Filomat 36:3 (2022), 1061–1067
https://doi.org/10.2298/FIL2203061Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Studies on Partial Isometry in Rings with Involution

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Abstract. This paper mainly gives some sufficient and necessary conditions for an element in a ring with involution to be partial isometry and strongly EP element by using some invertible elements and solutions of certain equations.

1. Introduction

Let R be an associative ring with 1. An element $a \in R$ is said to be group invertible if there is $a^{\#} \in R$ satisfying the following conditions:

$$aa^{\#}a = a, \quad a^{\#}aa^{\#} = a, \quad aa^{\#} = a^{\#}a.$$

If $a^{\#}$ exists, it is unique. Denote by $R^{\#}$ the set of group invertible elements of R [1].

An involution $* : a \longmapsto a^*$ in R is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

An element $a^+ \in R$ is called the Moore-Penrose inverse (or MP-inverse) of a [5], if

$$aa^+a = a, \quad a^+aa^+ = a^+, \quad (aa^+)^* = aa^+, \quad (a^+a)^* = a^+a.$$

Also, if a^+ exists, it is unique. Denote by R^+ the set of all MP-invertible elements of R [5].

If $a \in R^{\#} \bigcap R^+$ and $a^{\#} = a^+$, then a is called an EP element [2]. Denote by R^{EP} the set of all EP elements of R.

If $a = aa^*a$, then a is called a partial isometry element of R [4]. Denote by R^{PI} the set of all partial isometry elements of R.

If $a \in R^{EP} \cap R^{PI}$, then a is called a strongly partial isometry element. Denote by R^{SEP} the set of all strongly partial isometry elements of R.

In [9], by discussing the solutions of some equations in a fixed set, we give some new characterizations of EP element. In [8], EP elements are studies by using principally one-sided ideals and annihilators; More results on EP elements can be founded in [3, 4].

In [4, 6, 7], many characterizations of partial isometry elements are given. Motivated by the above results, this paper is aimed to provide some equivalent conditions for an element a to be PI by using some invertible elements and the solutions of certain equations.

2020 *Mathematics Subject Classification*. 16B99; 16W10;15A09;46L05
Keywords. group inverse, MP-invertible element, partial isometry element, *EP* element, strongly *EP* element
Received: 18 February 2021; Revised: 13 September 2021; Accepted: 20 September 2021
Communicated by Dijana Mosić
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2. Partial isometry and construction of *EP* elements

Lemma 2.1. *Let $a \in R^{\#} \cap R^{+}$. Then*
(1) $a^*a^+a \in R^{EP}$ *and* $(a^*a^+a)^+ = \left(a^{\#}\right)^* a^+a$.
(2) $a^+a^+a \in R^{EP}$*and* $(a^+a^+a)^+ = \left(a^{\#}\right)^* a^*a$.

Proof. (1) Noting that $a^*(a^{\#})^*a^+ = a^+ = a^+a^*(a^{\#})^*$ and $a^+a(a^{\#})^* = (a^{\#})^* = (a^{\#})^*aa^+$. Then

$$(a^*a^+a)\left(\left(a^{\#}\right)^* a^+a\right)(a^*a^+a) = a^*\left(a^{\#}\right)^* a^+aa^*a^+a = a^*a^+a;$$

$$\left(\left(a^{\#}\right)^* a^+a\right)(a^*a^+a)\left(\left(a^{\#}\right)^* a^+a\right) = \left(a^{\#}\right)^* a^*a^+a\left(a^{\#}\right)^* a^+a = \left(a^{\#}\right)^* a^+a;$$

$$\left[(a^*a^+a)\left(\left(a^{\#}\right)^* a^+a\right)\right]^* = \left[a^*\left(a^{\#}\right)^* a^+a\right]^* = (a^+a)^* = a^+a = (a^*a^+a)\left(\left(a^{\#}\right)^* a^+a\right);$$

$$\left[\left(\left(a^{\#}\right)^* a^+a\right)(a^*a^+a)\right]^* = \left[\left(a^{\#}\right)^* a^*a^+a\right]^* = (a^+a)^* = a^+a = \left(\left(a^{\#}\right)^* a^+a\right)(a^*a^+a);$$

$$(a^*a^+a)\left(\left(a^{\#}\right)^* a^+a\right) = a^+a = \left(\left(a^{\#}\right)^* a^+a\right)(a^*a^+a).$$

Hence $a^*a^+a \in R^{EP}$ and $(a^*a^+a)^+ = \left(a^{\#}\right)^* a^+a$.
Similarly, we can show (2). □

Lemma 2.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if $a^*a^+a = a^+a^+a$.*

Proof. $\Longrightarrow$ It is evident because $a^* = a^+$.
$\Longleftarrow$ Assume that $a^*a^+a = a^+a^+a$. Post-multiplying the equality by $(aa^{\#})^*$, one gets $a^* = a^+$. Thus $a \in R^{PI}$. □

Lemma 2.1 and Lemma 2.2 imply the following theorem.

