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On Integral Generalization of Lupaş-Jain Operators

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Abstract. This paper mainly is a natural continuation of "On Lupaş-Jain Operators" constructed by Başcanbaz-Tunca et al. (Stud. Univ. Babeş-Bolyai Math. 63(4) (2018), 525-537) to approximate integrable functions on $[0,\infty)$. We first present the weighted uniform approximation and provide a quantitative estimate for integral generalization of Lupaş-Jain operators. We also scrutinize the order of approximation in regards to local approximation results in sense of a classical approach, Peetre's K-functional and Lipschitz class. Then, we prove that given operators can be approximated in terms of the Steklov means (Steklov averages). Lastly, a Voronovskaya-type asymptotic theorem is given.

1. Introduction

In 1972, Jain [11] generalized the well-known Szász-Mirakyan operators by constructing the positive linear operators given by

$$S_n^{\beta}(f)(x) = \sum_{k=0}^{\infty} \frac{nx\,(nx+k\beta)^{k-1}}{k!} e^{-(nx+k\beta)} f\left(\frac{k}{n}\right), \tag{1}$$

where $f : [0,\infty) \to \mathbb{R}$, $n \in \mathbb{N}$, $x > 0$ and $0 \le \beta < 1$, with β may depend only on n. For some interesting works related to Jain's operators, we refer to [1], [2], [4], [8], [13], [15], [19] and references cited therein. Moreover, numerous works meticulously collected by Agratini [3] which give a historical background on Jain's operators.

In 2015, Patel and Mishra [18] generalized Jain operators as a variant of the Lupaş operators [12] defined by

$$L_n^{\beta}(f)(x) = \sum_{k=0}^{\infty} \frac{(nx+k\beta)_k}{2^k k!} 2^{-(nx+k\beta)} f\left(\frac{k}{n}\right) \tag{2}$$

for real valued functions f on $[0,\infty)$, where they assumed that

$$(nx+k\beta)_0 = 1,\ (nx+k\beta)_1 = nx$$

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and $(nx+k\beta)_k = nx(nx+k\beta)(nx+k\beta+1)...(nx+k\beta+k-1)$, $k \geq 2$. Another variant of operators (2) was discussed by Patel in [17].

In 2018, Başcanbaz-Tunca et al. [5] introduced the slightly different generalization of the Jain operators just as

$$L_n^{\beta}(f)(x) = \sum_{k=0}^{\infty} \frac{nx(nx+1+k\beta)_{k-1}}{2^k k!} 2^{-(nx+k\beta)} f\left(\frac{k}{n}\right), \quad x \in (0,\infty) \tag{3}$$

and $L_n^{\beta}(f)(0) = f(0)$ for real valued functions f on $[0,\infty)$, where $0 \leq \beta < 1$, β depending only on n. From 2015, some paper's related to Lupaş-Jain operators [6], [16], [20] are contributed to literature of approximation theory.

Motivated by these works, for the purpose of approximating integrable functions on $[0,\infty)$, we introduce a new sequence of summation-integral type operators as

$$D_n^{\beta}(f;x) := D_n^{\beta}(f)(x) = n\sum_{k=0}^{\infty} l_{n,k}^{\beta}(x)\int_0^{\infty} s_{n,k}(t)f(t)\,dt \tag{4}$$

where

$$l_{n,k}^{\beta}(x) = \frac{nx(nx+1+k\beta)_{k-1}}{2^k k!}2^{-(nx+k\beta)},$$

$$s_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}$$

and $f : [0,\infty) \to \mathbb{R}$ is an integrable function such that $D_n^{\beta}(f;x)$ exists. Here, $x \in (0,\infty)$, $0 \leq \beta < 1$ and $D_n^{\beta}(f)(0) = f(0)$ for real valued functions.

The paper is organized as follows: The next section is devoted to preliminary results which are necessary to prove main results. Then, our aim is to discuss the weighted and local approximation results, respectively. The last section, we shall provide a Voronovskaya-type theorem, as well.

2. Preliminary Results

This section provides a quick overview on basic lemmas that will be necessary to prove the main results presented in the paper.

Lemma 2.1. *[6] For the Lupaş-Jain operators L_n^{β}, we have*

$$\begin{aligned}
L_n^{\beta}(e_0;x) &= 1,\\
L_n^{\beta}(e_1;x) &= \frac{x}{1-\beta},\\
L_n^{\beta}(e_2;x) &= \frac{x^2}{(1-\beta)^2} + \frac{2x}{n(1-\beta)^3},\\
L_n^{\beta}(e_3;x) &= \frac{x^3}{(1-\beta)^3} + \frac{6x^2}{n(1-\beta)^4} + \frac{6x(1+\beta)}{n^2(1-\beta)^5}\\
L_n^{\beta}(e_4;x) &= \frac{x^4}{(1-\beta)^4} + \frac{12x^3}{n(1-\beta)^5} + \frac{12x^2(2\beta+3)}{n^2(1-\beta)^6} + \frac{2x(13\beta^2+34\beta+13)}{n^3(1-\beta)^7}.
\end{aligned}$$

Depending on the above lemma, we can write the following lemmas for integral generalization of Lupaş-Jain operators.

