quasi-metric d_0 in Theorem 2.10, $(\overline{\mathbb{R}}_+,\mathcal{V}_{d_0})$ is a locally convex quasi-metric cone; indeed, for $v_0=0$ and $\lambda>0$, we have $a\leq b+\lambda v_0$ if and only if $d_0(a,b)=0\leq 1/\lambda$, i.e., $a\leq b+v_{\lambda d_0}$. That is, $\mathcal{V}_0$ is equivalent to $\mathcal{V}_{d_0}$. The function $d_0^s:\overline{\mathbb{R}}_+^2\to\overline{\mathbb{R}}_+$ is given by

$$d_0^s(x,y)=\begin{cases}0, & \text{if } x=y,\\ +\infty, & \text{if } x\neq y,\end{cases}$$

which induces the discrete topology on $\overline{\mathbb{R}}_+$.

Example 2.15. For $1\leq p<+\infty$, we define the $\overline{\ell}_p$*-norm* of a sequence $x=(x_i)_{i\in\mathbb{N}}$ in $\overline{\mathbb{R}}$ by

$$\|x\|_p=\begin{cases}\left(\sum_{i=1}^{\infty}|x_i|^p\right)^{\frac{1}{p}}, & \text{if } x\subset\mathbb{R},\\ +\infty, & \text{if } \exists i\in\mathbb{N},\ x_i=+\infty,\end{cases}$$

and for $p=+\infty$ as

$$\|x\|_\infty=\begin{cases}\sup_{i\in\mathbb{N}}|x_i|, & \text{if } x\subset\mathbb{R},\\ +\infty, & \text{if } \exists i\in\mathbb{N}, x_i=\infty.\end{cases}$$

If we set $\overline{\ell}_{\mathrm{p}}(\overline{\mathbb{R}}):=\{(x_i)_{i\in\mathbb{N}}\subset\overline{\mathbb{R}}:\|(x_i^-)_{i\in\mathbb{N}}\|_p<+\infty\}$, then with the following operation $\overline{\ell}_{\mathrm{p}}(\overline{\mathbb{R}})$ is a cone:

$$x+y=(x_i+y_i)_{i\in\mathbb{N}},\quad \lambda x=(\lambda x_i)_{i\in\mathbb{N}}\quad \text{for all}\quad x,y\in\overline{\ell}_{\mathrm{p}}(\overline{\mathbb{R}})\quad\text{and}\quad \lambda>0$$

(cf. [10, Ch I, 1.4 (f)]). We define the function $d_p:\overline{\ell}_p(\overline{\mathbb{R}})\times\overline{\ell}_p(\overline{\mathbb{R}})\to\overline{\mathbb{R}}_+$ for all $x,y\in\overline{\ell}_p(\overline{\mathbb{R}})$, $x=(x_i)_{i\in\mathbb{N}}$, $y=(y_i)_{i\in\mathbb{N}}$ by

$$d_p(x,y)=\begin{cases}\|((x_i-y_i)^+)_{i\in\mathbb{N}}\|_p, & \text{if } \exists i\in\mathbb{N}, y_i<+\infty,\\ 0, & \text{if } \forall i\in\mathbb{N}, y_i=+\infty,\\ +\infty, & \text{if } \exists i\in\mathbb{N},\ x_i=+\infty.\end{cases}$$

It is easy to verify that d_p satisfies (M_1)-(M_4). For every $x \in \overline{\ell}_p(\overline{\mathbb{R}})$, $x = (x_i)_{i\in\mathbb{N}}$, we have

$$d_p(0,x) = \|((0 - x_i)^+)_{i\in\mathbb{N}}\|_p = \|(x_i^-)_{i\in\mathbb{N}}\|_p < +\infty,$$

so d_p also satisfies (M_5). Thus $(\overline{\ell}_p(\overline{\mathbb{R}}), \mathcal{V}_{d_p})$ is a locally convex quasi-metric cone.

The function $d_p^s : \overline{\ell}_p(\overline{\mathbb{R}}) \times \overline{\ell}_p(\overline{\mathbb{R}}) \to \overline{\mathbb{R}}^+$ for all $x, y \in \overline{\ell}_p(\overline{\mathbb{R}})$, $x = (x_i)_{i\in\mathbb{N}}$ and $y = (y_i)_{i\in\mathbb{N}}$ is given by

$$d_p^s(x,y) = \begin{cases} \|x-y\|_p\,, & \text{if} \quad x, y \subset \mathbb{R}, \\ 0\,, & \text{if} \quad \forall i \in \mathbb{N}, x_i = y_i = +\infty, \\ +\infty, & \text{if} \quad \exists i \in \mathbb{N},\ x_i = +\infty \text{ or } y_i = +\infty \end{cases}$$

and satisfies (M_7) so $(\overline{\ell}_p(\overline{\mathbb{R}}), \mathcal{V}_{d_p^s})$ is a locally convex metric cone. In particular, $(\overline{\ell}_p(\mathbb{R}), \mathcal{V}_{d_p^s})$ and $(\overline{\ell}_p(\mathbb{R}_+), \mathcal{V}_{d_p^s})$ are identical to the usual spaces $\ell_p(\mathbb{R})$ and $\ell_p(\mathbb{R}_+)$.

