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Equivalence Among L-Closure (Interior) Operators, L-Closure (Interior) Systems and L-Enclosed (Internal) Relations

Fangfang Zhao[a], **Bin Pang**[a]

[a]*Beijing Key Laboratory on MCAACI, School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China*

Abstract. Closure (interior) operators and closure (interior) systems are important tools in many mathematical environments. Considering the logical sense of a complete residuated lattice L, this paper aims to present the concepts of L-closure (L-interior) operators and L-closure (L-interior) systems by means of infimums (supremums) of L-families of L-subsets and show their equivalence in a categorical sense. Also, two types of fuzzy relations between L-subsets corresponding to L-closure operators and L-interior operators are proposed, which are called L-enclosed relations and L-internal relations. It is shown that the resulting categories are isomorphic to that of L-closure spaces and L-interior spaces, respectively.

1. Introduction

Closure operators and closure systems play important roles in many research areas. As we all know, cotopologies are closure systems which are closed for finite unions and convex structures are closure systems which are closed for directed unions (equivalently, nested unions). Correspondingly, topological closure operators and algebraic closure operators can be used to characterize cotopologies and convex structures, respectively. In the dual situation, interior operators and interior systems are also investigated from different aspects.

Inspired by fuzzy set theory, closure operators and closure systems have been generalized to the fuzzy case in different approaches. Biacino and Gerla [5] studied fuzzy closure systems from L-subalgebras. Zhou [34] introduced the concept of L-closure spaces and studied many important properties for L-closure spaces. Luo and Fang [13] proposed the concepts of fuzzifying closure operators and fuzzifying closure systems. Shi and Pang [21] generalized fuzzifying closure operators and fuzzifying closure systems to L-fuzzy closure operators and L-fuzzy closure systems. Recently, Pang et al. [14–16] introduced fuzzy algebraic closure operators to characterize fuzzy convex structures (also called fuzzy algebraic closure systems). More research on fuzzy convex structures (fuzzy algebraic closure systems) can be found in [22, 23, 25, 31]. Recently, Wu et al. [28] proposed L-topological derived internal (resp. enclosed) relations to characterize L-topological derived interior (resp. closure) operators in L-topological spaces. Besides, many researcher studied fuzzy closure operators in matroid theory [12, 26, 29, 33]. Considering other

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Corresponding author: Bin Pang
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Email addresses: zhaoff827@126.com (Fangfang Zhao), pangbin1205@163.com (Bin Pang)

characterizations of fuzzy closure operators, Shi and Shi [24] proposed two new types of fuzzy relations between L-subsets in (L, M)-fuzzy topological spaces, which are called (L, M)-fuzzy enclosed relations and (L, M)-fuzzy internal relations. Xiu and Li [32] introduced the corresponding concepts in the framework of (L, M)-fuzzy convex spaces and showed their equivalence. Later, Liao and Wu [27] combined these relations to characterize (L, M)-fuzzy topological convex spaces. Up to now, fuzzy closure (interior) operators, fuzzy closure (interior) systems and fuzzy enclosed (internal) relations have been composed to a unified universe and are being investigated in a common framework.

Considering complete residuated lattice L as the lattice background, Bělohlávek [2] started a general theory of fuzzy closure systems and fuzzy closure operators. Later, Georgescu and Popescu [6] introduced the concepts of closure operators and closure systems in a non-commutative lattice valued environment, where the lattice valued environment come form a generalized residuated lattice. In [7], Fang and Yue discussed the categorical relationship between L-fuzzy closure operators and L-fuzzy closure systems. Moreover, Bělohlávek and Funioková [4] proved that there is a one-to-one correspondence between fuzzy interior systems and fuzzy interior operators. Afterwards, Ramadan [19] showed the existence of a Galois correspondence between the category of L-fuzzy interior system spaces and that of L-fuzzy interior spaces. Recently, Li and Wang [11] introduced L-ordered interior operators and L-ordered neighborhood systems to characterize stratified L-concave spaces (also called stratified algebraic L-interior systems). Besides, Xiu [31] showed that the category of L-concave spaces can be embedded in that of L-convergence spaces as a reflective subcategory.

As a basic and important algebraic structure, a complete residuated lattice $(L, *, \to, \vee, \wedge)$ possesses plentiful logical flavor with the logical tensor $*$ and the logical implication $\to$. This provides a more logical environment for the research on fuzzy mathematical structures. As generalizations of infimums and supremums of L-subsets and subsethoods between L-subsets, infimums and supremums of L-families of L-subsets [3] and subsethood degrees between L-subsets [3] are introduced by means of the logical tensor $*$ and the logical implication $\to$ on L. This motivates us to combine infimums and supremums of L-families and subsethood degrees between L-subsets with fuzzy closure (interior) operators, fuzzy closure (interior) systems and fuzzy enclosed (internal) relations.

This structure of this paper is organized as follows. In Section 2, we review some notions and notations. In Section 3, we use the infimums of L-families of L-subsets and subsethood degrees between L-subsets as tools to define L-closure systems and L-closure operators, respectively. And we study their relationship in a categorical aspect. In Section 4, we present a new type of fuzzy relations between L-subsets, which is called L-enclosed relations. In addition, we study its relationship with L-closure operators. In Section 5 and Section 6, we introduce the notions of L-interior operators, L-interior systems by means of supremums of L-families of L-subsets and L-internal relations between L-subsets. And we show the categories of L-interior system spaces and L-internal relational spaces are both isomorphic to that of L-interior spaces. In Section 7, we discuss the relationship between L-closure operators and L-interior operators.

2. Preliminaries

Let L be a complete lattice. The largest element and the smallest element in L are denoted by $\top$ and $\bot$, respectively.

Definition 2.1. A complete residuated lattice is a structure $(L, *, \to, \vee, \wedge)$, where $(L, \vee, \wedge, \bot, \top)$ is a complete lattice with the largest element $\top$ and the smallest element $\bot$; $(L, *, \top)$ is a commutative monoid, i.e., $*$ is commutative, associative, and $x * \top = x$ holds for each $x \in L$; and $(*, \to)$ is an adjoint pair, i.e., $x * y \leq z \Longleftrightarrow x \leq y \to z$ holds for each $x, y, z \in L$.

The basic properties of the complete residuated lattice are referred to [3, 8, 9].

Given a complete residuated lattice $(L, *, \to, \vee, \wedge)$, a unary operator $\neg$ which is called precomplement is defined as $\neg x = x \to 0$.

Definition 2.2. ([18]) Let $(L, *, \to, \vee, \wedge)$ be a complete residuated lattice and $\neg$ be the precomplement operator. If $\neg\neg x = x$ holds for each $x \in L$, then $(L, *, \to, \vee, \wedge)$ is called a regular complete residuated lattice.

Lemma 2.3. ([18]) *Let* $(L, *, \to, \vee, \wedge)$ *be a regular complete residuated lattice, then the following conditions hold.*

(R1) $x \to y = \neg y \to \neg x$;
(R2) $x \to y = \neg(x * \neg y)$;
(R3) $\neg\left(\bigwedge_{i\in I} x_i\right) = \bigvee_{i\in I}(\neg x_i)$;
(R4) If $x \leq y$, then $\neg y \leq \neg x$.

Let X be a nonempty set. An L-subset on X is a mapping from X to L, and L^X denotes the set of all L-subsets on X. All algebraic operations on L can be extended pointwise to L^X. The largest element and the smallest element in L^X are denoted by $\underline{\top}$ and $\underline{\perp}$, respectively. We say $\{A_i\}_{i\in I}$ is a directed (resp. co-directed) subset of L^X if for each $A_{i1}, A_{i2} \in \{A_i\}_{i\in I}$, there exists $A_{i3} \in \{A_i\}_{i\in I}$ such that $A_{i1}, A_{i2} \leq A_{i3}$ (resp. $A_{i1}, A_{i2} \geq A_{i3}$).

Let X, Y be two nonempty sets and $f : X \longrightarrow Y$ be a mapping. Define $f_L^{\to} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ by $f_L^{\to}(A)(y) = \bigvee_{f(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $f_L^{\leftarrow}(B) = B \circ f$ for $B \in L^Y$, respectively.

In [3], the supremum and infimum of an L-family $\mathcal{A} : L^X \longrightarrow L$ on X are defined as follows:

$$\sqcup\mathcal{A} = \bigvee_{B\in L^X} (\mathcal{A}(B) * B); \ \sqcap\mathcal{A} = \bigwedge_{B\in L^X} (\mathcal{A}(B) \to B).$$

Given $A, B \in L^X$, the subsethood degree [3] of A and B is defined by

$$\mathcal{S}(A, B) = \bigwedge_{x\in X} (A(x) \to B(x)).$$

In the following lemma, we collect some properties of subsethood degrees which will be used in the sequel.

Lemma 2.4. ([2, 3, 16]) *Let* $A, B, C \in L^X$ *and* $f : X \longrightarrow Y$ *be a mapping. Then*

(S1) $\mathcal{S}(A, B) = \top \Longleftrightarrow A \leq B$;
(S2) $\mathcal{S}(A, B) \leq \mathcal{S}(C, A) \to \mathcal{S}(C, B)$;
(S3) $\mathcal{S}(A, B) \leq \mathcal{S}(B, C) \to \mathcal{S}(A, C)$;
(S4) $A \leq \mathcal{S}(A, B) \to B$;
(S5) $\mathcal{S}(A, B) * \mathcal{S}(B, C) \leq \mathcal{S}(A, C)$;
(S6) $\mathcal{S}(f_L^{\to}(A), B) = \mathcal{S}(A, f_L^{\leftarrow}(B))$.

Terminologies of category theory used this paper can be seen in [1].

3. *L*-Closure Operators and *L*-Closure Systems

In this section, we will introduce the concepts of L-closure operators by means of the subsethood degree between L-subsets and L-closure systems by means of the infimum of L-families, and then we will study their relationship from a categorical aspect. Considering their algebraic and topological counterparts, we will also propose the concepts of algebraic (topological) L-closure operators and algebraic (topological) L-closure systems.

Definition 3.1. A mapping $\mathfrak{c}$: $L^X \longrightarrow L^X$ is called an L-closure operator if it satisfies:

(LCL1) $\mathfrak{c}(\underline{\perp}) = \underline{\perp}$;
(LCL2) $\mathfrak{c}(A) \geqslant A$ for each $A \in L^X$;
(LCL3) $\mathfrak{c}(a \to A) \leqslant a \to \mathfrak{c}(A)$ for each $A \in L^X$ and $a \in L$;
(LCL4) $\mathcal{S}(A, \mathfrak{c}(B)) \leqslant \mathcal{S}(\mathfrak{c}(A), \mathfrak{c}(B))$ for each $A, B \in L^X$.

For an L-closure operator $\mathfrak{c}$ on X, the pair $(X, \mathfrak{c})$ is called an L-closure space.

A mapping $f\colon (X, \mathfrak{c}_X) \longrightarrow (Y, \mathfrak{c}_Y)$ between L-closure spaces is called L-closure-preserving provided that $f_L^{\rightarrow}(\mathfrak{c}_X(A)) \leqslant \mathfrak{c}_Y(f_L^{\rightarrow}(A))$ for each $A \in L^X$.

It is easy to verify that all L-closure spaces as objects and all L-closure-preserving mappings as morphisms form a category, which is denoted by L-**CLS**.

Usually, closure operators (classical or fuzzy) are defined in terms of monotone law and the idempotent law. In the following result, we will show (LCL4) can be decomposed into these two laws.

Proposition 3.2. *Let* $\mathfrak{c} : L^X \longrightarrow L^X$ *be a mapping satisfying* (LCL2). *Then* (LCL4) *is equivalent to the following two statements.*

(LCL5) $\mathcal{S}(A, B) \leqslant \mathcal{S}(\mathfrak{c}(A), \mathfrak{c}(B))$;
(LCL6) $\mathfrak{c}(\mathfrak{c}(A)) \leqslant \mathfrak{c}(A)$.

Proof. ($\Longrightarrow$) Take each $A, B \in L^X$. Then

$$\mathcal{S}(A, B) \leqslant \mathcal{S}(A, \mathfrak{c}(B)) \leqslant \mathcal{S}(\mathfrak{c}(A), \mathfrak{c}(B)).$$

and

$$\top = \mathcal{S}(\mathfrak{c}(A), \mathfrak{c}(A)) \leqslant \mathcal{S}(\mathfrak{c}(\mathfrak{c}(A)), \mathfrak{c}(A)).$$

This means $\mathfrak{c}(\mathfrak{c}(A)) \leqslant \mathfrak{c}(A)$.

($\Longleftarrow$) Take each $A, B \in L^X$. Then

$$\mathcal{S}(A, \mathfrak{c}(B)) \leqslant \mathcal{S}(\mathfrak{c}(A), \mathfrak{c}(\mathfrak{c}(B))) \leqslant \mathcal{S}(\mathfrak{c}(A), \mathfrak{c}(B)),$$

as desired. $\square$

By means of the infimum of L-families, we propose the concept of L-closure systems as follows.

Definition 3.3. A subset $\mathcal{C} \subseteq L^X$ is called an L-closure system on X if it satisfies:

(LCS1) $\underline{\perp} \in \mathcal{C}$;
(LCS2) $\sqcap \mathcal{A} \in \mathcal{C}$ for each $\mathcal{A} : \mathcal{C} \longrightarrow L$.

For an L-closure system $\mathcal{C}$ on X, the pair $(X, \mathcal{C})$ is called an L-closure system space.