Lemma 2.2. *Let $e_r(t) := t^r$, $r = \overline{0,4}$. Then, integral generalization of Lupaş-Jain operators D_n^β satisfy the followings:*

$$\begin{aligned}
D_n^\beta(e_0;x) =& 1,\\
D_n^\beta(e_1;x) =& \frac{x}{1-\beta}+\frac{1}{n},\\
D_n^\beta(e_2;x) =& \frac{x^2}{(1-\beta)^2}+\frac{x(3\beta^2-6\beta+5)}{n(1-\beta)^3}+\frac{2}{n^2},\\
D_n^\beta(e_3;x) =& \frac{x^3}{(1-\beta)^3}+\frac{6x^2(\beta^2-2\beta+2)}{n(1-\beta)^4}+\frac{x(11\beta^4-44\beta^3+78\beta^2-62\beta+29)}{n^2(1-\beta)^5}+\frac{6}{n^3}\\
D_n^\beta(e_4;x) =& \frac{x^4}{(1-\beta)^4}+\frac{2x^3(5\beta^2-10\beta+11)}{n(1-\beta)^5}+\frac{x^2(35\beta^4-140\beta^3+270\beta^2-236\beta+131)}{n^2(1-\beta)^6}\\
&+\frac{2x(25\beta^6-150\beta^5+410\beta^4-610\beta^3+568\beta^2-286\beta+103)}{n^3(1-\beta)^7}+\frac{24}{n^4}.
\end{aligned}$$

Proof. Remembering the definition of Gamma functions, we possess

$$\int_0^\infty s_{n,k}(t)t^r dt = \int_0^\infty e^{-nt}\frac{(nt)^k}{k!}t^r dt = \frac{\Gamma(k+r+1)}{n^{r+1}k!}, \qquad r = 0,1,2,\ldots. \tag{5}$$

Using $L_n^\beta(e_0;x) = 1$ and $L_n^\beta(e_1;x) = \dfrac{x}{1-\beta}$

$$D_n^\beta(e_0;x) = n\sum_{k=0}^\infty l_{n,k}^\beta(x)\int_0^\infty s_{n,k}(t)dt = n\sum_{k=0}^\infty l_{n,k}^\beta(x)\frac{1}{n} = 1,$$

$$D_n^\beta(e_1;x) = n\sum_{k=0}^\infty l_{n,k}^\beta(x)\int_0^\infty s_{n,k}(t)tdt = n\sum_{k=0}^\infty l_{n,k}^\beta(x)\frac{k+1}{n^2} = L_n^\beta(e_1;x)+\frac{1}{n}L_n^\beta(e_0;x) = \frac{x}{1-\beta}+\frac{1}{n}.$$

Hence, the 2nd, 3rd and 4th moments can be obtained similarly. □

Lemma 2.3. *Given $D_n^\beta((t-x)^r,x), r = 1,2,4$. Then the following relations are satisfied.*

$$\begin{aligned}
D_n^\beta(t-x;x) =& \frac{x\beta}{1-\beta}+\frac{1}{n},\\
D_n^\beta((t-x)^2;x) =& \frac{x^2\beta^2}{(1-\beta)^2}+\frac{x(2\beta^3-3\beta^2+3)}{n(1-\beta)^3}+\frac{2}{n^2} = \lambda_n^\beta(x), \qquad (6)\\
D_n^\beta((t-x)^4;x) =& \frac{x^4\beta^4}{(1-\beta)^4}+\frac{2x^3\beta^2(2\beta^3-\beta^2-4\beta+9)}{n(1-\beta)^5}+\frac{x^2(12\beta^6-28\beta^5-5\beta^4+108\beta^3-110\beta^2+56\beta+27)}{n^2(1-\beta)^6}\\
&+\frac{2x(12\beta^7-59\beta^6+102\beta^5-10\beta^4-190\beta^3+316\beta^2-202\beta+91)}{n^3(1-\beta)^7}+\frac{24}{n^4}.
\end{aligned}$$

Remark 2.4. *One should note that, for* $0 < \beta < 1$,

$$\begin{aligned} D_n^{\beta}(t^2+1;x) =& \frac{x^2}{(1-\beta)^2} + \frac{x(3\beta^2-6\beta+5)}{n(1-\beta)^3} + \frac{2}{n^2} + 1 \\ =& \frac{x^2n^2 + nx(1-\beta)(3\beta^2-6\beta+5) + 2(1-\beta)^3 + n^2(1-\beta)^3}{n^2(1-\beta)^3} \\ \leq& \frac{(1+x^2)}{(1-\beta)^3}\left[\frac{x^2}{1+x^2} + \frac{8x}{n(1+x^2)} + \frac{2}{n^2(1+x^2)} + 1\right] \\ \leq& \frac{(1+x^2)}{(1-\beta)^3}\left[2 + \frac{8}{n} + \frac{2}{n^2}\right] = M_n(1+x^2), \end{aligned}$$

where $M_n = \dfrac{1}{(1-\beta)^3}\left[2 + \dfrac{8}{n} + \dfrac{2}{n^2}\right]$.