We note that a locally convex cone is not necessary to be quasi-metrizable:

Example 2.16. Let $\mathcal{P}$ be the cone of all sequences $x = (x_i)_{i\in\mathbb{N}}$ in $\overline{\mathbb{R}}$ with the pointwise operations of addition and scalar multiplication by non-negative scalars $\lambda \geq 0$. For every $n \in \mathbb{N}$, we set

$$v_n := (\epsilon_i)_{i\in\mathbb{N}} \in \mathcal{P}, \quad \text{where} \quad \epsilon_i = \begin{cases} \frac{1}{n}, & i = 1, 2, ..., n, \\ 0, & \text{otherwise}, \end{cases}$$

and $\mathcal{V}_{\mathbb{N}} := \{v_n : n \in \mathbb{N}\}$. For elements $x, y \in \mathcal{P}$ and $n \in \mathbb{N}$, we define

$$x \leq y + v_n \quad \text{if} \quad x_i \leq y_i + \frac{1}{n} \quad \text{for} \quad i = 1, 2, ..., n.$$

If we set $U_n = \{(x,y) \in \mathcal{P}^2 : x \leq y + v_n\}$ for all $n \in \mathbb{N}$ then $\mathrm{U}_{\mathbb{N}} = \{U_n : n \in \mathbb{N}\}$ forms a convex quasi-uniform structure on $\mathcal{P}$ with condition (U_5); indeed, for every $x \in \mathcal{P}, x = (x_i)_{i\in\mathbb{N}}$ and $n \in \mathbb{N}$, we have $0 \leq x + \lambda v_n$, where $\lambda = \max\{|x_i| : i = 1, 2, ..., n\}$, i.e., $(0, x) \in \lambda U_n$. Thus according to [1, Ch I, 5.4], there exists a full cone $\mathcal{P} \oplus \mathcal{V}_{\mathbb{N}_0}$ with abstract neighborhood system $V_{\mathbb{N}} = \{0\} \oplus \mathcal{V}_{\mathbb{N}}$, whose neighborhoods yield the same quasi-uniform structure on $\mathcal{P}$. The elements of $\mathcal{V}_{\mathbb{N}}$ form a basis for $V_{\mathbb{N}}$ in the following sense: For every $a, b \in \mathcal{P}$ and $n \in \mathbb{N}$, $a \leq b + v_n$ implies that $a \leq b \oplus \mathrm{v}_n$. Therefore $(\mathcal{P}, \mathcal{V}_{\mathbb{N}})$ is a locally convex cone with the countable base $\mathcal{V}_{\mathbb{N}}$ (cf. [2, Example 2.3.25]).

We claim that $\mathcal{V}_n$ does not have any upper bounded neighborhood and $(\mathcal{P}, \mathcal{V}_{\mathbb{N}})$ is not quasi-metrizable. For every $n \in \mathbb{N}$, if we choose $x, y \in \mathcal{P}$, $x = (x_i)_{i\in\mathbb{N}}$, $y = (y_i)_{i\in\mathbb{N}}$ such that

$$x_i = \begin{cases} 2, & \text{for } \ i = n+1, \\ 0, & \text{for } \ i \neq n+1, \end{cases} \quad \text{and} \quad y_i = \begin{cases} 1, & \text{for } \ i = n+1, \\ 0, & \text{for } \ i \neq n+1, \end{cases}$$

then

$$x \leq y + v_n \qquad \text{but} \qquad x \nleq y + v_{n+1} \tag{1}$$

i.e., v_n is not upper bounded. Now, assume to the contrary that $(\mathcal{P}, \mathcal{V}_{\mathbb{N}})$ is quasi-metrizable and let d be a sublinear quasi-metric on $\mathcal{P}$ satisfying (M_5) such that $\mathcal{V}_{\mathbb{N}}$ is equivalent to $\mathcal{V}_d$. If we choose $n \in \mathbb{N}$ such that $v_n \leq v_d$, then (1) yields $0 < d(x,y) \leq 1$. On the other hand, for every $\lambda > 0$, we have

$$\lambda x \leq \lambda y + v_n \leq \lambda y + v_d,$$

so $d(\lambda(x,y) \leq 1$. Thus $\lambda d(x,y) \neq d(\lambda(x,y))$ for all $\lambda \geq \frac{2}{d(x,y)}$ which is a contradiction.

References

[1] K. Keimel, W. Roth, Ordered Cones and Approximation, Lecture Notes in Mathematics, vol. 1517, Springer Verlag, Heidelberg-Berlin-New York, 1992.
[2] R.E. Megginson, An Introduction to Banach Space Theory, Springer-Verlag, New York, 1998.
[3] M.R. Motallebi, Locally convex product and direct sum cones, Mediterr. J. Math. 11 (2014) 913–927.
[4] M.R. Motallebi, Locally convex projective limit cones, Math. Slovaca 66 (2016) 1387–1398.
[5] M.R. Motallebi, On weak completeness of products and direct sums in locally convex cones, Period. Math. Hung. 75 (2017) 322–329.
[6] M.R. Motallebi, Weak compactness of direct sums in locally convex cones, Stud. Sci. Math. Hung. 55 (2018) 487–497.
[7] M.R. Motallebi, H. Saiflu, Duality on locally convex cones, J. Math. Anal. Appl. 337 (2008) 888–905.
[8] W. Roth, Boundedness and connectedness components for locally convex cones, N.Z.J. Math. 34 (2005) 143–158
[9] W. Roth, Locally convex lattice cones, J. Convex Anal. 16 (2009) 1–31.
[10] W. Roth, Operator-Valued Measures and Integrals for Cone-Valued Functions, Lecture Notes in Mathematics, vol. 1964, Springer Verlag, Heidelberg-Berlin-New York, 2009.