Theorem 2.3. *Let $a \in R^{\#} \bigcap R^{+}$, then the following conclusions are equivalent.*
1) $a \in R^{PI}$;
2) $\left(a^{\#}\right)^* a^+a = \left(a^{\#}\right)^* a^*a$;
3) $(a^*a^+a)^+ = \left(a^{\#}\right)^* a^*a$;
4) $(a^+a^+a)^+ = \left(a^{\#}\right)^* a^+a$.

.

It is well known that $a \in R^{+}$ is partial isometry if and only if $a = (a^+)^*$. Hence Lemma 2.2 also implies the following corollary.

Corollary 2.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if $a^*a^+(a^+)^* = a^+a^+(a^+)^*$.*

The following lemma can be obtained by routine verification.

Lemma 2.5. *Let $a \in R^{\#} \cap R^{+}$. Then*
(1) $a^*a^+(a^+)^* \in R^{EP}$ *and* $(a^*a^+(a^+)^*)^+ = a^*a(a^{\#})^*a^+a$.
(2) $a^+a^+(a^+)^* \in R^{EP}$ *and* $(a^+a^+(a^+)^*)^+ = a^*a(a^{\#})^*a^*a$.

Hence Corollary 2.4 and Lemma 2.5 leads to the following theorem.

Theorem 2.6. *Let $a \in R^{\#} \bigcap R^{+}$. Then the following conclusions are equivalent.*
1) $a \in R^{PI}$;
2) $a^*a(a^{\#})^*a^+a = a^*a(a^{\#})^*a^*a$;
3) $(a^*a^+(a^+)^*)^+ = a^*a(a^{\#})^*a^*a$;
4) $(a^+a^+(a^+)^*)^+ = a^*a(a^{\#})^*a^+a$.

It is evident that for $a \in R^{\#} \bigcap R^{+}$, $a \in R^{PI}$ if and only if $a^*a(a^{\#})^* = (a^{\#})^*$. Hence Lemma 2.5 and Theorem 2.6 imply the following corollary.

Corollary 2.7. *Let $a \in R^{\#} \bigcap R^{+}$. Then the following conclusions are equivalent.*
1) $a \in R^{PI}$;
2) $(a^{\#})^*a^+a = (a^{\#})^*a^*a$;
3) $(a^*a^+(a^+)^*)^+ = (a^{\#})^*a^+a$;
4) $(a^+a^+(a^+)^*)^+ = (a^{\#})^*a^*a$.

It is easy to see that for $a \in R^{+}$, $a \in R^{PI}$ if and only if $a^+a = a^+(a^+)^*$. Hence we have the following corollary by Corollary 2.7.

Corollary 2.8. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if $(a^*a^+a)^+ = (a^{\#})^*a^+(a^+)^*$.*

Lemma 2.9. *Let $a \in R^{\#} \cap R^{+}$. Then $(a^{\#})^*a^+(a^+)^* \in R^{EP}$ and $((a^{\#})^*a^+(a^+)^*)^+ = a^*aa^*a^+a$.*

Proof. Routine verification is enough. □

Corollary 2.8 and Lemma 2.9 give the following corollary.

Corollary 2.10. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if $a^*a^+a = a^*aa^*a^+a$.*

3. Partial isometry and construction of invertible elements

The following two lemmas are well known.