Remark 2.5. *We need to make an adjustment to the parameter* β *by taking it as a sequence such as that* $\beta = \beta_n$, $0 \leq \beta_n < 1$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\lim_{n\to\infty} n\beta_n = 0$. *Based on the central moments the following limits hold for* $\beta = \{\beta_n\}_{n\geq 1}$.

$$\lim_{n\to\infty} n\Big(D_n^{\beta_n}(t-x;x)\Big) = 1,$$

$$\lim_{n\to\infty} n\Big(D_n^{\beta_n}((t-x)^2;x)\Big) = 3x$$

and

$$\lim_{n\to\infty} n^2\Big(D_n^{\beta_n}((t-x)^4;x)\Big) = 27x^2.$$

Lemma 2.6. *For* $f \in C_B[0,\infty)$ *(space of all real valued, bounded and continuous functions on* $[0,\infty)$ *endowed with the norm* $\|f\| = sup\{|f(x)| : x \in [0,\infty)\}$*), then* $\|D_n^{\beta}(f)\| \leq \|f\|$.

Proof. From definition of operators (4) and Lemma 2.2, the proof of this lemma easily follows. □

3. Weighted Approximation

In this section, we deal with the weighted uniform approximation result of integral generalization of the Lupaş-Jain operators D_n^{β} by using Gadjiev's theorem in [9], for which we have the following settings:

We take $\varphi(x) = 1 + x^2$ as the necessary weight function. Related to φ, we take the space

$$B_{\varphi}(\mathbb{R}^+) = \left\{ f : \mathbb{R}^+ \to \mathbb{R} : \left|f(x)\right| \leq M_f \varphi(x),\ x \in \mathbb{R}^+ \right\},$$

where $\mathbb{R}^+ := [0,\infty)$ and M_f is a constant depending on f. $B_{\varphi}(\mathbb{R}^+)$ is a normed space with the norm $\left\|f\right\|_{\varphi} = \sup_{x\in\mathbb{R}^+} \frac{|f(x)|}{\varphi(x)}$. Moreover, we denote, as usual, by $C_{\varphi}(\mathbb{R}^+)$, $C_{\varphi}^k(\mathbb{R}^+)$, $U_{\varphi}^k(\mathbb{R}^+)$ the following subspaces of $B_{\varphi}(\mathbb{R}^+)$

$$\begin{aligned} C_{\varphi}(\mathbb{R}^+) &= \left\{ f \in B_{\varphi}(\mathbb{R}^+) : f \text{ is continuous} \right\}, \\ C_{\varphi}^k(\mathbb{R}^+) &= \left\{ f \in C_{\varphi}(\mathbb{R}^+) : \lim_{x\to\infty} \frac{f(x)}{\varphi(x)} = k_f \right\}, \\ U_{\varphi}(\mathbb{R}^+) &= \left\{ f \in C_{\varphi}(\mathbb{R}^+) : \frac{f(x)}{\varphi(x)} \text{ is uniformly continuous} \right\}, \\ U_{\varphi}^k(\mathbb{R}^+) &= \left\{ f \in U_{\varphi}(\mathbb{R}^+) : \lim_{x\to\infty} \frac{f(x)}{\varphi(x)} = k_f \right\}, \end{aligned}$$

respectively, where k_f is a constant depending on f. It is clear that $C_{\varphi}^k(\mathbb{R}^+) \subset U_{\varphi}(\mathbb{R}^+) \subset C_{\varphi}(\mathbb{R}^+) \subset B_{\varphi}(\mathbb{R}^+)$. We have the following two results due to Gadjiev in [9]:

Lemma 3.1. *The positive linear operators T_n, $n \geq 1$, act from $C_\varphi(\mathbb{R}^+)$ to $B_\varphi(\mathbb{R}^+)$ if and only if*

$$\left|T_n(\varphi)(x)\right| \leq K_n \varphi(x),$$

where K_n is a positive constant.