A mapping $f\colon (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ between L-closure system spaces is called L-closure-preserving provided that $B \in \mathcal{C}_Y$ implies that $f_L^{\leftarrow}(B) \in \mathcal{C}_X$.

It is easy to check that all L-closure system spaces as objects and all L-closure-preserving mappings as morphisms form a category, which is denoted by L-**CSS**.

(LCS2) means L-closure systems are closed for infimums of L-families. Now we show it can also be characterized by infimums of classical subfamilies of L-subsets.

Proposition 3.4. *Let* $\mathcal{C} \subseteq L^X$ *be a subset satisfying* (LCS1). *Then* (LCS2) is equivalent to the following statements.

(LCS3) $\bigwedge_{i \in I} A_i \in \mathcal{C}$ for each $\{A_i\}_{i \in I} \subseteq \mathcal{C}$;
(LCS4) $a \to A \in \mathcal{C}$ for each $A \in \mathcal{C}$ and $a \in L$.

Proof. ($\Longrightarrow$) For each $\{A_i\}_{i \in I} \subseteq \mathcal{C}$, define $\mathcal{A}_1\colon \mathcal{C} \longrightarrow L$ as follows:

$$\forall A \in \mathcal{C}, \mathcal{A}_1(A) = \begin{cases} \top, & A \in \{A_i\}_{i \in I}; \\ \perp, & A \notin \{A_i\}_{i \in I}. \end{cases}$$

Then

$$\sqcap \mathcal{A}_1 = \bigwedge_{B \in \mathcal{C}} (\mathcal{A}_1(B) \to B) = \bigwedge_{i \in I} (\top \to A_i) = \bigwedge_{i \in I} A_i.$$

This implies $\bigwedge_{i\in I} A_i \in \mathcal{C}$.

For each $A \in \mathcal{C}$ and $a \in L$, define $\mathcal{A}_2\colon \mathcal{C} \longrightarrow L$ as follows:

$$\forall B \in \mathcal{C}, \mathcal{A}_2(B) = \begin{cases} a, & B = A; \\ \perp, & B \neq A. \end{cases}$$

Then

$$\sqcap \mathcal{A}_2 = \bigwedge_{B\in\mathcal{C}} (\mathcal{A}_2(B) \to B) = a \to A.$$

This means $a \to A \in \mathcal{C}$.

($\Longleftarrow$) For each $\mathcal{A} : \mathcal{C} \longrightarrow L$, it follows from (LCS3) and (LCS4) that

$$\sqcap \mathcal{A} = \bigwedge_{A\in\mathcal{C}} (\mathcal{A}(A) \to A) \in \mathcal{C},$$

as desired. □

Next we will discuss the relationship between L-closure operators and L-closure systems.

Proposition 3.5. *Let $\mathfrak{c}\colon L^X \longrightarrow L^X$ be an L-closure operator on X and define $\mathcal{C}^{\mathfrak{c}} \subseteq L^X$ by*

$$\mathcal{C}^{\mathfrak{c}} = \{A \in L^X \mid \mathfrak{c}(A) \leqslant A\}.$$

Then $\mathcal{C}^{\mathfrak{c}}$ is an L-closure system on X.

Proof. It suffices to show that $\mathcal{C}^{\mathfrak{c}}$ satisfies (LCS1) and (LCS2).

(LCS1) It follows immediately from (LCL1).

(LCS2) For each $\mathcal{A}\colon \mathcal{C}^{\mathfrak{c}} \longrightarrow L$, it follows that

$$\begin{array}{rcl} \mathfrak{c}(\sqcap\mathcal{A}) & = & \mathfrak{c}\left(\bigwedge_{A\in\mathcal{C}^{\mathfrak{c}}}(\mathcal{A}(A) \to A)\right) \\ & \leqslant & \bigwedge_{A\in\mathcal{C}^{\mathfrak{c}}} \mathfrak{c}(\mathcal{A}(A) \to A) \\ & \leqslant & \bigwedge_{\mathfrak{c}(A)\leqslant A}(\mathcal{A}(A) \to \mathfrak{c}(A)) \\ & \leqslant & \bigwedge_{\mathfrak{c}(A)\leqslant A}(\mathcal{A}(A) \to A) \\ & = & \bigwedge_{A\in\mathcal{C}^{\mathfrak{c}}}(\mathcal{A}(A) \to A) = \sqcap\mathcal{A}. \end{array}$$

This implies that $\sqcap A \in \mathcal{C}^{\mathfrak{c}}$. □

Proposition 3.6. *If $f\colon (X, \mathfrak{c}_X) \longrightarrow (Y, \mathfrak{c}_Y)$ is an L-closure-preserving mapping, then so is $f\colon (X, \mathcal{C}^{\mathfrak{c}_X}) \longrightarrow (Y, \mathcal{C}^{\mathfrak{c}_Y})$.*

Proof. Since $f\colon (X, \mathfrak{c}_X) \longrightarrow (Y, \mathfrak{c}_Y)$ is L-closure-preserving, it follows that

$$\forall A \in L^X, f_L^{\rightarrow}(\mathfrak{c}_X(A)) \leqslant \mathfrak{c}_Y(f_L^{\rightarrow}(A)).$$

Then for each $B \in \mathcal{C}^{\mathfrak{c}_Y}$, we have

$$\mathfrak{c}_X(f_L^{\leftarrow}(B)) \leqslant f_L^{\leftarrow}(\mathfrak{c}_Y(f_L^{\rightarrow}(f_L^{\leftarrow}(B)))) \leqslant f_L^{\leftarrow}(\mathfrak{c}_Y(B)) \leqslant f_L^{\leftarrow}(B).$$

This shows $f_L^{\leftarrow}(B) \in \mathcal{C}^{\mathfrak{c}_X}$, as desired. □

By Propositions 3.5 and 3.6, we obtain a functor $\mathcal{F}\colon L$-**CLS** $\longrightarrow L$-**CSS** by

$$\mathcal{F} : (X, \mathfrak{c}) \longmapsto (X, \mathcal{C}^{\mathfrak{c}}) \text{ and } f \longmapsto f.$$

Proposition 3.7. *Let $\mathcal{C}$ be an L-closure system on X and define $\mathfrak{c}^{\mathcal{C}}$: $L^X \longrightarrow L^X$ by*

$$\forall A \in L^X, \mathfrak{c}^{\mathcal{C}}(A) = \bigwedge_{B \in \mathcal{C}} (\mathcal{S}(A, B) \to B).$$

Then $\mathfrak{c}^{\mathcal{C}}$ is an L-closure operator on X.

Proof. By Proposition 3.2, we only need to show $\mathfrak{c}^{\mathcal{C}}$ satisfies (LCL1)-(LCL3) and (LCL5)-(LCL6). Actually, (LCL1) and (LCL2) are straightforward.

(LCL3) Take each $a \in L$ and $A \in L^X$. Then

$$\begin{array}{rcl} a \to \mathfrak{c}^{\mathcal{C}}(A) & = & a \to \bigwedge_{B \in \mathcal{C}} (\mathcal{S}(A, B) \to B) \\ & = & \bigwedge_{B \in \mathcal{C}} (a \to (\mathcal{S}(A, B) \to B)) \\ & = & \bigwedge_{B \in \mathcal{C}} ((a * \mathcal{S}(A, B)) \to B) \\ & \geqslant & \bigwedge_{B \in \mathcal{C}} (\mathcal{S}(a \to A, B) \to B) \\ & = & \mathfrak{c}^{\mathcal{C}}(a \to A). \end{array}$$

(LCL5) Take each $A, B \in L^X$. Then

$$\begin{array}{rcl} \mathcal{S}(\mathfrak{c}^{\mathcal{C}}(A), \mathfrak{c}^{\mathcal{C}}(B)) & = & \mathcal{S}\left(\bigwedge_{D \in \mathcal{C}} (\mathcal{S}(A, D) \to D), \bigwedge_{C \in \mathcal{C}} (\mathcal{S}(B, C) \to C)\right) \\ & \geqslant & \bigwedge_{C \in \mathcal{C}} \mathcal{S}(\mathcal{S}(A, C) \to C, \mathcal{S}(B, C) \to C) \\ & \geqslant & \bigwedge_{C \in \mathcal{C}} (\mathcal{S}(B, C) \to \mathcal{S}(A, C)) \\ & \geqslant & \mathcal{S}(A, B). \end{array}$$

(LCL6) Take each $A \in L^X$. Then by Proposition 3.4, we have

$$\mathfrak{c}^{\mathcal{C}}(A) = \bigwedge_{B \in \mathcal{C}} (\mathcal{S}(A, B) \to B) \in \mathcal{C}.$$

This implies

$$\begin{array}{rcl} \mathfrak{c}^{\mathcal{C}}(\mathfrak{c}^{\mathcal{C}}(A)) & = & \bigwedge_{B \in \mathcal{C}} (\mathcal{S}(\mathfrak{c}^{\mathcal{C}}(A), B) \to B) \\ & \leqslant & \mathcal{S}(\mathfrak{c}^{\mathcal{C}}(A), \mathfrak{c}^{\mathcal{C}}(A)) \to \mathfrak{c}^{\mathcal{C}}(A) \\ & = & \mathfrak{c}^{\mathcal{C}}(A). \end{array}$$

Hence $\mathfrak{c}^{\mathcal{C}}$ is an L-closure operator on X. □

Proposition 3.8. *If f: $(X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is an L-closure-preserving mapping, then so is f: $(X, \mathfrak{c}^{\mathcal{C}_X}) \longrightarrow (Y, \mathfrak{c}^{\mathcal{C}_Y})$.*

Proof. Since f: $(X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is L-closure-preserving, it follows that

$$\forall B \in L^Y, B \in \mathcal{C}_Y \text{ implies } f_L^{\leftarrow}(B) \in \mathcal{C}_X.$$

Then for each $A \in L^X$, we have

$$\begin{array}{rcl} f_L^{\leftarrow}(\mathfrak{c}^{\mathcal{C}_Y}(f_L^{\rightarrow}(A))) & = & f_L^{\leftarrow}\left(\bigwedge_{B \in \mathcal{C}_Y} (\mathcal{S}(f_L^{\rightarrow}(A), B) \to B)\right) \\ & = & \bigwedge_{B \in \mathcal{C}_Y} f_L^{\leftarrow}(\mathcal{S}(f_L^{\rightarrow}(A), B) \to B) \\ & = & \bigwedge_{B \in \mathcal{C}_Y} (\mathcal{S}(f_L^{\rightarrow}(A), B) \to f_L^{\leftarrow}(B)) \\ & = & \bigwedge_{B \in \mathcal{C}_Y} (\mathcal{S}(A, f_L^{\leftarrow}(B)) \to f_L^{\leftarrow}(B)) \\ & \geqslant & \bigwedge_{f_L^{\leftarrow}(B) \in \mathcal{C}_X} (\mathcal{S}(A, f_L^{\leftarrow}(B)) \to f_L^{\leftarrow}(B)) \\ & \geqslant & \bigwedge_{C \in \mathcal{C}_X} (\mathcal{S}(A, C) \to C) \\ & = & \mathfrak{c}^{\mathcal{C}_X}(A). \end{array}$$

This implies that $f_L^{\rightarrow}(\mathfrak{c}^{\mathcal{C}_X}(A)) \leqslant \mathfrak{c}^{\mathcal{C}_Y}(f_L^{\rightarrow}(A))$, as desired. □

By Propositions 3.7 and 3.8, we obtain a functor $\mathcal{G}$: L-**CSS**$\longrightarrow$ L-**CLS** by

$$\mathcal{G} : (X, \mathcal{C}) \longmapsto (X, \mathfrak{c}^{\mathcal{C}}) \text{ and } f \longmapsto f.$$

Now we will show L-closure operators and L-closure systems are one-to-one corresponding in a categorical sense. That is,

Theorem 3.9. *L-**CLS** and L-**CSS** are isomorphic.*

Proof. It suffices to verify that $\mathcal{G} \circ \mathcal{F} = \mathbb{I}_{L-\mathbf{CLS}}$ and $\mathcal{F} \circ \mathcal{G} = \mathbb{I}_{L-\mathbf{CSS}}$. That is to say, we only need to prove (1) $\mathfrak{c}^{\mathcal{C}^{\mathfrak{c}}} = \mathfrak{c}$ and (2) $\mathcal{C}^{\mathfrak{c}^{\mathcal{C}}} = \mathcal{C}$.

For (1), take each $A \in L^X$. Then

$$\begin{array}{rcl} \mathfrak{c}^{\mathcal{C}^{\mathfrak{c}}}(A) & = & \bigwedge_{B\in\mathcal{C}^{\mathfrak{c}}}(\mathcal{S}(A,B) \to B) \\ & = & \bigwedge_{\mathfrak{c}(B)\leqslant B}(\mathcal{S}(A,B) \to B) \\ & \leqslant & \mathcal{S}(A, \mathfrak{c}(A)) \to \mathfrak{c}(A) \quad \text{(by (LCL6))} \\ & = & \mathfrak{c}(A), \end{array}$$

and

$$\begin{array}{rcl} \mathfrak{c}^{\mathcal{C}^{\mathfrak{c}}}(A) & = & \bigwedge_{\mathfrak{c}(B)\leqslant B}(\mathcal{S}(A,B) \to B) \\ & \geqslant & \bigwedge_{\mathfrak{c}(B)\leqslant B}(\mathcal{S}(\mathfrak{c}(A),\mathfrak{c}(B)) \to B) \\ & \geqslant & \bigwedge_{\mathfrak{c}(B)\leqslant B}(\mathcal{S}(\mathfrak{c}(A),\mathfrak{c}(B)) \to \mathfrak{c}(B)) \\ & \geqslant & \mathfrak{c}(A). \end{array}$$

This means $\mathfrak{c}^{\mathcal{C}^{\mathfrak{c}}} = \mathfrak{c}$.