Lemma 3.1. *Let R be a ring and $a, b \in R$. If $1 - ab$ is invertible, then $1 - ba$ is also invertible, and $(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a$.*

Lemma 3.2. *Let $a \in R^{\#}$. Then $a + 1 - aa^{\#} \in R^{-1}$ and $(a + 1 - aa^{\#})^{-1} = a^{\#} + 1 - aa^{\#}$.*

Theorem 3.3. *Let $a \in R^{\#} \bigcap R^{+}$. Then $a^*a^+a + 1 - a^+a \in R^{-1}$ and $(a^*a^+a + 1 - a^+a)^{-1} = \left(a^{\#}\right)^* a^+a + 1 - a^+a$.*

Proof. By Lemma 2.1, we have $a^*a^+a \in R^{EP}$ with $(a^*a^+a)^{\#} = (a^{\#})^*a^+a$. Noting that $(a^*a^+a)((a^{\#})^*a^+a) = a^+a$. Then by Lemma 3.2, we have $a^*a^+a + 1 - a^+a \in R^{-1}$ and $(a^*a^+a + 1 - a^+a)^{-1} = \left(a^{\#}\right)^* a^+a + 1 - a^+a$. □

Corollary 3.4. *Let $a \in R^{\#} \bigcap R^{+}$. Then*
1) *$a \in R^{PI}$ if and only if $(a^*a^+a + 1 - a^+a)^{-1} = \left(a^{\#}\right)^* a^+ (a^+)^* + 1 - a^+a$.*
2) *$a \in R^{SEP}$ if and only if $(a^*a^+a + 1 - a^+a)^{-1} = a + 1 - a^+a$.*
3) *$a \in R^{SEP}$ if and only if $(a^*a^+a)^+ = a$.*

Proof. 1) It follows from Lemma 2.1, Corollary 2.8 and Theorem 3.3.
2) Noting that $a \in R^{SEP}$ if and only if $a \in R^{\#} \bigcap R^{+}$ and $a = (a^{\#})^*a^+a$. Then the result follows from Theorem 3.3.
3) It is evident. □

Theorem 3.5. *Let $a \in R^{\#} \bigcap R^{+}$. Then $a \in R^{PI}$ if and only if $(a^*a^+a + 1 - a^+a)^{-1} = (a^{\#})^*a^*a + 1 - a^+a$.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{PI}$. First, we have $a^* = a^+$ and $(a^+)^* = a$. Next, by Corollary 3.4, we have $(a^*a^+a + 1 - a^+a)^{-1} = (a^{\#})^*a^+(a^+)^* + 1 - a^+a$. Hence $(a^*a^+a + 1 - a^+a)^{-1} = (a^{\#})^*a^*a + 1 - a^+a$.
" $\Longleftarrow$ " Assume that $(a^*a^+a + 1 - a^+a)^{-1} = (a^{\#})^*a^*a + 1 - a^+a$. Then $(a^{\#})^*a^+a + 1 - a^+a = (a^{\#})^*a^*a + 1 - a^+a$ by Theorem 3.3, this gives $(a^{\#})^*a^+a = (a^{\#})^*a^*a$. Hence $a \in R^{PI}$ by Theorem 2.3. □

Theorem 3.6. *Let $a \in R^{\#} \bigcap R^{+}$. Then $a \in R^{PI}$ if and only if $(1 - a^+a + a^*)^{-1} = 1 + \left(a^{\#}\right)^* a^*a - \left(a^{\#}\right)^* a^*$.*

Proof. $'' \Longrightarrow ''$ Suppose that $a \in R^{PI}$, then $(a^*a^+a + 1 - a^+a)^{-1} = (a^\#)^*a^*a + 1 - a^+a$ by Theorem 3.5. Noting that $a^*a^+a + 1 - a^+a = 1 - (1 - a^*)a^+a$. Then $(1 - a^+a(1 - a^*))^{-1} = 1 + a^+a(a^*a^+a + 1 - a^+a)^{-1}(1 - a^*)$ by Lemma 3.1, e.g. $(1 - a^+a + a^*)^{-1} = 1 + a^+a((a^\#)^*a^*a + 1 - a^+a)(1 - a^*) = 1 + \left(a^\#\right)^* a^*a - \left(a^\#\right)^* a^*aa^*$. Since $a \in R^{PI}$, $a^* = a^*aa^*$. Hence $(1 - a^+a + a^*)^{-1} = 1 + \left(a^\#\right)^* a^*a - \left(a^\#\right)^* a^*$.