Theorem 3.2. *Let $\{T_n\}_{n\geq 1}$ be a sequence of positive linear operators mapping $C_\varphi(\mathbb{R}^+)$ into $B_\varphi(\mathbb{R}^+)$ and satisfying the conditions*

$$\lim_{n\to\infty} \|T_n(e_i) - e_i\|_\varphi = 0, \text{ for } i = 0, 1, 2.$$

Then for any $f \in C_\varphi^k(\mathbb{R}^+)$, we have

$$\lim_{n\to\infty} \left\|T_n(f) - f\right\|_\varphi = 0.$$

Here, we consider weighted approximation for integral generalization of Lupaş-Jain operators D_n^β acting on $C_\varphi(\mathbb{R}^+)$.

Theorem 3.3. *Let $\{\beta_n\}_{n\geq 1}$ be a sequence such that $0 \leq \beta_n < 1$ and $\lim_{n\to\infty}\beta_n = 0$. Then for each $f \in C_\varphi^k(\mathbb{R}^+)$ we have*

$$\lim_{n\to\infty} \left\|D_n^{\beta_n}(f) - f\right\|_\varphi = 0.$$

Proof. According to Lemma 2.2, Lemma 3.1 and Remark 2.4 the operators $D_n^{\beta_n}$ act from $C_\varphi(\mathbb{R}^+)$ to $B_\varphi(\mathbb{R}^+)$. Now, it is enough to show the sufficient conditions of the Theorem 3.2 for $D_n^{\beta_n}$. We obtain

$$\lim_{n\to\infty} \left\|D_n^{\beta_n}(e_0) - e_0\right\|_\varphi = 0$$

and

$$\left\|D_n^{\beta_n}(e_1) - e_1\right\|_\varphi \leq \sup_{x\in\mathbb{R}^+} \frac{\left|D_n^{\beta_n}(e_1) - e_1\right|}{1+x^2} = \sup_{x\in\mathbb{R}^+} \frac{\left|\frac{x}{1-\beta_n} + \frac{1}{n} - x\right|}{1+x^2} \leq \frac{\beta_n}{1-\beta_n} + \frac{1}{n}$$

which gives

$$\lim_{n\to\infty} \left\|D_n^{\beta_n}(e_1) - e_1\right\|_\varphi = 0.$$

Finally,

$$\begin{aligned}
\left\|D_n^{\beta_n}(e_2) - e_2\right\|_\varphi &\leq \sup_{x\in\mathbb{R}^+} \frac{\left|D_n^{\beta_n}(e_2) - e_2\right|}{1+x^2} \\
&= \sup_{x\in\mathbb{R}^+} \left| \frac{1}{1+x^2} \left(\frac{x^2}{(1-\beta_n)^2} + \frac{x(3\beta_n^2 - 6\beta_n + 5)}{n(1-\beta_n)^3} + \frac{2}{n^2} - x^2 \right) \right| \\
&= \sup_{x\in\mathbb{R}^+} \left| \frac{x^2}{1+x^2} \frac{2\beta_n - \beta_n^2}{(1-\beta_n)^2} + \frac{x}{1+x^2} \frac{3\beta_n^2 - 6\beta_n + 5}{n(1-\beta_n)^3} + \frac{2}{n^2} \right| \\
&\leq \frac{2\beta_n - \beta_n^2}{(1-\beta_n)^2} + \frac{3\beta_n^2 - 6\beta_n + 5}{n(1-\beta_n)^3} + \frac{2}{n^2}
\end{aligned}$$

which obviously gives us

$$\lim_{n\to\infty} \left\|D_n^{\beta_n}(e_2) - e_2\right\|_\varphi = 0.$$

□

4. A Quantitative Estimate

Now, we would like to show that the constructed operators are discussed linked with a uniform convergence and a quantitative estimate.

In 2008, Holhoş [10] represented a paper which includes some quantitative estimates in weighted spaces by proposing a new modulus of continuity. Now, we will give some notifications by his pioneering work.

For each $f \in C_{\varphi}(\mathbb{R}^+)$ for every $\delta \geq 0$ for all $x \in [0,\infty)$ defined the weighted modulus of continuity as

$$\omega_{\rho}(f,\delta) := \sup_{\substack{x,t\in\mathbb{R}^+ \\ |\rho(t)-\rho(x)|\leq\delta}} \frac{\left|f(t)-f(x)\right|}{\varphi(t)+\varphi(x)}.$$

We have to keep in mind that $\omega_{\rho}(f,0) = 0$ for every $f \in C_{\varphi}(\mathbb{R}^+)$ and also $\omega_{\rho}(f,\delta)$ is a nonnegative and increasing function with respect to δ for $f \in C_{\varphi}(\mathbb{R}^+)$. The above weighted modulus of continuity is defined in [10] for the unbounded strictly increasing continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such there exist $M > 0$ and $\alpha \in (0,1]$ with the property

$$|x-y| \leq M|\rho(x)-\rho(y)|^{\alpha} \text{ for every } x, y \geq 0.$$

Here we consider the $\rho(x) = x$. Using the properties of $\omega_{\rho}(f,\delta)$ which is showed in [10] elegantly, Holhoş gave the following theorem and remark.