For (2), take each $A \in L^X$. Then

$$A \in \mathcal{C}^{\mathfrak{c}^{\mathcal{C}}} \Longleftrightarrow \mathfrak{c}^{\mathcal{C}}(A) \leqslant A \Longleftrightarrow \mathfrak{c}^{\mathcal{C}}(A) = A \Longleftrightarrow A \in \mathcal{C}.$$

This shows $\mathcal{C}^{\mathfrak{c}^{\mathcal{C}}} = \mathcal{C}$. □

In the classical case, algebraic closure operators and topological closure operators are obtained by equipping closure operators with some "algebraic" and "topological" conditions. For L-closure operators, we can also introduce the following two concepts.

Definition 3.10. An L-closure operator $\mathfrak{c}$ on X is called algebraic if it satisfies:

(ALCL) $\mathfrak{c}\left(\bigvee_{j\in J} A_j\right) \leqslant \bigvee_{j\in J} \mathfrak{c}(A_j)$ for each directed subfamily $\{A_j\}_{j\in J} \subseteq L^X$.

For an algebraic L-closure operator $\mathfrak{c}$ on X, the pair $(X, \mathfrak{c})$ is called an algebraic L-closure space.

Definition 3.11. An L-closure operator $\mathfrak{c}$ on X is called topological if it satisfies:

(TLCL) $\mathfrak{c}(A \vee B) = \mathfrak{c}(A) \vee \mathfrak{c}(B)$ for each $A, B \in L^X$.

For a topological L-closure operator $\mathfrak{c}$ on X, the pair $(X, \mathfrak{c})$ is called a topological L-closure space.

The full subcategories of L-**CLS** consisting of algebraic and topological L-closure spaces are denoted by L-**ACLS** and L-**TCLS**, respectively.

Example 3.12. (1) Define $\mathfrak{c} : L^X \longrightarrow L^X$ as follows $\mathfrak{c}(A) = A$. Then it is easy to check that $\mathfrak{c}$ satisfies (LCL1)-(LCL4), (ALCL) and (TLCL). Hence $\mathfrak{c}$ is an algebraic and topological L-closure operator on X.

(2) Define $\mathfrak{c} : L^X \longrightarrow L^X$ as follows $\mathfrak{c}(A) = \bigvee_{x\in X} A(x)$. Then it is easy to check that $\mathfrak{c}$ satisfies (LCL1)-(LCL3) and (LCL5)-(LCL6), (ALCL) and (TLCL). Hence $\mathfrak{c}$ is an algebraic and topological L-closure operator on X.

By Theorem 3.9, we know L-closure operators and L-closure systems are one-to-one corresponding. Then a natural question has risen: what kind of L-closure systems are corresponding to algebraic and topological L-closure operators?

Definition 3.13. An L-closure system $\mathcal{C}$ on X is called algebraic if it satisfies:

(ALCS) $\bigvee_{j\in J} C_j \in \mathcal{C}$ for each directed subfamily $\{C_j\}_{j\in J} \subseteq \mathcal{C}$.

For an algebraic L-closure system $\mathcal{C}$ on X, the pair $(X, \mathcal{C})$ is called an algebraic L-closure system space.

Example 3.14. ([25]) Let X be a vector space over a real number field $\mathbf{K}$ and let $A : X \longrightarrow ([0,1], \wedge)$ be a fuzzy set in X. A is called convex [10] if $A(x) \wedge A(y) \leq A(kx + (1-k)y)$ for any $x, y \in X$ and $k \in \mathbf{K}$ with $0 \leq k \leq 1$. Then

$$\mathcal{C} = \{A \in [0,1]^X \mid A \text{ is convex}\}$$

is an algebraic L-closure systems on X. It is easy to verify that $\mathcal{C}$ satisfies (LCS1), (LCS3)-(LCS4) and (ALCS). Hence $\mathcal{C}$ is an algebraic L-closure systems on X.

Definition 3.15. An L-closure system $\mathcal{C}$ on X is called topological if it satisfies:

(TLCS) $A, B \in \mathcal{C}$ implies $A \vee B \in \mathcal{C}$.

For a topological L-closure system $\mathcal{C}$ on X, the pair $(X, \mathcal{C})$ is called a topological L-closure system space.

The full subcategories of L-**CSS** consisting of algebraic and topological L-closure system spaces are denoted by L-**ACSS** and L-**TCSS**, respectively.

Remark 3.16. By Proposition 3.4, we know that algebraic L-closure systems are L-convex structures with (LCS4) and topological L-closure systems are L-cotopologies with (LCS4). In the classical case, (LCS4) holds naturally. This means algebraic L-closure systems and topological L-closure systems can be considered as reasonable generalizations of convex structures and cotopologies, respectively.

Next we will study the relationship between algebraic (topological) L-closure operators and algebraic (topological) L-closure systems.

Proposition 3.17. *If $\mathfrak{c}$ is an algebraic (a topological) L-closure operator on X, then $\mathcal{C}^{\mathfrak{c}}$ is an algebraic (a topological) L-closure system on X.*

Proof. It follows immediately from Proposition 3.5 and Definition 3.10 (3.11). □

Proposition 3.18. *If $\mathcal{C}$ is an algebraic (a topological) L-closure system on X, then $\mathfrak{c}^{\mathcal{C}}$ is an algebraic (a topological) L-closure operator on X.*

Proof. It is enough to show that

(1) (LCS3), (LCS4) and (ALCS) imply (ALCL) and

(2) (LCS3), (LCS4) and (TLCS) imply (TLCL).

For (1), take each directed subfamily $\{A_j\}_{j\in J} \subseteq L^X$. Then it follows from (LCS3) and (LCS4) that for each $j \in J$,

$$\mathfrak{c}^{\mathcal{C}}(A_j) = \bigwedge_{B_j \in \mathcal{C}} (\mathcal{S}(A_j, B_j) \to B_j) \in \mathcal{C}.$$

Since $\{\mathfrak{c}^{\mathcal{C}}(A_j)\}_{j\in J}$ is directed, it follows from (ALCS) that $\bigvee_{j\in J} \mathfrak{c}^{\mathcal{C}}(A_j) \in \mathcal{C}$. Then we have

$$\begin{array}{rcl} \mathfrak{c}^{\mathcal{C}}(\bigvee_{j\in J} A_j) & = & \bigwedge_{B\in\mathcal{C}} \left(\mathcal{S}(\bigvee_{j\in J} A_j, B) \to B\right) \\ & \leqslant & \mathcal{S}(\bigvee_{j\in J} A_j, \bigvee_{j\in J} \mathfrak{c}^{\mathcal{C}}(A_j)) \to \bigvee_{j\in J} \mathfrak{c}^{\mathcal{C}}(A_j) \\ & = & \bigvee_{j\in J} \mathfrak{c}^{\mathcal{C}}(A_j). \end{array}$$

This means (ALCL) holds.

For (2), it is similar to (1). □

By Propositions 3.17, 3.18 and Theorem 3.9, we have the following corollary.

Corollary 3.19. (1) L-**ACLS** is isomorphic to L-**ACSS**.

(2) L-**TCLS** is isomorphic to L-**TCSS**.

4. *L*-Enclosed Relations

A binary relation between two subsets, usually called an enclosed relation, can be induced by closure operators. In this section, we introduce a fuzzy counterpart of this relation between L-subsets, which is called L-enclosed relation. And we will establish its relationship with L-closure operators.

Definition 4.1. A mapping $\mathfrak{r} : L^X \times L^X \longrightarrow L$ is called an L-enclosed relation on X if it satisfies:

(ELR1) $\mathfrak{r}(\underline{\perp}, \underline{\perp}) = \top$;
(ELR2) $\mathfrak{r}(A, B) \leqslant \mathcal{S}(A, B)$;
(ELR3) $\mathfrak{r}(B, \sqcap\mathcal{A}) = \bigwedge_{A\in L^X}(\mathcal{A}(A) \to \mathfrak{r}(B, A))$;
(ELR4) $\mathcal{S}(A, B) \leqslant \mathfrak{r}(B, C) \to \mathfrak{r}(A, C)$;
(ELR5) $\mathfrak{r}(A, B) \leqslant \bigvee_{C\in L^X}(\mathfrak{r}(A, C) * \mathfrak{r}(C, B))$;
(ELR6) $\mathfrak{r}(a \to A, B) \geqslant a * \mathfrak{r}(A, B)$.

For an L-enclosed relation $\mathfrak{r}$ on X, the pair $(X, \mathfrak{r})$ is called an L-enclosed relational space.

A mapping $f : (X, \mathfrak{r}_X) \longrightarrow (Y, \mathfrak{r}_Y)$ between L-enclosed relational spaces is called L-enclosed-relation-preserving (L-ERP, in short) provided that $\mathfrak{r}_X(f_L^{\leftarrow}(A), f_L^{\leftarrow}(B)) \geqslant \mathfrak{r}_Y(A, B)$ for each $A, B \in L^Y$.

It is easy to check that all L-enclosed relational spaces as objects and all L-ERP mappings as morphisms form a category, denoted by L-**ERS**.

Example 4.2. It is easy to verify that the subsethood degree $\mathcal{S}(-, -) : L^X \times L^X \longrightarrow L$ is an L-enclosed relation on X.

In order to establish the relationship between L-enclosed relations and L-closure operators, we first give the following lemma.

Lemma 4.3. *Let $\mathfrak{r}$ be an L-enclosed relation on X. Then*

(1) $\mathfrak{r}(B, \bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} \mathfrak{r}(B, A_i)$;
(2) $\mathfrak{r}(B, a \to A) = a \to \mathfrak{r}(B, A)$;
(3) $\mathcal{S}(A, B) \leqslant \mathfrak{r}(C, A) \to \mathfrak{r}(C, B)$;
(4) $A \leqslant B$ implies $\mathfrak{r}(A, C) \geqslant \mathfrak{r}(B, C)$.

Proof. (1) Take each $\{A_i\}_{i\in I} \subseteq L^X$ and $B \in L^X$. Define $\mathcal{A} : L^X \longrightarrow L$ by

$$\forall A \in L^X, \mathcal{A}(A) = \begin{cases} \top, & A \in \{A_i\}_{i\in I}; \\ \perp, & \textit{otherwise}. \end{cases}$$

Then

$$\sqcap\mathcal{A} = \bigwedge_{A\in L^X}(\mathcal{A}(A) \to A) = \bigwedge_{A\in\{A_i\}_{i\in I}} A = \bigwedge_{i\in I} A_i.$$

This implies

$$\begin{aligned} \mathfrak{r}(B, \textstyle\bigwedge_{i\in I} A_i) &= \mathfrak{r}(B, \sqcap\mathcal{A}) \\ &= \textstyle\bigwedge_{A\in L^X}(\mathcal{A}(A) \to \mathfrak{r}(B, A)) \\ &= \textstyle\bigwedge_{A\in\{A_i\}_{i\in I}} \mathfrak{r}(B, A) = \bigwedge_{i\in I} \mathfrak{r}(B, A_i). \end{aligned}$$

(2) Take each $A, B \in L^X$ and $a \in L$. Define $\mathcal{A} : L^X \longrightarrow L$ by

$$\forall C \in L^X, \mathcal{A}(C) = \begin{cases} a, & C = A; \\ \perp, & C \neq A. \end{cases}$$

Then

$$\sqcap\mathcal{A} = \bigwedge_{C\in L^X}(\mathcal{A}(C) \to C) = \bigwedge_{C\in\{A\}}(\mathcal{A}(C) \to C) = a \to A.$$

This implies

$$\mathfrak{r}(B, a \to A) = \mathfrak{r}(B, \sqcap A) = \bigwedge_{C\in L^X}(\mathcal{A}(C) \to \mathfrak{r}(B, C)) = a \to \mathfrak{r}(B, A).$$

(3) Take each $A, B, C \in L^X$. Then

$$\begin{array}{rcl} \mathcal{S}(A, B) \to \mathfrak{r}(C, B) & = & \mathfrak{r}(C, \mathcal{S}(A, B) \to B) \ \text{(by (2))} \\ & \geqslant & \mathfrak{r}(C, A). \end{array}$$

This means

$$\mathcal{S}(A, B) * \mathfrak{r}(C, A) \leqslant \mathfrak{r}(C, B).$$

That is,

$$\mathcal{S}(A, B) \leqslant \mathfrak{r}(C, A) \to \mathfrak{r}(C, B).$$

(4) It follows immediately from (ELR4). □

Now let us show how to induce an L-enclosed relation from an L-closure operator.

Proposition 4.4. *Let $\mathfrak{c}$ be an L-closure operator on X and define $\mathfrak{r}^{\mathfrak{c}} : L^X \times L^X \longrightarrow L$ by*

$$\forall A, B \in L^X, \mathfrak{r}^{\mathfrak{c}}(A, B) = \mathcal{S}(\mathfrak{c}(A), B).$$

Then $\mathfrak{r}^{\mathfrak{c}}$ is an L-enclosed relation on X.