$'' \Longleftarrow ''$ If $(1 - a^+a + a^*)^{-1} = 1 + \left(a^\#\right)^* a^*a - \left(a^\#\right)^* a^*$, then $(1 - a^+a + a^*)(1 + \left(a^\#\right)^* a^*a - \left(a^\#\right)^* a^* = 1$. Noting that $(1 - a^+a)\left(a^\#\right)^* a^*a = 0 = (1 - a^+a)\left(a^\#\right)^* a^*$. Then by simple calculation, we obtain $-a^+a + a^*a = 0$. Hence $a \in R^{PI}$ by [4, Theorem 2.2]. □

Corollary 3.7. *Let $a \in R^\# \bigcap R^+$. Then*

1) *$a \in R^{PI}$ if and only if $(1 - aa^+ + a^*)^{-1} = 1 + (a^+)^* \left(aa^\#\right)^* aa^* - \left(aa^\#\right)^*$.*

2) *$a \in R^{SEP}$ if and only if $(1 - aa^+ + a^*)^{-1} = 1 + a - \left(aa^\#\right)^*$.*

Proof. 1) $'' \Longrightarrow ''$ Assume that $a \in R^{PI}$, then $(1 - a^+a + a^*)^{-1} = 1 + \left(a^\#\right)^* a^*a - \left(a^\#\right)^* a^*$ by Theorem 3.6. Noting that $1 - a^+a + a^* = 1 - a^*\left((a^+)^* - 1\right)$. Then

$$(1 - aa^+ + a^*)^{-1} = [1 - \left((a^+)^* - 1\right) a^*]^{-1} = 1 + \left((a^+)^* - 1\right)(1 - a^+a + a^*)^{-1} a^*$$

$$= 1 + \left((a^+)^* - 1\right)\left[1 + \left(a^\#\right)^* a^*a - \left(a^\#\right)^* a^*\right] a^*$$

$$= 1 + ((a^+)^* - 1)(aa^\#)^*aa^* = 1 + (a^+)^*(aa^\#)^*aa^* - (aa^\#)^*aa^*.$$

Since $a \in R^{PI}$, $aa^* = aa^+$, it follows that $(aa^\#)^*aa^* = aa^\#$. Hence $(1 - aa^+ + a^*)^{-1} = 1 + (a^+)^* \left(aa^\#\right)^* aa^* - \left(aa^\#\right)^*$.

$'' \Longleftarrow ''$ If $(1 - aa^+ + a^*)^{-1} = 1 + (a^+)^* \left(aa^\#\right)^* aa^* - \left(aa^\#\right)^*$, then

$$(1 - aa^+ + a^*)(1 + (a^+)^* \left(aa^\#\right)^* aa^* - \left(aa^\#\right)^*) = 1.$$

Nothing that $(1 - aa^+)\left((a^+)^* \left(a\, a^\#\right)^* a\, a^*\right) = 0$. Then again by a simple calculation, we obtain $(aa^\#)^* = (aa^\#)^*aa^*$. Pre-multiplying the equality by a^+, we have $a^+ = a^*$. Therefore $a \in R^{PI}$.

2) It is easy to show that $a \in R^{SEP}$ if and only $a = (a^+)^*(aa^\#)^*aa^+$.

$'' \Longrightarrow ''$ Since $a \in R^{SEP}$, $a^* = a^+$. Hence $(1 - aa^+ + a^*)^{-1} = 1 + a - \left(aa^\#\right)^*$ by 1).

$'' \Longleftarrow ''$ If $(1 - aa^+ + a^*)^{-1} = 1 + a - \left(aa^\#\right)^*$, then $(1 - aa^+ + a^*)(1 + a - \left(aa^\#\right)^*) = 1$, this gives $a^*a = \left(aa^\#\right)^*$. Hence $a^*a = aa^\#$, by [4, Theorem 2.3], we have $a \in R^{SEP}$. □

4. Partial isometry and the solution of equations

Observing Lemma 2.2, we can construct the following equation

$$a^*xa = xa^+a. \tag{1}$$

The following theorem follows from [7, Theorem 2.9].

Theorem 4.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the equation* (1) *has at least one solution in χ_a, where $\chi_a = \{a, a^\#, a^+, a^*, (a^\#)^*, (a^+)^*\}$.*

Variable a in Equation (1), we can obtain the following equation.

$$a^*xy = xa^+y. \tag{2}$$

Lemma 4.2. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equivalent:*
(1) $a \in R^{PI}$;
(2) $a^*a^* = a^*a^+$;
(3) $a^*a^* = a^+a^*$;
(4) $a^*a^+ = a^+a^+$;
(5) $a^+a^* = a^+a^+$.