Theorem 4.1. *[10] Let $\{L_n\}_{n\geq1}$ be a sequence of positive linear operators mapping $C_{\varphi}(\mathbb{R}^+)$ into $B_{\varphi}(\mathbb{R}^+)$ with*

$$\left\|L_n\left(\rho^0\right)-\rho^0\right\|_{\varphi^0} = a_n$$
$$\left\|L_n\left(\rho\right)-\rho\right\|_{\varphi^{1/2}} = b_n$$
$$\left\|L_n\left(\rho^2\right)-\rho^2\right\|_{\varphi} = c_n$$
$$\left\|L_n\left(\rho^3\right)-\rho^3\right\|_{\varphi^{3/2}} = d_n,$$

where a_n, b_n, c_n and d_n, tend to zero as n goes to the infinity. Then

$$\left\|L_n(f)-f\right\|_{\varphi^{3/2}} \leq (7+4a_n+2c_n)\omega_{\rho}(f;\delta_n) + \left\|f\right\|_{\varphi} a_n$$

for all $f \in C_{\varphi}(\mathbb{R}^+)$, where

$$\delta_n = 2\sqrt{(a_n+2b_n+c_n)(1+a_n)} + a_n + 3b_n + 3c_n + d_n.$$

Remark 4.2. *Under the conditions of the Theorem 4.1 and using the fact $\lim_{\delta\to0}\omega_{\rho}(f;\delta) = 0$, we have*

$$\lim_{n\to\infty}\left\|L_n(f)-f\right\|_{\varphi^{3/2}} = 0$$

for all $f \in U^k_{\varphi^{3/2}}(\mathbb{R}^+)$.

Theorem 4.3. *Let $0 \leq \beta_n < 1$, $\lim_{n\to\infty}\beta_n = 0$, $\lim_{n\to\infty} n\beta_n = 0$ and $\left\{D_n^{\beta_n}\right\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators. Then for all $f \in C_{\varphi}(\mathbb{R}^+)$, we have*

$$\left\|D_n^{\beta_n}(f)-f\right\|_{\varphi^{3/2}} \leq \left(7 + \frac{-2\beta_n^2+4\beta_n}{(1-\beta_n)^2} + \frac{6\beta_n^2-12\beta_n+10}{n(1-\beta_n)^3} + \frac{2}{n^2}\right)\omega_{\rho}(f;\delta_n),$$

where

$$\delta := \delta_n = 2\left(\frac{2\beta_n}{1-\beta_n} + \frac{-\beta_n^2+2\beta_n}{(1-\beta_n)^2} + \frac{3\beta_n^2-6\beta_n+5}{n(1-\beta_n)^3} + \frac{2(n+1)}{n^2}\right)^{1/2} + \frac{3\beta_n}{1-\beta_n} + \frac{-3\beta_n^2+6\beta_n}{(1-\beta_n)^2} + \frac{\beta_n^3-3\beta_n^2+3\beta_n}{(1-\beta_n)^3}$$
$$+ \frac{9\beta_n^2-18\beta_n+15}{n(1-\beta_n)^3} + \frac{6\beta_n^2-12\beta_n+12}{n(1-\beta_n)^4} + \frac{11\beta_n^4-44\beta_n^3-78\beta_n^2-62\beta_n+29}{n^2(1-\beta_n)^5} + \frac{3(n+1)^2}{n^3}.$$

Proof. From Lemma 2.2 and selecting $\varphi(x) = 1 + x^2$, $\rho(x) = x$ for Theorem 4.1, one can be written

$$a_n = \left\| D_n^{\beta_n}\left(\rho^0\right) - \rho^0 \right\|_{\varphi^0} = 0,$$

$$b_n = \left\| D_n^{\beta_n}(\rho) - \rho \right\|_{\varphi^{1/2}} \leq \frac{\beta_n}{1-\beta_n} + \frac{1}{n}$$

$$c_n = \left\| D_n^{\beta_n}\left(\rho^2\right) - \rho^2 \right\|_{\varphi} \leq \frac{-\beta_n^2 + 2\beta_n}{(1-\beta_n)^2} + \frac{3\beta_n^2 - 6\beta_n + 5}{n(1-\beta_n)^3} + \frac{2}{n^2}$$

and

$$d_n = \left\| D_n^{\beta_n}\left(\rho^3\right) - \rho^3 \right\|_{\varphi^{3/2}} \leq \frac{\beta_n^3 - 3\beta_n^2 + 3\beta_n}{(1-\beta_n)^3} + \frac{6\beta_n^2 - 12\beta_n + 12}{n(1-\beta_n)^4} + \frac{11\beta_n^4 - 44\beta_n^3 - 78\beta_n^2 - 62\beta_n + 29}{n^2(1-\beta_n)^5} + \frac{6}{n^3}.$$