Proof. It suffices to show that $\mathfrak{r}^{\mathfrak{c}}$ satisfies (ELR1)-(ELR6). Indeed,
(ELR1) Straightforward.
(ELR2) For each $A, B \in L^X$, it follows that

$$\mathfrak{r}^{\mathfrak{c}}(A, B) = \mathcal{S}(\mathfrak{c}(A), B) \leqslant \mathcal{S}(A, B).$$

(ELR3) For each $B \in L^X$ and $\mathcal{A} : L^X \longrightarrow L$, it follows that

$$\begin{array}{rcl} \mathfrak{r}^{\mathfrak{c}}(B, \sqcap\mathcal{A}) & = & \mathcal{S}(\mathfrak{c}(B), \sqcap\mathcal{A}) \\ & = & \mathcal{S}\left(\mathfrak{c}(B), \bigwedge_{A\in L^X}(\mathcal{A}(A) \to A)\right) \\ & = & \bigwedge_{A\in L^X}\mathcal{S}(\mathfrak{c}(B), \mathcal{A}(A) \to A) \\ & = & \bigwedge_{A\in L^X}(\mathcal{A}(A) \to \mathcal{S}(\mathfrak{c}(B), A)) \\ & = & \bigwedge_{A\in L^X}(\mathcal{A}(A) \to \mathfrak{r}^{\mathfrak{c}}(B, A)). \end{array}$$

(ELR4) For each $A, B, C \in L^X$, it follows that

$$\begin{array}{rcl} \mathfrak{r}^{\mathfrak{c}}(B, C) \to \mathfrak{r}^{\mathfrak{c}}(A, C) & = & \mathcal{S}(\mathfrak{c}(B), C) \to \mathcal{S}(\mathfrak{c}(A), C) \\ & \geqslant & \mathcal{S}(\mathfrak{c}(A), \mathfrak{c}(B)) \\ & \geqslant & \mathcal{S}(A, B). \end{array}$$

(ELR5) For each $A, B \in L^X$, we have

$$\begin{aligned} \bigvee_{C\in L^X}(\mathfrak{r}^{\mathfrak{c}}(A, C) * \mathfrak{r}^{\mathfrak{c}}(C, B)) &= \bigvee_{C\in L^X}(\mathcal{S}(\mathfrak{c}(A), C) * \mathcal{S}(\mathfrak{c}(C), B) \\ &\geqslant \mathcal{S}(\mathfrak{c}(A), \mathfrak{c}(A)) * \mathcal{S}(\mathfrak{c}(\mathfrak{c}(A)), B)) \\ &= \mathcal{S}(\mathfrak{c}(\mathfrak{c}(A)), B) \\ &\geqslant \mathcal{S}(\mathfrak{c}(A), B) = \mathfrak{r}^{\mathfrak{c}}(A, B). \end{aligned}$$

(ELR6) For each $A, B \in L^X$ and $a \in L$, we have

$$\begin{array}{rcl} \mathfrak{r}^{\mathfrak{c}}(A,B) \to \mathfrak{r}^{\mathfrak{c}}(a \to A, B) & = & \mathcal{S}(\mathfrak{c}(A), B) \to \mathcal{S}(\mathfrak{c}(a \to A), B) \\ & \geqslant & \mathcal{S}(\mathfrak{c}(a \to A), \mathfrak{c}(A)) \\ & \geqslant & \mathcal{S}(a \to A, A) \\ & \geqslant & a. \end{array}$$

This implies $a * \mathfrak{r}^{\mathfrak{c}}(A,B) \leqslant \mathfrak{r}^{\mathfrak{c}}(a \to A, B)$. □

Proposition 4.5. *If $f : (X, \mathfrak{c}_X) \longrightarrow (Y, \mathfrak{c}_Y)$ is L-closure-preserving, then $f : (X, \mathfrak{r}^{\mathfrak{c}_X}) \longrightarrow (Y, \mathfrak{r}^{\mathfrak{c}_Y})$ is L-ERP.*

Proof. Since $f : (X, \mathfrak{c}_X) \longrightarrow (Y, \mathfrak{c}_Y)$ is L-closure-preserving, it follows that

$$\forall A \in L^X, f_L^{\rightarrow}(\mathfrak{c}_X(A)) \leqslant \mathfrak{c}_Y(f_L^{\rightarrow}(A)).$$

Then for each $B, C \in L^Y$, we have

$$\begin{array}{rcl} \mathfrak{r}^{\mathfrak{c}_X}(f_L^{\leftarrow}(B), f_L^{\leftarrow}(C)) & = & \mathcal{S}(\mathfrak{c}_X(f_L^{\leftarrow}(B)), f_L^{\leftarrow}(C)) \\ & = & \mathcal{S}(f_L^{\rightarrow}(\mathfrak{c}_X(f_L^{\leftarrow}(B))), C) \\ & \geqslant & \mathcal{S}(\mathfrak{c}_Y(f_L^{\rightarrow}(f_L^{\leftarrow}(B))), C) \\ & \geqslant & \mathcal{S}(\mathfrak{c}_Y(B), C) \\ & = & \mathfrak{r}^{\mathfrak{c}_Y}(B, C). \end{array}$$

Thus $f : (X, \mathfrak{r}^{\mathfrak{c}_X}) \longrightarrow (Y, \mathfrak{r}^{\mathfrak{c}_Y})$ is L-ERP. □

By Propositions 4.4 and 4.5, we obtain a functor $\mathcal{H} : L\text{-}\mathbf{CLS} \longrightarrow L\text{-}\mathbf{ERS}$ by

$$\mathcal{H} : (X, \mathfrak{c}) \longmapsto (X, \mathfrak{r}^{\mathfrak{c}}) \textit{ and } f \longmapsto f.$$

Conversely, we can also construct an L-closure operator from an L-enclosed relation.

Proposition 4.6. *Let $\mathfrak{r}$ be an L-enclosed relation on X and define $\mathfrak{c}^{\mathfrak{r}} : L^X \longrightarrow L^X$ by*

$$\forall A \in L^X, \mathfrak{c}^{\mathfrak{r}}(A) = \bigwedge_{B \in L^X} (\mathfrak{r}(A,B) \to B).$$

Then $\mathfrak{c}^{\mathfrak{r}}$ is an L-closure operator on X.

Proof. By Proposition 3.2, it suffices to verify that $\mathfrak{c}^{\mathfrak{r}}$ satisfies (LCL1)-(LCL3), (LCL5) and (LCL6). Indeed,
(LCL1) It follows from (ELR1) that

$$\mathfrak{c}^{\mathfrak{r}}(\underline{\perp}) = \bigwedge_{B \in L^X} (\mathfrak{r}(\underline{\perp}, B) \to B) \leqslant \mathfrak{r}(\underline{\perp}, \underline{\perp}) \to \underline{\perp} = \top \to \underline{\perp} = \underline{\perp}.$$

(LCL2) It follows from (ELR2) that

$$\mathfrak{c}^{\mathfrak{r}}(A) = \bigwedge_{B \in L^X} (\mathfrak{r}(A,B) \to B) \geqslant \bigwedge_{B \in L^X} (\mathcal{S}(A,B) \to B) \geqslant A.$$

(LCL3) Take each $A \in L^X$ and $a \in L$. Then

$$\begin{aligned} a \to \mathfrak{c}^{\mathfrak{r}}(A) &= a \to \bigwedge_{B \in L^X} (\mathfrak{r}(A,B) \to B) \\ &= \bigwedge_{B \in L^X} (a \to (\mathfrak{r}(A,B) \to B)) \end{aligned}$$

$$= \bigwedge_{B\in L^X}((a * \mathfrak{r}(A,B)) \to B)$$
$$\geqslant \bigwedge_{B\in L^X}(\mathfrak{r}(a\to A,B)\to B) \quad \text{(by (ELR6))}$$
$$= \mathfrak{c}^{\mathfrak{r}}(a\to A).$$

(LCL5) Take each $A, B \in L^X$. Then

$$\begin{array}{rcl} \mathcal{S}(\mathfrak{c}^{\mathfrak{r}}(A), \mathfrak{c}^{\mathfrak{r}}(B)) & = & \mathcal{S}\left(\bigwedge_{D\in L^X}(\mathfrak{r}(A,D)\to D), \bigwedge_{C\in L^X}(\mathfrak{r}(B,C)\to C)\right) \\ & \geqslant & \bigwedge_{C\in L^X}\mathcal{S}(\mathfrak{r}(A,C)\to C, \mathfrak{r}(B,C)\to C) \\ & \geqslant & \bigwedge_{C\in L^X}(\mathfrak{r}(B,C)\to \mathfrak{r}(A,C)) \\ & \geqslant & \mathcal{S}(A,B). \end{array}$$

(LCL6) Take each $A \in L^X$. Then

$$\begin{array}{rcl} \mathfrak{r}(\mathfrak{c}^{\mathfrak{r}}(A), B) & = & \mathfrak{r}\left(\bigwedge_{C\in L^X}(\mathfrak{r}(A,C)\to C), B\right) \\ & \geqslant & \bigvee_{C\in L^X}\mathfrak{r}(\mathfrak{r}(A,C)\to C, B) \quad \text{(by Lemma 4.3)} \\ & \geqslant & \bigvee_{C\in L^X}(\mathfrak{r}(A,C) * \mathfrak{r}(C,B)) \quad \text{(by (ELR6))} \\ & \geqslant & \mathfrak{r}(A,B). \quad \text{(by(ELR5))} \end{array}$$

This implies

$$\begin{array}{rcl} \mathfrak{c}^{\mathfrak{r}}(\mathfrak{c}^{\mathfrak{r}}(A)) & = & \bigwedge_{B\in L^X}(\mathfrak{r}(\mathfrak{c}^{\mathfrak{r}}(A),B)\to B) \\ & \leqslant & \bigwedge_{B\in L^X}(\mathfrak{r}(A,B)\to B) \\ & = & \mathfrak{c}^{\mathfrak{r}}(A). \end{array}$$

Thus $\mathfrak{c}^{\mathfrak{r}}$ is an L-closure operator on X. □

Proposition 4.7. *If $f : (X, \mathfrak{r}_X) \longrightarrow (Y, \mathfrak{r}_Y)$ is L-ERP, then $f : (X, \mathfrak{c}^{\mathfrak{r}_X}) \longrightarrow (Y, \mathfrak{c}^{\mathfrak{r}_Y})$ is L-closure-preserving.*

Proof. Since $f : (X, \mathfrak{r}_X) \longrightarrow (Y, \mathfrak{r}_Y)$ is L-ERP, it follows that

$$\forall B, C \in L^Y, \mathfrak{r}_X(f_L^{\leftarrow}(B), f_L^{\leftarrow}(C)) \geqslant \mathfrak{r}_Y(B,C).$$

Then for each $A \in L^X$, we have

$$\begin{array}{rcl} f_L^{\leftarrow}(\mathfrak{c}^{\mathfrak{r}_Y}(f_L^{\rightarrow}(A))) & = & \bigwedge_{B\in L^Y} f_L^{\leftarrow}(\mathfrak{r}_Y(f_L^{\rightarrow}(A),B)\to B) \\ & = & \bigwedge_{B\in L^Y}(\mathfrak{r}_Y(f_L^{\rightarrow}(A),B)\to f_L^{\leftarrow}(B)) \\ & \geqslant & \bigwedge_{B\in L^Y}(\mathfrak{r}_X(f_L^{\leftarrow}(f_L^{\rightarrow}(A)), f_L^{\leftarrow}(B))\to f_L^{\leftarrow}(B)) \\ & \geqslant & \bigwedge_{B\in L^Y}(\mathfrak{r}_X(A, f_L^{\leftarrow}(B))\to f_L^{\leftarrow}(B)) \\ & \geqslant & \bigwedge_{D\in L^X}(\mathfrak{r}_X(A,D)\to D) \\ & = & \mathfrak{c}^{\mathfrak{r}_X}(A). \end{array}$$

Thus, we obtain $f_L^{\rightarrow}(\mathfrak{c}^{\mathfrak{r}_X}(A)) \leqslant \mathfrak{c}^{\mathfrak{r}_Y}(f_L^{\rightarrow}(A))$. □

By Propositions 4.6 and 4.7, we obtain a functor $\mathcal{K} : L$-**ERS**$\longrightarrow L$-**CLS** by

$$\mathcal{K} : (X, \mathfrak{r}) \longmapsto (X, \mathfrak{c}^{\mathfrak{r}}) \text{ and } f \longmapsto f.$$

Next we will show L-closure operators and L-enclosed relations are one-to-one corresponding. That is,

Theorem 4.8. *L-**CLS** is isomorphic to L-**ERS**.*

Proof. It suffices to verify that $\mathcal{K}\circ\mathcal{H} = \mathbb{I}_{L-\mathbf{CLS}}$ and $\mathcal{H}\circ\mathcal{K} = \mathbb{I}_{L-\mathbf{ERS}}$. That is to say, we only need to prove (1) $\mathfrak{c}^{\mathfrak{r}^{\mathfrak{c}}} = \mathfrak{c}$ and (2) $\mathfrak{r}^{\mathfrak{c}^{\mathfrak{r}}} = \mathfrak{r}$.