Proof. (1) $\Longrightarrow$ (2) It is trivial.
(2) $\Longrightarrow$ (5) Assume that $a^*a^* = a^*a^+$. Pre-multiplying the equality by $a^+(a^+)^*$, one yields $a^+a^* = a^+a^+$.
(5) $\Longrightarrow$ (1) If $a^+a^* = a^+a^+$. Pre-multiplying the equality by $(aa^{\#})^*a$, one has $a^* = a^+$. Hence $a \in R^{PI}$.
Similarly, we can show (1) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1). $\square$

Theorem 4.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if the equation* (2) *has at least one solution in $\chi_a^2 =: \{(x, y)|x, y \in \chi_a\}$.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{PI}$, then $(x, y) = (a^+, a)$ is a solution by Lemma 2.2 and Theorem 4.1.
" $\Longleftarrow$ " (1) If $y = a$, then we have the equation (1). Hence $a \in R^{PI}$ by Theorem 4.1;
(2) If $y = a^{\#}$, then we have the following equation

$$a^*xa^{\#} = xa^+a^{\#}. \tag{3}$$

Post-multiplying the equation (3) by a^2, we have the equation (1). Thus $a \in R^{PI}$ by Theorem 4.1;
(3) If $y = a^+$, then we obtain the following equation

$$a^*xa^+ = xa^+a^+. \tag{4}$$

(a) If $x = a$, then $a^* = a^*aa^+ = aa^+a^+$. Pre-multiplying the equality by $(aa^{\#})^*$, one yields $a^* = a^+$. Hence $a \in R^{PI}$;

(b) If $x = a^{\#}$, then $a^*a^{\#}a^+ = a^{\#}a^+a^+$. Pre-multiplying the equality by aa^+, one yields $a^{\#}a^+a^+ = aa^+a^{\#}a^+a^+ = aa^+a^*a^{\#}a^+$, it follows that $a^*a^{\#}a^+ = aa^+a^*a^{\#}a^+$. Post-multiplying the last equality by $a^2(a^+)^*$, one has $a^+a = aa^+a^+a$. Hence $a \in R^{EP}$, this gives $a^*a^{\#}a^{\#} = a^{\#}a^{\#}a^{\#}$. Post-multiplying the equality by a^4, one gets $a^*a^2 = a$. Hence $a \in R^{PI}$ by [4, Theorem 2.3];

(c) If $x = a^+$, then $a^*a^+a^+ = a^+a^+a^+$. Post-multiplying the equality by $aa^*(a^{\#})^*a$, one has $a^*a^+a = a^+a^+a$. Hence $a \in R^{PI}$ by Lemma 2.2;

(d) If $x = a^*$, then $a^*a^*a^+ = a^*a^+a^+$. Pre-multiplying the equality by $(a^{\#})^*$, one has $a^*a^+ = a^+a^+$. Hence $a \in R^{PI}$ by Lemma 4.2;

(e) If $x = (a^{\#})^*$, then $a^+ = a^*(a^{\#})^*a^+ = (a^{\#})^*a^+a^+$, this gives $a^*a^+ = a^+a^+$. Hence $a \in R^{PI}$ by Lemma 4.2;

(f) If $x = (a^+)^*$, then $a^+ = a^*(a^+)^*a^+ = (a^+)^*a^+a^+$, so $a^*a^+ = a^+a^+$. Thus $a \in R^{PI}$ by Lemma 4.2;

(4) If $y = a^*$, we get the following equation

$$a^*xa^* = xa^+a^*. \tag{5}$$

Post-multiplying the equation (5) by $(a^+)^*a^+$, one obtains the equation (4). Hence $a \in R^{PI}$ by (4).
(5) If $y = \left(a^{\#}\right)^*$, then we have the following equation

$$a^*x\left(a^{\#}\right)^* = xa^+\left(a^{\#}\right)^*. \tag{6}$$

Post-multiplying the equation (6) by $(a^*)^2$, we obtain the equation (5). Thus $a \in R^{PI}$ by (4);
(6) If $y = (a^+)^*$, then we have the following equation

$$a^*x\,(a^+)^* = xa^+\,(a^+)^* \tag{7}$$

Post-multiplying the equation (7) by a^*a, one yields the equation (1). Thus $a \in R^{PI}$ by Theorem 4.1. $\square$

It is well known that $a \in R^{PI}$ if and only if $a^* \in R^{PI}$. Hence substitute a^* for a in the equation (1), we obtain the following equation.

$$axa^* = xaa^+. \tag{8}$$

Post-multiplying the equation (8) by $(a^+)^*$, we have the following equation.

$$axa^+a = x(a^+)^*. \tag{9}$$

Noting that the equation (8) and (9) have the same solution. Hence we have the following corollary which follows from [7, Theorem 2.9].