Since $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} n\beta_n = 0.$, it is clear that a_n, b_n, c_n and d_n, tend to zero as n goes to the infinity. Lastly, choosing $\delta := \delta_n$

$$\left\| D_n^{\beta_n}(f) - f \right\|_{\varphi^{3/2}} \leq \left(7 + \frac{4\beta_n - 2\beta_n^2}{(1-\beta_n)^2} + \frac{6\beta_n^2 - 12\beta_n + 10}{n(1-\beta_n)^3} + \frac{2}{n^2} \right) \omega_\rho(f;\delta_n),$$

Hence, the proof is completed. □

Remark 4.4. *Let* $0 \leq \beta_n < 1$ *such that* $\lim_{n\to\infty} \beta_n = 0$. *Then depending on the Remark 4.2 for all* $f \in U^k_{\varphi^{3/2}}(\mathbb{R}^+)$, *we possess*

$$\lim_{n\to\infty} \left\| D_n^{\beta_n}(f) - f \right\|_{\varphi^{3/2}} = 0.$$

5. Local Approximation Results

By $\tilde{C}_B[0,\infty)$, we denote the class of real valued, bounded and uniformly continuous functions defined on $[0,\infty)$ with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. For $f \in \tilde{C}_B[0,\infty)$ and $\delta > 0$ the m-th order modulus of continuity is defined as

$$\omega_m(f,\delta) = \sup_{0\leq h\leq\delta} \sup_{x\in[0,\infty)} \left| \Delta_h^m f(x) \right|,$$

where Δ_h^m is the forward difference given by [7] (Chapter 2, p. 40-44). In case $m = 1$, we mean the usual modulus of continuity denoted by $\omega(f;\delta)$. The Peetre's K-functional of the function is defined by

$$K_2(f,\delta) := \inf_{g\in C_B^2[0,\infty)} \{\|f-g\| + \delta\|g''\|\},$$

where

$$C_B^2[0,\infty) := \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}.$$

The following inequality

$$K_2(f,\delta) \leq M\{\omega_2(f,\sqrt{\delta})\} \tag{7}$$

is valid, for all $\delta > 0$ [7]. The positive constant M is independent of f and δ (for detailed reading see [7]).

For $f \in C_B[0,\infty)$, the Steklov mean is defined as

$$f_h(x) = \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} \left[2f(x+u+v) - f(x+2(u+v)) \right] du dv \tag{8}$$

and verifies the following inequalities:

(i) $\|f_h - f\| \le \omega_2(f,h)$,
(ii) $f_h', f_h'' \in C_B[0,\infty)$ and $\|f_h'\| \le \frac{5}{h}\omega(f,h)$, $\|f_h''\| \le \frac{9}{h^2}\omega_2(f,h)$.

Theorem 5.1. *Let $f \in \tilde{C}_B[0,\infty), n \in \mathbb{N}, 0 \le \beta < 1$ and $x \in [0,\infty)$. Then for the operators D_n^β, we possess*

$$\left|D_n^\beta(f;x) - f(x)\right| \le 2\omega(f,\delta),$$

where $\delta := \sqrt{\lambda_n^\beta(x)}$ and $\lambda_n^\beta(x)$ are defined as (6).

Proof. By using Lemma 2.3 and recognizing the following property of modulus of continuity

$$|f(t) - f(x)| \le \omega(f,\delta)\left(\frac{(t-x)^2}{\delta^2} + 1\right),$$

we obtain

$$|D_n^\beta(f;x) - f(x)| \le D_n^\beta(|f(t) - f(x)|;x) \le \omega(f,\delta)\left(1 + \frac{1}{\delta^2}D_n^\beta((t-x)^2;x)\right).$$

Lastly, choosing $\delta := \sqrt{\lambda_n^\beta(x)}$, so

$$\left|D_n^\beta(f;x) - f(x)\right| \le 2\omega(f,\delta).$$

Hence, the proof is completed. □

Theorem 5.2. *Let $f \in \tilde{C}_B[0,\infty)$ and $0 \le \beta < 1$. Then for every $x \ge 0$, the following inequality holds*

$$\left|D_n^\beta(f;x) - f(x)\right| \le 5\omega(f, \sqrt{\lambda_n^\beta(x)}) + \frac{13}{2}\omega_2(f, \sqrt{\lambda_n^\beta(x)})$$

where $\lambda_n^\beta(x)$ is defined as (6).