For (1), take each $A \in L^X$. Then

$$\mathfrak{c}^{\mathfrak{r}^{\mathfrak{c}}}(A) = \bigwedge_{B \in L^X} (\mathfrak{r}^{\mathfrak{c}}(A, B) \to B) = \bigwedge_{B \in L^X} (\mathcal{S}(\mathfrak{c}(A), B) \to B) \geq \mathfrak{c}(A).$$

Conversely,

$$\mathfrak{c}^{\mathfrak{r}^{\mathfrak{c}}}(A) = \bigwedge_{B \in L^X} (\mathcal{S}(\mathfrak{c}(A), B) \to B) \leq \mathcal{S}(\mathfrak{c}(A), \mathfrak{c}(A)) \to \mathfrak{c}(A) = \mathfrak{c}(A).$$

For (2), take each $A \in L^X$. Then

$$\begin{aligned} \mathfrak{r}(A, \mathfrak{c}^{\mathfrak{r}}(A)) &= \mathfrak{r}\left(A, \textstyle\bigwedge_{B \in L^X} (\mathfrak{r}(A, B) \to B)\right) \\ &= \textstyle\bigwedge_{B \in L^X} \mathfrak{r}(A, \mathfrak{r}(A, B) \to B) \\ &= \textstyle\bigwedge_{B \in L^X} (\mathfrak{r}(A, B) \to \mathfrak{r}(A, B)) \\ &= \top. \end{aligned}$$

This implies for each $A, B \in L^X$,

$$\begin{aligned} \mathfrak{r}^{\mathfrak{c}^{\mathfrak{r}}}(A, B) = \mathcal{S}(\mathfrak{c}^{\mathfrak{r}}(A), B) &\leqslant \mathfrak{r}(A, \mathfrak{c}^{\mathfrak{r}}(A)) \to \mathfrak{r}(A, B) \\ &= \top \to \mathfrak{r}(A, B) \\ &= \mathfrak{r}(A, B). \end{aligned}$$

Further, we have

$$\begin{aligned} \mathfrak{r}^{\mathfrak{c}^{\mathfrak{r}}}(A, B) = \mathcal{S}(\mathfrak{c}^{\mathfrak{r}}(A), B) &= \mathcal{S}\left(\textstyle\bigwedge_{C \in L^X} (\mathfrak{r}(A, C) \to C), B\right) \\ &\geqslant \mathcal{S}(\mathfrak{r}(A, B) \to B, B) \\ &\geqslant \mathfrak{r}(A, B). \end{aligned}$$

This means $\mathfrak{r}^{\mathfrak{c}^{\mathfrak{r}}} = \mathfrak{r}$, as desired. $\square$

In order to characterize algebraic and topological L-closure operators, we propose algebraic and topological counterparts of L-enclosed relations as follows:

Definition 4.9. An L-enclosed relation $\mathfrak{r}$ on X is called algebraic if it satisfies

(AELR) $\mathfrak{r}(\bigvee_{j \in J} A_j, B) = \bigwedge_{j \in J} \mathfrak{r}(A_j, B)$ for each directed subfamily $\{A_j\}_{j \in J}$.

For an algebraic L-enclosed relation $\mathfrak{r}$ on X, the pair $(X, \mathfrak{r})$ is called an algebraic L-enclosed relational space.

Definition 4.10. An L-enclosed relation $\mathfrak{r}$ on X is called topological if it satisfies

(TELR) $\mathfrak{r}(A \vee B, C) = \mathfrak{r}(A, C) \wedge \mathfrak{r}(B, C)$.

For a topological L-enclosed relation $\mathfrak{r}$ on X, the pair $(X, \mathfrak{r})$ is called a topological L-enclosed relational space.

The full subcategories of L-**ERS** consisting of algebraic and topological L-enclosed relational spaces are denoted by L-**AERS** and L-**TERS**, respectively.

As pointed in Example 4.2, it can be further verified that the subsethood degree $\mathcal{S}(-,-) : L^X \times L^X \longrightarrow L$ is an algebraic and topological L-enclosed relation on X.

Proposition 4.11. *If $\mathfrak{c}$ is an algebraic (a topological) L-closure operator on X, then $\mathfrak{r}^{\mathfrak{c}}$ is an algebraic (a topological) L-enclosed relation on X.*

Proof. It follows immediately from Proposition 4.4 and Definition 3.10 (3.11). $\square$

Proposition 4.12. *If $\mathfrak{r}$ is an algebraic (a topological) L-enclosed relation on X, then $\mathfrak{c}^{\mathfrak{r}}$ is an algebraic (a topological) L-closure operator on X.*

Proof. It is enough to show that
(1) (ELR1)-(ELR6) and (AELR) imply (ALCL) and
(2) (ELR1)-(ELR6) and (TELR) imply (TLCL).
For (1), take each directed subfamily $\{A_j\}_{j\in J} \subseteq L^X$. Then for each $j \in J$, it follows that

$$\begin{array}{rcl} \mathfrak{r}(A_j, \mathfrak{c}^{\mathfrak{r}}(A_j)) & = & \mathfrak{r}\left(A_j, \bigwedge_{B\in L^X}(\mathfrak{r}(A_j, B) \to B)\right) \\ & = & \bigwedge_{B\in L^X} \mathfrak{r}(A_j, \mathfrak{r}(A_j, B) \to B) \quad \text{(by Lemma 4.3)} \\ & = & \bigwedge_{B\in L^X}(\mathfrak{r}(A_j, B) \to \mathfrak{r}(A_j, B)) \quad \text{(by Lemma 4.3)} \\ & = & \top. \end{array}$$

This implies

$$\begin{array}{rl} & \mathcal{S}\left(\mathfrak{c}^{\mathfrak{r}}(\bigvee_{j\in J} A_j), \bigvee_{j\in J} \mathfrak{c}^{\mathfrak{r}}(A_j)\right) \\ = & \mathcal{S}\left(\bigwedge_{B\in L^X}\left(\mathfrak{r}\left(\bigvee_{j\in J} A_j, B\right) \to B\right), \bigvee_{j\in J} \mathfrak{c}^{\mathfrak{r}}(A_j)\right) \\ \geqslant & \mathcal{S}\left(\mathfrak{r}\left(\bigvee_{j\in J} A_j, \bigvee_{j\in J} \mathfrak{c}^{\mathfrak{r}}(A_j)\right) \to \bigvee_{j\in J} \mathfrak{c}^{\mathfrak{r}}(A_j), \bigvee_{j\in J} \mathfrak{c}^{\mathfrak{r}}(A_j)\right) \\ \geqslant & \mathfrak{r}\left(\bigvee_{j\in J} A_j, \bigvee_{j\in J} \mathfrak{c}^{\mathfrak{r}}(A_j)\right) \\ = & \bigwedge_{j\in J} \mathfrak{r}\left(A_j, \bigvee_{j\in J} \mathfrak{c}^{\mathfrak{r}}(A_j)\right) \\ \geqslant & \bigwedge_{j\in J} \mathfrak{r}(A_j, \mathfrak{c}^{\mathfrak{r}}(A_j)) \\ = & \top. \end{array}$$

This means $\mathfrak{c}^{\mathfrak{r}}(\bigvee_{j\in J} A_j) \leqslant \bigvee_{j\in J} \mathfrak{c}^{\mathfrak{r}}(A_j)$.
For (2), it can be shown similarly. □

By Propositions 4.11, 4.12 and Theorem 4.8, we obtain the following corollary.

Corollary 4.13. (1) L-**ACLS** is isomorphic to L-**AERS**.
(2) L-**TCLS** is an isomorphic to L-**TERS**.

5. *L*-Interior Operators and *L*-Interior sSystems

As the dual concepts of L-closure operators and L-closure systems, we will propose the concepts of L-interior operators and L-interior systems via supremums of L-families and study their relationship. Further, we will equip L-interior operators and L-interior systems with algebraic and topological conditions, respectively.

Definition 5.1. A mapping $\mathcal{I}\colon L^X \longrightarrow L^X$ is called an L-interior operator if it satisfies:

(LIL1) $\mathcal{I}(\underline{\top}) = \underline{\top}$;
(LIL2) $\mathcal{I}(A) \leqslant A$ for each $A \in L^X$;
(LIL3) $a * \mathcal{I}(A) \leqslant \mathcal{I}(a * A)$ for each $A \in L^X$ and $a \in L$;
(LIL4) $\mathcal{S}(\mathcal{I}(A), B) \leqslant \mathcal{S}(\mathcal{I}(A), \mathcal{I}(B))$ for each $A, B \in L^X$.

For an L-interior operator $\mathcal{I}$ on X, the pair $(X, \mathcal{I})$ is called an L-interior space.

A mapping $f\colon (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$ between L-interior spaces is called L-interior-preserving provided that $f_L^{\leftarrow}(\mathcal{I}_Y(A)) \leqslant \mathcal{I}_X(f_L^{\leftarrow}(A))$ for each $A \in L^Y$.
It is easy to verify that all L-interior spaces as objects and all L-interior-preserving mappings as morphisms form a category, which is denoted by L-**ILS**.

Proposition 5.2. *Let* $\mathcal{I} : L^X \longrightarrow L^X$ *be a mapping satisfying* (LIL2). Then (LIL4) is equivalent to the following two statements.

(LIL5) $\mathcal{S}(A, B) \leqslant \mathcal{S}(\mathcal{I}(A), \mathcal{I}(B))$ for each $A, B \in L^X$;
(LIL6) $\mathcal{I}(\mathcal{I}(A)) \geqslant \mathcal{I}(A)$ for each $A \in L^X$.

Proof. The proof of Proposition 3.2 can be adopted. □

Next, we present the concept of L-interior systems by means of supremums of L-families of L-subsets.

Definition 5.3. A subset $\Im \subseteq L^X$ is called an L-interior system on X if it satisfies:

(LIS1) $\underline{\top} \in \Im$;
(LIS2) $\sqcup\mathcal{A} \in \Im$ for each $\mathcal{A} : \Im \longrightarrow L$.

For an L-interior system $\Im$ on X, the pair $(X, \Im)$ is called an L-interior system space.

A mapping $f\colon (X, \Im_X) \longrightarrow (Y, \Im_Y)$ between L-interior system spaces is called L-interior-preserving provided that $B \in \Im_Y$ implies that $f_L^{\leftarrow}(B) \in \Im_X$.

It is easy to check that all L-interior system spaces as objects and all L-interior-preserving mappings as morphisms form a category, which is denoted by L-**ISS**.

Proposition 5.4. *Let $\Im \subseteq L^X$ be a subset satisfying* (LIS1). Then (LIS2) is equivalent to the following statements.

(LIS3) $\bigvee_{i\in I} A_i \in \Im$ for each $\{A_i\}_{i\in I} \subseteq \Im$;
(LIS4) $a * A \in \Im$ for each $A \in \Im$ and $a \in L$.

Proof. The proof of Proposition 3.4 can be adopted. □

Now let us establish the relationship between L-interior operators and L-interior systems.

Proposition 5.5. *Let $\mathcal{I}\colon L^X \longrightarrow L^X$ be an L-interior operator on X and define $\Im^{\mathcal{I}} \subseteq L^X$ by*

$$\Im^{\mathcal{I}} = \{A \in L^X \mid \mathcal{I}(A) \geqslant A\}.$$

Then $\Im^{\mathcal{I}}$ is an L-interior system on X.

Proof. It suffices to show that $\Im^{\mathcal{I}}$ satisfies (LIS1) and (LIS2).

(LIS1) It follows immediately from (LIL1).

(LIS2) For each $\mathcal{A}\colon \Im^{\mathcal{I}} \longrightarrow L$, it follows that

$$\begin{aligned} \mathcal{I}(\sqcup\mathcal{A}) &= \mathcal{I}\left(\bigvee_{A\in\Im^{\mathcal{I}}}(\mathcal{A}(A) * A)\right) \\ &\geqslant \bigvee_{A\in\Im^{\mathcal{I}}} \mathcal{I}(\mathcal{A}(A) * A) \\ &\geqslant \bigvee_{\mathcal{I}(A)\geqslant A}(\mathcal{A}(A) * \mathcal{I}(A)) \\ &\geqslant \bigvee_{\mathcal{I}(A)\geqslant A}(\mathcal{A}(A) * A) \\ &= \bigvee_{A\in\Im^{\mathcal{I}}}(\mathcal{A}(A) * A) \\ &= \sqcup\mathcal{A}. \end{aligned}$$

This implies that $\sqcup\mathcal{A} \in \Im^{\mathcal{I}}$, as desired. □

Proposition 5.6. *If $f\colon (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$ is an L-interior-preserving mapping, then so is $f\colon (X, \Im^{\mathcal{I}_X}) \longrightarrow (Y, \Im^{\mathcal{I}_Y})$.*

Proof. Since $f\colon (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$ is an L-interior-preserving, it follows that

$$\forall A \in L^Y, f_L^{\leftarrow}(\mathcal{I}_Y(A)) \leqslant \mathcal{I}_X(f_L^{\leftarrow}(A)).$$

Then for each $B \in \Im^{\mathcal{I}_Y}$, i.e., $\mathcal{I}_Y(B) \geqslant B$, we have

$$\mathcal{I}_X(f_L^{\leftarrow}(B)) \geqslant f_L^{\leftarrow}(\mathcal{I}_Y(B)) \geqslant f_L^{\leftarrow}(B).$$

This shows $f_L^{\leftarrow}(B) \in \Im^{\mathcal{I}_X}$, as desired. □

By Propositions 5.5 and 5.6, we obtain a functor $\mathcal{M}$: L-**ILS** $\longrightarrow$ L-**ISS** by

$$\mathcal{M} : (X, \mathcal{I}) \longmapsto (X, \Im^{\mathcal{I}}) \text{ and } f \longmapsto f.$$

Proposition 5.7. *Let $\Im$ be an L-interior system on X and define $\mathcal{I}^{\Im}$: $L^X \longrightarrow L^X$ by*

$$\forall A \in L^X, \mathcal{I}^{\Im}(A) = \bigvee_{B\in\Im}(\mathcal{S}(B, A) * B).$$

Then $\mathcal{I}^{\Im}$ is an L-interior operator on X.

Proof. By Proposition 5.2, we only need to show $\mathcal{I}^{\Im}$ satisfies (LIL1)-(LIL3) and (LIL5)-(LIL6). Actually, (LIL1) and (LIL2) are straightforward.