Corollary 4.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the equation* (4.9) *has at least one solution in χ_a.*

5. Partial isometry and generalized equations

We can change the equation (1) as follows

$$a^*xa = ya^+a. \tag{10}$$

Proposition 5.1. *The general solution of equation* (10) *is given by*

$$\begin{cases} x = pa^+ + u - aa^+uaa^+ \\ y = a^*p + z - za^+a \qquad \text{where } p, u, z \in R. \end{cases} \tag{11}$$

Proof. First, by a simple calculation, we obtain that the formula (11) is the solution of equation (10). Next, if $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$ is a solution, then $a^*x_0a = y_0a^+a$. Then

$$\begin{aligned} &\left((a^+)^* y_0a^+a\right)a^+ + x_0 - aa^+x_0aa^+ \\ &= \left((a^+)^* a^*x_0a\right)a^+ + x_0 - aa^+x_0aa^+ \\ &= aa^+x_0aa^+ + x_0 - aa^+x_0aa^+ = x_0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &a^*\left((a^+)^* y_0a^+a\right) + y_0 - y_0a^+a \\ &= a^+aa^*x_0a + y_0 - y_0a^+a \\ &= a^*x_0a + y_0 - y_0a^+a = y_0. \end{aligned}$$

Hence the general solution of the equation (10) is given by the formula (11). □

Theorem 5.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the general solution of the equation* (10) *is given by*

$$\begin{cases} x = pa^+ + u - aa^+uaa^+ \\ y = a^+p + z - za^+a \qquad \text{where } p, u, z \in R. \end{cases} \tag{12}$$

Proof. $'' \Longrightarrow ''$ Assume that $a \in R^{PI}$. Then $a^+ = a^*$, it follows from Proposition 5.1 that the general solution of the equation (10) is given by the formula (12).

$'' \Longleftarrow ''$ Suppose the formula (12) is the general solution of the equation (10), then we have

$$a^*(pa^+ + u - aa^+uaa^+)a = (a^+p + z - za^+a)a^+a,$$

this gives $a^*pa^+a = a^+pa^+a$ for each $p \in R$. Especially, we choose $p = a$, one yields $a^*a = a^+a$. Hence $a \in R^{PI}$. □

Now we give the following equations.

$$a^*xa = yaa^+. \tag{13}$$

$$a^+xa = ya^+a. \tag{14}$$

$$a^\# xa = ya^+a. \tag{15}$$

Theorem 5.3. *Let $a \in R^\# \cap R^+$. Then*

(1) $a \in R^{EP}$ *if and only if the general solution of the equation* (13) *is given by the formula* (11).

2) $a \in R^{PI}$ *if and only if the general solution of the equation* (14) *is given by the formula* (11).

3) $a \in R^{SEP}$ *if and only if the general solutions of the equation* (15) *is given by the formula* (11).

Proof. (1) $'' \Longrightarrow ''$ Assume that $a \in R^{EP}$, then $aa^+ = a^+a$, this infers that the equation (13) is same as the equation (10). Hence the general solution of the equation (13) is given by the formula (11) by Proposition 5.1.

$\Longleftarrow ''$ Suppose that the general solution of the equation (13) is given by the formula (11), then $a^*(pa^+ + u - aa^+uaa^+)a = (a^*p + z - za^+a)aa^+$. Choosing $z = 0$ and $p = (a^\#)^*$, one obtains $a^+a = aa^\#$. Hence $a \in R^{EP}$.

(2) $'' \Longrightarrow ''$ Assume that $a \in R^{PI}$, then $a^+ = a^*$, this infers the equation (14) is same as the equation (10) and the formula (12) is same as the formula (11). Hence we are done by Theorem 5.2.

$'' \Longleftarrow ''$ If the general solution of the equation (14) is given by the formula (11), then $a^+(pa^+ + u - aa^+uaa^+)a = (a^*p + z - za^+a)a^+a$. Choosing $p = a$, one yields $a^+a = a^*a$. Hence $a \in R^{PI}$.

(3) It is an immediate result of (1) and (2). □

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