Proof. Applying the Steklov mean f_h that is given by (8), we obtain

$$\left|D_n^\beta(f;x) - f(x)\right| \le D_n^\beta(\left|f - f_h\right|;x) + \left|D_n^\beta(f_h - f_h(x);x)\right| + |f_h(x) - f(x)|. \tag{9}$$

Also

$$|D_n^\beta(f)| \le n\sum_{k=0}^{\infty} l_{n,k}^\beta(x)\int_0^\infty s_{n,k}(t)|f(t)|dt \le \|f\| \tag{10}$$

Using property (i) of Steklov mean and (10)

$$D_n^\beta(\left|f - f_h\right|;x) \le \|f - f_h\| \le \omega_2(f,h).$$

By Taylor's expansion and Cauchy-Schwarz inequality, we possess

$$\left|D_n^\beta(f_h - f_h(x);x)\right| \le \|f_h'\|\sqrt{D_n^\beta((t-x)^2;x)} + \frac{1}{2}\|f_h''\|D_n^\beta((t-x)^2;x).$$

By Lemma 2.3 and property (ii) of Steklov mean, we obtain

$$\left|D_n^\beta(f_h - f_h(x);x)\right| \le \frac{5}{h}\omega(f,h)\sqrt{\lambda_n^\beta(x)} + \frac{9}{2h^2}\omega_2(f,h)\lambda_n^\beta(x).$$

Lastly, choosing $h = \sqrt{\lambda_n^\beta(x)}$ and substituting the values of the above estimates in (9), then the proof is ended. □

Theorem 5.3. *Let $f \in \tilde{C}_B[0,\infty)$ and $n > 1$*

$$|D_n^{\beta}(f;x) - f(x)| \leq C\omega_2(f, \sqrt{\delta_n^{\beta}(x)}) + \omega\Big(f, \frac{x}{n-1}\Big),$$

where $\delta_n^{\beta}(x) = \Big[\dfrac{2x^2\beta^2}{(1-\beta)^2} + \dfrac{xn(4\beta^3 - 7\beta^2 + 2\beta + 3)}{n^2(1-\beta)^3} + \dfrac{3}{n^2}\Big]$.

Proof. We introduce an auxiliary operators $\tilde{D}_n^{\beta} : \tilde{C}_B[0,\infty) \to \tilde{C}_B[0,\infty)$ as follows

$$\tilde{D}_n^{\beta}(f;x) = D_n^{\beta}(f;x) - f\Big(\frac{x}{1-\beta} + \frac{1}{n}\Big) + f(x). \tag{11}$$

These operators are linear and preserve linear functions in view of Lemma 2.3. Let $g \in C_B^2[0,\infty)$ and x, $t \in [0,\infty)$. By Taylor's expansion

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du,$$

we have

$$\begin{aligned}
|\tilde{D}_n^{\beta}(g;x) - g(x)| \leq& \tilde{D}_n^{\beta}\Big(\Big|\int_x^t (t-u)g''(u)du, x\Big|\Big)\\
\leq& D_n^{\beta}\Big(\Big|\int_x^t (t-u)g''(u)du, x\Big|\Big) + \Big|\int_x^{\frac{x}{1-\beta}+\frac{1}{n}} \Big(\frac{x}{1-\beta} + \frac{1}{n} - u\Big)g''(u)du\Big|\\
\leq& D_n^{\beta}((t-x)^2;x)\|g''\| + \Big(\frac{x}{1-\beta} + \frac{1}{n} - x\Big)^2\|g''\|.
\end{aligned}$$

Next, using central moments of operators, we have

$$\begin{aligned}
|\tilde{D}_n^{\beta}(g;x) - g(x)| \leq& \Big[D_n^{\beta}((t-x)^2;x) + \Big(\frac{x\beta}{1-\beta} + \frac{1}{n}\Big)^2\Big]\|g''\|\\
\leq& \Big[\frac{x^2\beta^2}{(1-\beta)^2} + \frac{x(2\beta^3 - 3\beta^2 + 3)}{n(1-\beta)^3} + \frac{2}{n^2} + \Big(\frac{x\beta}{1-\beta} + \frac{1}{n}\Big)^2 \|\|\Big]\|g''\|\\
\leq& \Big[\frac{2x^2\beta^2}{(1-\beta)^2} + \frac{xn(4\beta^3 - 7\beta^2 + 2\beta + 3)}{n^2(1-\beta)^3} + \frac{3}{n^2}\Big]\|g''\|.
\end{aligned} \tag{12}$$

On the other hand, relying on the Lemma 2.6 for $f \in \tilde{C}_B[0,\infty)$ and $x \in [0,\infty)$,

$$\|D_n^{\beta}(f)\| \leq \|f\|.$$

Now, for the operators $\tilde{D}_n^{\beta}$, we possess,

$$\|\tilde{D}_n^{\beta}(f;x)\| \leq \|D_n^{\beta}(f;x)\| + 2\|f\| \leq 3\|f\| \tag{13}$$