(LIL3) Take each $a \in L$ and $A \in L^X$. Then

$$\begin{array}{rcl} a * \mathcal{I}^{\Im}(A) & = & a * \bigvee_{B\in\Im}(\mathcal{S}(B, A) * B) \\ & = & \bigvee_{B\in\Im}(a * \mathcal{S}(B, A) * B) \\ & \leqslant & \bigvee_{B\in\Im}(\mathcal{S}(B, a * A) * B) \\ & = & \mathcal{I}^{\Im}(a * A). \end{array}$$

(LIL5) Take each $A, B \in L^X$. Then

$$\begin{array}{rcl} \mathcal{S}(\mathcal{I}^{\Im}(A), \mathcal{I}^{\Im}(B)) & = & \mathcal{S}\left(\bigvee_{C\in\Im}(\mathcal{S}(C, A) * C), \bigvee_{D\in\Im}(\mathcal{S}(D, B) * D)\right) \\ & \geqslant & \bigwedge_{C\in\Im} \mathcal{S}(\mathcal{S}(C, A) * C, \mathcal{S}(C, B) * C) \\ & \geqslant & \bigwedge_{C\in\Im}(\mathcal{S}(C, A) \to \mathcal{S}(C, B)) \\ & \geqslant & \mathcal{S}(A, B). \end{array}$$

(LIL6) Take each $A \in L^X$. Then by Proposition 5.4, we have

$$\mathcal{I}^{\Im}(A) = \bigvee_{B\in\Im}(\mathcal{S}(B, A) * B) \in \Im.$$

This implies

$$\begin{array}{rcl} \mathcal{I}^{\Im}(\mathcal{I}^{\Im}(A)) & = & \bigvee_{B\in\Im}(\mathcal{S}(B, \mathcal{I}^{\Im}(A)) * B) \\ & \geqslant & \mathcal{S}(\mathcal{I}^{\Im}(A), \mathcal{I}^{\Im}(A)) * \mathcal{I}^{\Im}(A) \\ & = & \mathcal{I}^{\Im}(A). \end{array}$$

Hence $\mathcal{I}^{\Im}$ is an L-interior operator on X. □

Proposition 5.8. *If f: $(X, \Im_X) \longrightarrow (Y, \Im_Y)$ is an L-interior-preserving mapping, then so is f: $(X, \mathcal{I}^{\Im_X}) \longrightarrow (Y, \mathcal{I}^{\Im_Y})$.*

Proof. Since f: $(X, \Im_X) \longrightarrow (Y, \Im_Y)$ is an L-interior-preserving, it follows that

$$\forall B \in L^Y, B \in \Im_Y \text{ implies } f_L^{\leftarrow}(B) \in \Im_X.$$

Then for each $A \in L^Y$, we have

$$\begin{array}{rcl} f_L^{\leftarrow}(\mathcal{I}^{\Im_Y}(A)) & = & f_L^{\leftarrow}\left(\bigvee_{B\in\Im_Y} \mathcal{S}(B, A) * B\right) \\ & = & \bigvee_{B\in\Im_Y} f_L^{\leftarrow}(\mathcal{S}(B, A) * B) \\ & = & \bigvee_{B\in\Im_Y}(\mathcal{S}(B, A) * f_L^{\leftarrow}(B)) \\ & \leqslant & \bigvee_{f_L^{\leftarrow}(B)\in\Im_X}(\mathcal{S}(f_L^{\leftarrow}(B), f_L^{\leftarrow}(A)) * f_L^{\leftarrow}(B)) \\ & \leqslant & \bigvee_{D\in\Im_X}(\mathcal{S}(D, f_L^{\leftarrow}(A)) * D) \\ & = & \mathcal{I}^{\Im_X}(f_L^{\leftarrow}(A)). \end{array}$$

Thus f: $(X, \mathcal{I}^{\Im_X}) \longrightarrow (Y, \mathcal{I}^{\Im_Y})$ is an L-interior-preserving mapping. □

By Propositions 5.7 and 5.8, we obtain a functor $\mathcal{N}$: L-**ISS** $\longrightarrow$ L-**ILS** by

$$\mathcal{N} : (X, \Im) \longmapsto (X, \mathcal{I}^{\Im}) \text{ and } f \longmapsto f.$$

Now we will show L-interior operators and L-interior systems are one-to-one corresponding in a categorical sense. That is,

Theorem 5.9. *L-**ILS** and L-**ISS** are isomorphic.*

Proof. It suffices to verify that $\mathcal{N} \circ \mathcal{M} = \mathbb{I}_{L-\mathbf{ILS}}$ and $\mathcal{M} \circ \mathcal{N} = \mathbb{I}_{L-\mathbf{ISS}}$. That is to say, we only need to prove (1) $\mathcal{I}^{\Im^{\mathcal{I}}} = \mathcal{I}$ and (2) $\Im^{\mathcal{I}^{\Im}} = \Im$.

For (1), take each $A \in L^X$. Then

$$\begin{array}{rcl} \mathcal{I}^{\Im^{\mathcal{I}}}(A) & = & \bigvee_{B \in \Im^{\mathcal{I}}} (\mathcal{S}(B, A) * B) \\ & = & \bigvee_{B \leq \mathcal{I}(B)} (\mathcal{S}(B, A) * B) \\ & \geqslant & \mathcal{S}(\mathcal{I}(A), A) * \mathcal{I}(A) \ \text{(by (LIL6))} \\ & = & \mathcal{I}(A) \end{array}$$

and

$$\begin{array}{rcl} \mathcal{I}^{\Im^{\mathcal{I}}}(A) & = & \bigvee_{B \in \Im^{\mathcal{I}}} (\mathcal{S}(B, A) * B) \\ & = & \bigvee_{B \leq \mathcal{I}(B)} (\mathcal{S}(B, A) * B) \\ & \leqslant & \bigvee_{B \leq \mathcal{I}(B)} (\mathcal{S}(\mathcal{I}(B), \mathcal{I}(A)) * B) \\ & \leqslant & \bigvee_{B \leq \mathcal{I}(B)} (\mathcal{S}(\mathcal{I}(B), \mathcal{I}(A)) * \mathcal{I}(B)) \\ & \leqslant & \mathcal{I}(A). \end{array}$$

This means $\mathcal{I}^{\Im^{\mathcal{I}}} = \mathcal{I}$.

For (2), take each $A \in L^X$. Then

$$A \in \Im^{\mathcal{I}^{\Im}} \Longleftrightarrow A \leqslant \mathcal{I}^{\Im}(A) \Longleftrightarrow A = \mathcal{I}^{\Im}(A) \Longleftrightarrow A \in \Im.$$

This shows $\Im^{\mathcal{I}^{\Im}} = \Im$. $\square$

We can also introduce the concepts of algebraic L-interior operators and topological L-interior operators and will discuss what kind of L-interior systems are corresponding to algebraic and topological L-interior operators, respectively.

Definition 5.10. An L-interior operator $\mathcal{I}$ on X is called algebraic if it satisfies:

(ALIL) $\mathcal{I}(\bigwedge_{j \in J} A_j) \geqslant \bigwedge_{j \in J} \mathcal{I}(A_j)$ for each co-directed subfamily $\{A_j\}_{j \in J} \subseteq L^X$.

For an algebraic L-interior operator $\mathcal{I}$ on X, the pair $(X, \mathcal{I})$ is called an algebraic L-interior space.

Definition 5.11. An L-interior operator $\mathcal{I}$ on X is called topological if it satisfies:

(TLIL) $\mathcal{I}(A \wedge B) = \mathcal{I}(A) \wedge \mathcal{I}(B)$ for each $A, B \in L^X$.

For a topological L-interior operator $\mathcal{I}$ on X, the pair $(X, \mathcal{I})$ is called a topological L-interior space.

The full subcategories of L-**ILS** consisting of algebraic and topological L-interior spaces are denoted by L-**AILS** and L-**TILS**, respectively.

Example 5.12. ([20]) (1) Define $\mathcal{I} : L^X \longrightarrow L^X$ as follows $\mathcal{I}(A) = A$. Then it is easy to check that $\mathcal{I}$ satisfies (LIL1)-(LIL4), (ALIL) and (TLIL). Hence $\mathcal{I}$ is an algebraic and topological L-interior operator on X.

(2) Define $\mathcal{I} : L^X \longrightarrow L^X$ as follows $\mathcal{I}(A) = \bigwedge_{x \in X} A(x)$. Then it is easy to check that $\mathcal{I}$ satisfies (LIL1)-(LIL3), (LIL5)-(LIL6), (ALIL) and (TLIL). Hence $\mathcal{I}$ is an algebraic and topological L-interior operator on X.

Definition 5.13. An L-interior system $\Im$ on X is called algebraic if it satisfies:

(ALIS) $\bigwedge_{j\in J} C_j \in \mathfrak{I}$ for each co-directed subfamily $\{C_j\}_{j\in J} \subseteq \mathfrak{I}$.

For an algebraic L-interior system $\mathfrak{I}$ on X, the pair $(X, \mathfrak{I})$ is called an algebraic L-interior system space.

Definition 5.14. An L-interior system $\mathfrak{I}$ on X is called topological if it satisfies:

(TLIS) $A, B \in \mathfrak{I}$ implies $A \wedge B \in \mathfrak{I}$.

For a topological L-interior system $\mathfrak{I}$ on X, the pair $(X, \mathfrak{I})$ is called a topological L-interior system space.

The full subcategories of L-**ISS** consisting of algebraic and topological L-interior system spaces are denoted by L-**AISS** and L-**TISS**, respectively.

Example 5.15. Define $\mathfrak{I} \subseteq L^X$ as follows:

$$\mathfrak{I} = \{A \in L^X \mid \bigvee_{x\in X} A(x) \leq \bigwedge_{x\in X} A(x)\}.$$

Then it is easy to check that $\mathfrak{I}$ satisfies (LIS1), (LIS3)-(LIS4) and (ALIS). Hence $\mathfrak{I}$ is an algebraic L-interior system on X.

Remark 5.16. By Proposition 5.4, we know that algebraic L-interior systems are L-concave structures [17] with (LIS4) and topological L-interior systems are L-topologies with (LIS4). In the classical case, (LIS4) holds naturally. This means algebraic L-interior systems and topological L-interior systems can be considered as L-concave structures and L-topologies, respectively.

Next, we will study the relationship between algebraic (topological) L-interior operators and algebraic (topological) L-interior systems.

Proposition 5.17. *If $\mathcal{I}$ is an algebraic (a topological) L-interior operator on X, then $\mathfrak{I}^{\mathcal{I}}$ is an algebraic (a topological) L-interior system on X.*

Proof. It follows immediately from Proposition 5.5 and Definition 5.10 (5.11). □

Proposition 5.18. *If $\mathfrak{I}$ is an algebraic (a topological) L-interior system on X, then $\mathcal{I}^{\mathfrak{I}}$ is an algebraic (a topological) L-interior operator on X.*

Proof. The proof of Proposition 3.18 can be adopted. □

By Propositions 5.17, 5.18 and Theorem 5.9, we have the following corollary.

Corollary 5.19. (1) L-**AILS** is isomorphic to L-**AISS**.
(2) L-**TILS** is isomorphic to L-**TISS**.

6. *L*-Internal Relations

In this section, we will propose a new kind of fuzzy relations between L-subsets, which is called L-internal relations. And we will show its compatible relationship with L-interior operators.

Definition 6.1. A mapping $\mathfrak{T} : L^X \times L^X \longrightarrow L$ is called an L-internal relation on X if it satisfies:

(ILR1) $\mathfrak{T}(\underline{\top}, \underline{\top}) = \top$;
(ILR2) $\mathfrak{T}(A, B) \leqslant \mathcal{S}(A, B)$;
(ILR3) $\mathfrak{T}(\sqcup\mathcal{A}, B) = \bigwedge_{A\in L^X}(\mathcal{A}(A) \rightarrow \mathfrak{T}(A, B))$;
(ILR4) $\mathcal{S}(A, B) \leqslant \mathfrak{T}(C, A) \rightarrow \mathfrak{T}(C, B)$;
(ILR5) $\mathfrak{T}(A, B) \leqslant \bigvee_{C\in L^X}(\mathfrak{T}(A, C) * \mathfrak{T}(C, B))$;
(ILR6) $a * \mathfrak{T}(A, B) \leqslant \mathfrak{T}(A, a * B)$.

For an L-internal relation $\mathfrak{I}$ on X, the pair $(X, \mathfrak{I})$ is called an L-internal relational space.

A mapping $f : (X, \mathfrak{I}_X) \longrightarrow (Y, \mathfrak{I}_Y)$ between L-internal relational spaces is called L-internal-relation-preserving (L-IRP, in short) provided that $\mathfrak{I}_X(f_L^{\leftarrow}(A), f_L^{\leftarrow}(B)) \geqslant \mathfrak{I}_Y(A, B)$ for each $A, B \in L^Y$.

It is easy to check that all L-internal relational spaces as objects and all L-IRP mappings as morphisms form a category, denoted by L-**IRS**.

In order to establish the relationship between L-internal relations and L-interior operators, we first give the following lemma.

Lemma 6.2. *Let $\mathfrak{I}$ be an L-internal relation on X. Then*

(1) $\mathfrak{I}(\bigvee_{i\in I} A_i, B) = \bigwedge_{i\in I} \mathfrak{I}(A_i, B)$;
(2) $\mathfrak{I}(a * A, B) = a \to \mathfrak{I}(A, B)$;
(3) $\mathcal{S}(A, B) \leqslant \mathfrak{I}(B, C) \to \mathfrak{I}(A, C)$;
(4) $A \leqslant B$ implies $\mathfrak{I}(C, A) \geqslant \mathfrak{I}(C, B)$.

Proof. The proof of Lemma 4.3 can be adopted. □

Now let us show how to induce an L-internal relation from an L-interior operator.

Proposition 6.3. *Let $\mathcal{I}$ be an L-interior operator on X and define $\mathfrak{I}^{\mathcal{I}} : L^X \times L^X \longrightarrow L$ by*

$$\forall A, B \in L^X, \mathfrak{I}^{\mathcal{I}}(A, B) = \mathcal{S}(A, \mathcal{I}(B)).$$

Then $\mathfrak{I}^{\mathcal{I}}$ is an L-internal relation on X.

Proof. It suffices to show that $\mathfrak{I}^{\mathcal{I}}$ satisfies (ILR1)-(ILR6). Indeed,

(ILR1) Straightforward.

(ILR2) For each $A, B \in L^X$, it follows that

$$\mathfrak{I}^{\mathcal{I}}(A, B) = \mathcal{S}(A, \mathcal{I}(B)) \leqslant \mathcal{S}(A, B).$$

(ILR3) For each $B \in L^X$ and $\mathcal{A} : L^X \longrightarrow L$, it follows that

$$\begin{array}{rcl} \mathfrak{I}^{\mathcal{I}}(\sqcup\mathcal{A}, B) & = & \mathcal{S}(\sqcup\mathcal{A}, \mathcal{I}(B)) \\ & = & \mathcal{S}\left(\bigvee_{A\in L^X}(\mathcal{A}(A) * A), \mathcal{I}(B)\right) \\ & = & \bigwedge_{A\in L^X} \mathcal{S}(\mathcal{A}(A) * A, \mathcal{I}(B)) \\ & = & \bigwedge_{A\in L^X}(\mathcal{A}(A) \to \mathcal{S}(A, \mathcal{I}(B))) \\ & = & \bigwedge_{A\in L^X}(\mathcal{A}(A) \to \mathfrak{I}^{\mathcal{I}}(A, B)). \end{array}$$

(ILR4) For each $A, B, C \in L^X$, it follows that

$$\begin{array}{rcl} \mathfrak{I}^{\mathcal{I}}(C, A) \to \mathfrak{I}^{\mathcal{I}}(C, B) & = & \mathcal{S}(C, \mathcal{I}(A)) \to \mathcal{S}(C, \mathcal{I}(B)) \\ & \geqslant & \mathcal{S}(\mathcal{I}(A), \mathcal{I}(B)) \\ & \geqslant & \mathcal{S}(A, B). \end{array}$$

(ILR5) For each $A, B \in L^X$, we have

$$\begin{array}{rl} \bigvee\limits_{C\in L^X} (\mathfrak{I}^{\mathcal{I}}(A, C) * \mathfrak{I}^{\mathcal{I}}(C, B)) = & \bigvee\limits_{C\in L^X} (\mathcal{S}(A, \mathcal{I}(C)) * \mathcal{S}(C, \mathcal{I}(B))) \\ & \geqslant \mathcal{S}(A, \mathcal{I}(\mathcal{I}(B))) * \mathcal{S}(\mathcal{I}(B), \mathcal{I}(B)) \\ & = \mathcal{S}(A, \mathcal{I}(\mathcal{I}(B))) \\ & \geqslant \mathcal{S}(A, \mathcal{I}(B)) \quad \text{(by (LIL6))} \\ & = \mathfrak{I}^{\mathcal{I}}(A, B). \end{array}$$

(ILR6) For each $A, B \in L^X$ and $a \in L$, we have

$$\begin{array}{rcl} \mathfrak{I}^{\mathcal{I}}(A, a * B) & = & \mathcal{S}(A, \mathcal{I}(a * B)) \\ & \geqslant & \mathcal{S}(A, a * \mathcal{I}(B)) \\ & \geqslant & a * \mathcal{S}(A, \mathcal{I}(B)) \\ & = & a * \mathfrak{I}^{\mathcal{I}}(A, B). \end{array}$$

Hence $\mathfrak{I}^{\mathcal{I}}$ is an L-internal relation on X. □

Proposition 6.4. *If $f : (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$ is L-interior-preserving, then $f : (X, \mathfrak{I}^{\mathcal{I}_X}) \longrightarrow (Y, \mathfrak{I}^{\mathcal{I}_Y})$ is L-IRP.*

Proof. Since $f : (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$ is L-interior-preserving, it follows that

$$\forall A \in L^Y, f_L^{\leftarrow}(\mathcal{I}_Y(A)) \leqslant \mathcal{I}_X(f_L^{\leftarrow}(A)).$$

Take each $B, C \in L^Y$. Then

$$\begin{array}{rcl} \mathfrak{I}^{\mathcal{I}_X}(f_L^{\leftarrow}(B), f_L^{\leftarrow}(C)) & = & \mathcal{S}(f_L^{\leftarrow}(B), \mathcal{I}_X(f_L^{\leftarrow}(C))) \\ & \geqslant & \mathcal{S}(f_L^{\leftarrow}(B), f_L^{\leftarrow}(\mathcal{I}_Y(C))) \\ & \geqslant & \mathcal{S}(B, \mathcal{I}_Y(C)) \\ & = & \mathfrak{I}^{\mathcal{I}_Y}(B, C). \end{array}$$

Hence $f : (X, \mathfrak{I}^{\mathcal{I}_X}) \longrightarrow (Y, \mathfrak{I}^{\mathcal{I}_Y})$ is L-IRP. □

By Propositions 6.3 and 6.4, we obtain a functor $\mathcal{P} : L$-**ILS**$\longrightarrow L$-**IRS** by

$$\mathcal{P} : (X, \mathcal{I}) \longmapsto (X, \mathfrak{I}^{\mathcal{I}}) \textit{ and } f \longmapsto f.$$

Conversely, we can also construct an L-interior operator from an L-internal relation.

Proposition 6.5. *Let $\mathfrak{I}$ be an L-internal relation on X and define $\mathcal{I}^{\mathfrak{I}} : L^X \longrightarrow L^X$ by*

$$\forall A \in L^X, \mathcal{I}^{\mathfrak{I}}(A) = \bigvee_{B \in L^X} (\mathfrak{I}(B, A) * B).$$

Then $\mathcal{I}^{\mathfrak{I}}$ is an L-interior operator on X.

Proof. By Proposition 5.2, it suffices to verify that $\mathcal{I}^{\mathfrak{I}}$ satisfies (LIL1)-(LIL3), (LIL5) and (LIL6). Indeed,
(LIL1) It follows from (ILR1) that

$$\mathcal{I}^{\mathfrak{I}}(\underline{\top}) = \bigvee_{B \in L^X} (\mathfrak{I}(B, \underline{\top}) * B) \geqslant \mathfrak{I}(\underline{\top}, \underline{\top}) * \underline{\top} = \top * \underline{\top} = \underline{\top}.$$

(LIL2) It follows from (ILR2) that

$$\mathcal{I}^{\mathfrak{I}}(A) = \bigvee_{B \in L^X} (\mathfrak{I}(B, A) * B) \leqslant \bigvee_{B \in L^X} (\mathcal{S}(B, A) * B) \leqslant A.$$

(LIL3) Take each $A \in L^X$ and $a \in L$. Then

$$\begin{array}{rcll} \mathcal{I}^{\mathfrak{I}}(a * A) & = & \bigvee_{B \in L^X} (\mathfrak{I}(B, a * A) * B) & \\ & \geqslant & \bigvee_{B \in L^X} (a * \mathfrak{I}(B, A) * B) & \text{(by (ILR6))} \\ & = & a * \bigvee_{B \in L^X} (\mathfrak{I}(B, A) * B) & \\ & = & a * \mathcal{I}^{\mathfrak{I}}(A). & \end{array}$$

(LIL5) Take each $A, B \in L^X$. Then

$$\begin{array}{rcl} \mathcal{S}(\mathcal{I}^{\mathfrak{T}}(A), \mathcal{I}^{\mathfrak{T}}(B)) & = & \mathcal{S}\left(\bigvee_{C\in L^X}(\mathfrak{T}(C,A) * C), \bigvee_{D\in L^X}(\mathfrak{T}(D,B) * D)\right) \\ & \geqslant & \bigwedge_{C\in L^X} \mathcal{S}(\mathfrak{T}(C,A) * C, \mathfrak{T}(C,B) * C) \\ & \geqslant & \bigwedge_{C\in L^X}(\mathfrak{T}(C,A) \to \mathfrak{T}(C,B)) \\ & \geqslant & \mathcal{S}(A,B). \end{array}$$

(LIL6) Take each $A, B \in L^X$. Then

$$\begin{array}{rcl} \mathfrak{T}(B, \mathcal{I}^{\mathfrak{T}}(A)) & = & \mathfrak{T}\left(B, \bigvee_{C\in L^X}(\mathfrak{T}(C,A) * C)\right) \\ & \geqslant & \bigvee_{C\in L^X} \mathfrak{T}(B, \mathfrak{T}(C,A) * C) \quad \text{(by Lemma 6.2)} \\ & \geqslant & \bigvee_{C\in L^X}(\mathfrak{T}(C,A) * \mathfrak{T}(B,C)) \quad \text{(by (ILR6))} \\ & \geqslant & \mathfrak{T}(B,A). \quad \text{(by (ILR5))} \end{array}$$

This implies

$$\begin{array}{rcl} \mathcal{I}^{\mathfrak{T}}(\mathcal{I}^{\mathfrak{T}}(A)) & = & \bigvee_{B\in L^X}(\mathfrak{T}(B, \mathcal{I}^{\mathfrak{T}}(A)) * B) \\ & \geqslant & \bigvee_{B\in L^X}(\mathfrak{T}(B,A) * B) \\ & = & \mathcal{I}^{\mathfrak{T}}(A). \end{array}$$

Thus $\mathcal{I}^{\mathfrak{T}}$ is an L-interior operator on X. $\square$

Proposition 6.6. *If $f : (X, \mathfrak{T}_X) \longrightarrow (Y, \mathfrak{T}_Y)$ is L-IRP, then $f : (X, \mathcal{I}^{\mathfrak{T}_X}) \longrightarrow (Y, \mathcal{I}^{\mathfrak{T}_Y})$ is L-interior-preserving.*

Proof. Since $f : (X, \mathfrak{T}_X) \longrightarrow (Y, \mathfrak{T}_Y)$ is L-IRP, it follows that

$$\forall B, C \in L^Y, \mathfrak{T}_X(f_L^{\leftarrow}(B), f_L^{\leftarrow}(C)) \geqslant \mathfrak{T}_Y(B, C).$$

Then for each $A \in L^Y$, we have

$$\begin{array}{rcl} f_L^{\leftarrow}(\mathcal{I}^{\mathfrak{T}_Y}(A)) & = & f_L^{\leftarrow}\left(\bigvee_{B\in L^Y}(\mathfrak{T}_Y(B,A) * B)\right) \\ & = & \bigvee_{B\in L^Y}(\mathfrak{T}_Y(B,A) * f_L^{\leftarrow}(B)) \\ & \leqslant & \bigvee_{B\in L^Y}(\mathfrak{T}_X(f_L^{\leftarrow}(B), f_L^{\leftarrow}(A)) * f_L^{\leftarrow}(B)) \\ & \leqslant & \bigvee_{D\in L^X}(\mathfrak{T}_X(D, f_L^{\leftarrow}(A)) * D) \\ & = & \mathcal{I}^{\mathfrak{T}_X}(f_L^{\leftarrow}(A)). \end{array}$$

Therefore $f : (X, \mathcal{I}^{\mathfrak{T}_X}) \longrightarrow (Y, \mathcal{I}^{\mathfrak{T}_Y})$ is L-interior-preserving. $\square$

By Propositions 6.5 and 6.6, we obtain a functor $\mathcal{R}$:L-**IRS**⟶ L-**ILS** by

$$\mathcal{R} : (X, \mathfrak{T}) \longmapsto (X, \mathcal{I}^{\mathfrak{T}}) \textit{ and } f \longmapsto f.$$

Next we will show L-interior operators and L-internal relations are one-to-one corresponding. That is,

Theorem 6.7. *L-**ILS** is isomorphic to L-**IRS**.*

Proof. It suffices to verify that $\mathcal{R} \circ \mathcal{P} = \mathbb{I}_{L-\mathbf{ILS}}$ and $\mathcal{P} \circ \mathcal{R} = \mathbb{I}_{L-\mathbf{IRS}}$. That is to say, we only need to prove (1) $\mathcal{I}^{\mathfrak{T}^{\mathcal{I}}} = \mathcal{I}$ and (2) $\mathfrak{T}^{\mathcal{I}^{\mathfrak{T}}} = \mathfrak{T}$.

For (1), take each $A \in L^X$. Then

$$\mathcal{I}^{\mathfrak{T}^{\mathcal{I}}}(A) = \bigvee_{B\in L^X} (\mathfrak{T}^{\mathcal{I}}(B,A) * B) = \bigvee_{B\in L^X} (\mathcal{S}(B, \mathcal{I}(A)) * B) \leq \mathcal{I}(A).$$

Conversely,

$$\mathcal{I}^{\mathfrak{T}^{\mathcal{I}}}(A) = \bigvee_{B\in L^X} (\mathcal{S}(B, \mathcal{I}(A)) * B) \geqslant \mathcal{S}(\mathcal{I}(A), \mathcal{I}(A)) * \mathcal{I}(A) = \mathcal{I}(A).$$

This proves that $\mathcal{I}^{\mathfrak{I}^{\mathcal{I}}} = \mathcal{I}$.

For (2), take each $B \in L^X$. Then

$$\begin{array}{rcl} \mathfrak{I}(\mathcal{I}^{\mathfrak{I}}(B), B) & = & \mathfrak{I}\left(\bigvee_{C\in L^X}(\mathfrak{I}(C,B) * C), B\right) \\ & = & \bigwedge_{C\in L^X} \mathfrak{I}(\mathfrak{I}(C,B) * C), B) \\ & = & \bigwedge_{C\in L^X}(\mathfrak{I}(C,B) \rightarrow \mathfrak{I}(C,B)) \\ & = & \top. \end{array}$$

This implies for each $A, B \in L^X$,

$$\begin{array}{rcl} \mathfrak{I}^{\mathcal{I}^{\mathfrak{I}}}(A,B) & = & \mathcal{S}(A, \mathcal{I}^{\mathfrak{I}}(B)) \\ & \leqslant & \mathfrak{I}(\mathcal{I}^{\mathfrak{I}}(B), B) \rightarrow \mathfrak{I}(A,B) \\ & = & \top \rightarrow \mathfrak{I}(A,B) \\ & = & \mathfrak{I}(A,B). \end{array}$$

Further, we have

$$\begin{array}{rcl} \mathfrak{I}^{\mathcal{I}^{\mathfrak{I}}}(A,B) & = & \mathcal{S}(A, \mathcal{I}^{\mathfrak{I}}(B)) \\ & = & \mathcal{S}\left(A, \bigvee_{C\in L^X}(\mathfrak{I}(C,B) * C)\right) \\ & \geqslant & \mathcal{S}(A, \mathfrak{I}(A,B) * A) \\ & \geqslant & \mathfrak{I}(A,B). \end{array}$$

This proves that $\mathfrak{I}^{\mathcal{I}^{\mathfrak{I}}} = \mathfrak{I}$, as desired. □

Now let us introduce algebraic and topological counterparts of L-internal relations and study their relationship with algebraic and topological L-interior operators.

Definition 6.8. An L-internal relation $\mathfrak{I}$ on X is called algebraic if it satisfies

(AILR) $\mathfrak{I}(A, \bigwedge_{j\in J} B_j) = \bigwedge_{j\in J} \mathfrak{I}(A, B_j)$ for each co-directed subfamily $\{B_j\}_{j\in J}$.

For an algebraic L-internal relation $\mathfrak{I}$ on X, the pair $(X, \mathfrak{I})$ is called an algebraic L-internal relational space.

Definition 6.9. An L-internal relation $\mathfrak{I}$ on X is called topological if it satisfies

(TILR) $\mathfrak{I}(A, B \wedge C) = \mathfrak{I}(A,B) \wedge \mathfrak{I}(A,C)$.

For a topological L-internal relation $\mathfrak{I}$ on X, the pair $(X, \mathfrak{I})$ is called topological L-internal relational space.

The full subcategories of L-**IRS** consisting of algebraic and topological L-internal relational spaces are denoted by L-**AIRS** and L-**TIRS**.

Proposition 6.10. *If $\mathcal{I}$ is an algebraic (a topological) L-interior operator on X, then $\mathfrak{I}^{\mathcal{I}}$ is an algebraic (a topological) L-internal relation on X.*

Proof. It follows immediately from Proposition 6.3 and Definition 5.10 (5.11). □

Proposition 6.11. *If $\mathfrak{I}$ is an algebraic (a topological) L-internal relation on X, then $\mathcal{I}^{\mathfrak{I}}$ is an algebraic (a topological) L-interior operator on X.*

Proof. The proof of Proposition 4.12 can be adopted. □

By Propositions 6.10, 6.11 and Theorem 6.7, we obtain the following corollary.

Corollary 6.12. (1) L-**AILS** is isomorphic to L-**AIRS**.

(2) L-**TILS** is an isomorphic to L-**TIRS**.

7. Equivalence Between *L*-Closure Operators and *L*-Interior Operators

In the classical case, closure operators and interior operators are dual concepts and they are equivalent. In this section, we will focus on the equivalence of L-closure operators and L-interior operators. To this end, the lattice background L is always assumed to be a regular complete residuated lattice.

By means of the precomplement operator $\neg$ on L, we will establish the relationship between L-closure operators and L-interior operators.

Proposition 7.1. *Let $\mathcal{I}$ be an L-interior operator on X and define $\mathfrak{c}^{\mathcal{I}}\colon L^X \longrightarrow L^X$ by*

$$\forall A \in L^X, \mathfrak{c}^{\mathcal{I}}(A) = \neg\mathcal{I}(\neg A).$$

Then $\mathfrak{c}^{\mathcal{I}}$ is an L-closure operator on X.

Proof. By Proposition 3.2, we only need to show $\mathfrak{c}^{\mathcal{I}}$ satisfies (LCL1)-(LCL3) and (LCL5)-(LCL6). Indeed,

(LCL1) $\mathfrak{c}^{\mathcal{I}}(\underline{\bot}) = \neg\mathcal{I}(\neg\underline{\bot}) = \neg\mathcal{I}(\underline{\top}) = \neg\underline{\top} = \underline{\bot}$.

(LCL2) $\mathfrak{c}^{\mathcal{I}}(A) = \neg\mathcal{I}(\neg A) \geq \neg(\neg A) = A$.

(LCL3) Take each $a \in L$ and $A \in L^X$. Then

$$\begin{aligned}\mathfrak{c}^{\mathcal{I}}(a \to A) = \neg\mathcal{I}(\neg(a \to A)) &= \neg\mathcal{I}(a * \neg A)\\ &\leq \neg(a * \mathcal{I}(\neg A)) = a \to \neg\mathcal{I}(\neg A) = a \to \mathfrak{c}^{\mathcal{I}}(A).\end{aligned}$$

(LCL5) Take each $A, B \in L^X$. Then

$$\begin{aligned}\mathcal{S}(\mathfrak{c}^{\mathcal{I}}(A), \mathfrak{c}^{\mathcal{I}}(B)) = \mathcal{S}(\neg\mathcal{I}(\neg A), \neg\mathcal{I}(\neg B)) &= \mathcal{S}(\mathcal{I}(\neg B), \mathcal{I}(\neg A))\\ &\geq \mathcal{S}(\neg B, \neg A) = \mathcal{S}(A, B).\end{aligned}$$

(LCL6) Take each $A \in L^X$. Then

$$\mathfrak{c}^{\mathcal{I}}(A) = \neg\mathcal{I}(\neg A) \geq \neg\mathcal{I}(\mathcal{I}(\neg A)) = \mathfrak{c}^{\mathcal{I}}(\neg\mathcal{I}(\neg A)) = \mathfrak{c}^{\mathcal{I}}(\mathfrak{c}^{\mathcal{I}}(A)).$$

Thus $\mathfrak{c}^{\mathcal{I}}$ is an L-closure operator on X. □

Proposition 7.2. *If $f\colon (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$ is an L-interior-preserving mapping, then $f\colon (X, \mathfrak{c}^{\mathcal{I}_X}) \longrightarrow (Y, \mathfrak{c}^{\mathcal{I}_Y})$ is an L-closure-preserving mapping.*

Proof. Since $f\colon (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$ is an L-interior-preserving, it follows that

$$\forall B \in L^Y, f_L^{\leftarrow}(\mathcal{I}_Y(B)) \leqslant \mathcal{I}_X(f_L^{\leftarrow}(B)).$$

Then for each $A \in L^X$, we have

$$\begin{aligned}f_L^{\leftarrow}(\mathfrak{c}^{\mathcal{I}_Y}(f_L^{\rightarrow}(A))) &= f_L^{\leftarrow}(\neg\mathcal{I}_Y(\neg f_L^{\rightarrow}(A)))\\ &= \neg f_L^{\leftarrow}(\mathcal{I}_Y(\neg f_L^{\rightarrow}(A)))\\ &\geq \neg\mathcal{I}_X(f_L^{\leftarrow}(\neg f_L^{\rightarrow}(A)))\\ &= \neg\mathcal{I}_X(\neg f_L^{\leftarrow}(f_L^{\rightarrow}(A)))\\ &\geq \neg\mathcal{I}_X(\neg A)\\ &= \mathfrak{c}^{\mathcal{I}_X}(A).\end{aligned}$$

This implies that $f_L^{\rightarrow}(\mathfrak{c}^{\mathcal{I}_X}(A)) \leqslant \mathfrak{c}^{\mathcal{I}_Y}(f_L^{\rightarrow}(A))$, as desired. □

Proposition 7.3. *Let $\mathfrak{c}$ be an L-closure operator on X and define $\mathcal{I}^{\mathfrak{c}} : L^X \longrightarrow L^X$ by*

$$\forall A \in L^X, \mathcal{I}^{\mathfrak{c}}(A) = \neg\mathfrak{c}(\neg A).$$

Then $\mathcal{I}^{\mathfrak{c}}$ is an L-interior operator on X.

Proof. The proof of Proposition 7.1 can be adopted. □

Proposition 7.4. *If $f : (X, \mathfrak{c}_X) \longrightarrow (Y, \mathfrak{c}_Y)$ is an L-closure-preserving mapping, then $f\colon (X, \mathcal{I}^{\mathfrak{c}_X}) \longrightarrow (Y, \mathcal{I}^{\mathfrak{c}_Y})$ is an L-interior-preserving mapping.*

Proof. Since $f : (X, \mathfrak{c}_X) \longrightarrow (Y, \mathfrak{c}_Y)$ is L-closure-preserving, it follows that

$$\forall A \in L^X, f_L^{\rightarrow}(\mathfrak{c}_X(A)) \leqslant \mathfrak{c}_Y(f_L^{\rightarrow}(A)).$$

Then for each $B \in L^X$, we have

$$\begin{array}{rcl} f_L^{\leftarrow}(\mathcal{I}^{\mathfrak{c}_Y}(B)) & = & f_L^{\leftarrow}(\neg\mathfrak{c}_Y(\neg B)) \\ & = & \neg f_L^{\leftarrow}(\mathfrak{c}_Y(\neg B)) \\ & \leq & \neg f_L^{\leftarrow}(\mathfrak{c}_Y(f_L^{\rightarrow}(f_L^{\leftarrow}(\neg B)))) \\ & \leq & \neg f_L^{\leftarrow}(f_L^{\rightarrow}(\mathfrak{c}_X(f_L^{\leftarrow}(\neg B)))) \\ & \leq & \neg\mathfrak{c}_X(f_L^{\leftarrow}(\neg B)) \\ & = & \neg\mathfrak{c}_X(\neg f_L^{\leftarrow}(B)) \\ & = & \mathcal{I}^{\mathfrak{c}_X}(f_L^{\leftarrow}(B)). \end{array}$$

Hence $f\colon (X, \mathcal{I}^{\mathfrak{c}_X}) \longrightarrow (Y, \mathcal{I}^{\mathfrak{c}_Y})$ is an L-interior-preserving mapping. □

Theorem 7.5. *If $\mathfrak{c}$ is an L-closure operator on X and $\mathcal{I}$ is an L-interior operator on X, then $\mathfrak{c}^{\mathcal{I}^{\mathfrak{c}}} = \mathfrak{c}$ and $\mathcal{I}^{\mathfrak{c}^{\mathcal{I}}} = \mathcal{I}$.*

Proof. Straightforward. □

By Propositions 7.1, 7.2, 7.3, 7.4 and Theorem 7.5, we have the following result.

Theorem 7.6. *L-**CLS** and L-**ILS** are isomorphic.*

The following diagram collects the main results of the previous sections in this paper.

$$\begin{tikzcd} \boxed{L\text{-}\mathbf{CSS}} \arrow[r, equal, "\mathbf{iso}"] & \boxed{L\text{-}\mathbf{CLS}} \arrow[r, equal, "\mathbf{iso}"] \arrow[d, equal, "\mathbf{iso}"', "L\ \textbf{is regular}"] & \boxed{L\text{-}\mathbf{ERS}} \\ \boxed{L\text{-}\mathbf{ISS}} \arrow[r, equal, "\mathbf{iso}"] & \boxed{L\text{-}\mathbf{ILS}} \arrow[r, equal, "\mathbf{iso}"] & \boxed{L\text{-}\mathbf{IRS}} \end{tikzcd}$$

Here, "**iso**" stands for "isomorphic". "L **is regular**" stands for "L is a regular complete residuated lattice".

8. Conclusions

In this paper, we presented the notions of L-closure (L-interior) operators, L-closure (L-interior) systems based on fuzzy infimums (supremums) of L-families of L-subsets and showed there is a one-to-one correspondence between L-closure (L-interior) operators and L-closure (L-interior) systems. In addition, we introduced two types of fuzzy relations between L-subsets, called L-enclosed relations and L-internal relations. And we proved that there is a one-to-one correspondence between L-enclosed (L-internal) relations and L-closure (L-interior) operators. Finally, we showed the category of L-closure spaces and that of L-interior spaces are isomorphic, whenever L is a regular complete residuated lattice. Following this paper, we will consider the following problems in the future.

• Enclosed relations and internal relations have been generalized to (L, M)-fuzzy convex structures [32]. Hence, we will consider fuzzy counterparts of enclosed (internal) relations in the framework of L-fuzzy convex structures, where L is a complete residuated lattice.

• We will consider other characterizations of fuzzy closure systems and fuzzy interior systems, such as fuzzy neighborhood systems, fuzzy remote-neighborhood systems and so on.

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