Combining (12) and (13), we can write

$$\begin{aligned}
|D_n^{\beta}(f;x) - f(x)| \leq& |\tilde{D}_n^{\beta}(f-g;x) - (f-g)(x)| + |\tilde{D}_n^{\beta}(g;x) - g(x)| + \Big|f\Big(\frac{x}{1-\beta} + \frac{1}{n}\Big) - f(x)\Big|\\
\leq& 4\|f-g\| + \Big[\frac{2x^2\beta^2}{(1-\beta)^2} + \frac{xn(4\beta^3 - 7\beta^2 + 2\beta + 3)}{n^2(1-\beta)^3} + \frac{3}{n^2}\Big]\|g''\| + \Big|f(x) - f\Big(\frac{x}{1-\beta} + \frac{1}{n}\Big)\Big|\\
\leq& C\Big\{\|f-g\| + \Big[\frac{2x^2\beta^2}{(1-\beta)^2} + \frac{xn(4\beta^3 - 7\beta^2 + 2\beta + 3)}{n^2(1-\beta)^3} + \frac{3}{n^2}\Big]\|g''\|\Big\} + \omega\Big(f, \frac{x}{1-\beta} + \frac{1}{n}\Big).
\end{aligned}$$

Taking infimum over all $g \in C_B^2[0,\infty)$, and using the inequality (7), we get the desired assertion. □

A necessary definition for the following theorem belongs to Lipschitz type space. A Lipschitz type space with two parameters $\eta_1 \geq 0$ and $\eta_2 > 0$ is defined as [14]

$$Lip_M^{(\eta_1,\eta_2)}(\mu) := \left\{ f \in C_B[0,\infty) : |f(t) - f(x)| \leq M \frac{|t-x|^\mu}{\left(t + \eta_1 x^2 + \eta_2 x\right)^{\mu/2}}; \quad x, t \in (0,\infty) \right\}$$

where M is any positive constant and $0 < \mu \leq 1$.

Theorem 5.4. *Let $f \in Lip_M^{(\eta_1,\eta_2)}(\mu)$. Then, we have*

$$|D_n^\beta(f;x) - f(x)| \leq M\left(\frac{\lambda_n^\beta(x)}{\eta_1 x^2 + \eta_2 x}\right)^{\mu/2}.$$

Proof. Let us first consider the case $\mu = 1$. We can write

$$|D_n^\beta(f;x) - f(x)| \leq D_n^\beta(|f(t) - f(x)|;x) \leq M D_n^\beta\left(\frac{|t-x|}{\sqrt{t + \eta_1 x^2 + \eta_2 x}};x\right)$$

Using the fact that $\dfrac{1}{\sqrt{t + \eta_1 x^2 + \eta_2 x}} \leq \dfrac{1}{\sqrt{\eta_1 x^2 + \eta_2 x}}$, the Cauchy-Schwarz inequality and applying Lemma 2.3

$$\begin{aligned} |D_n^\beta(f;x) - f(x)| &\leq M \frac{1}{\sqrt{\eta_1 x^2 + \eta_2 x}} D_n^\beta\left(|t-x|;x\right) \\ &\leq \frac{M}{\sqrt{\eta_1 x^2 + \eta_2 x}}\left(D_n^\beta((t-x)^2;x)\right)^{1/2} \leq M\left(\frac{\lambda_n^\beta(x)}{\eta_1 x^2 + \eta_2 x}\right)^{1/2} \end{aligned}$$

Thus, the assertion holds for the case $\mu = 1$.

Let us now take $0 < \mu < 1$. Then, using the Hölder inequality for $p = \dfrac{2}{\mu}$ and $q = \dfrac{2}{2-\mu}$ and considering Lemma 2.3 we obtain

$$\begin{aligned} |D_n^\beta(f;x) - f(x)| \leq& n \sum_{k=0}^\infty l_{n,k}^\beta(x) \int_0^\infty s_{n,k}(t)|f(t) - f(x)|dt \\ \leq& \left(n \sum_{k=0}^\infty l_{n,k}^\beta(x) \int_0^\infty s_{n,k}(t)\left(|f(t) - f(x)|dt\right)^{2/\mu}\right)^{\mu/2} \left(n \sum_{k=0}^\infty l_{n,k}^\beta(x) \int_0^\infty s_{n,k}(t)dt\right)^{(2-\mu)/2} \\ \leq& M\left(n \sum_{k=0}^\infty l_{n,k}^\beta(x) \int_0^\infty s_{n,k}(t) \frac{(t-x)^2}{(t + \eta_1 x^2 + \eta_2 x)} dt\right)^{\mu/2} \\ \leq& \frac{M}{(\eta_1 x^2 + \eta_2 x)^{\mu/2}}\left(D_n^\beta((t-x)^2;x)\right)^{\mu/2} \\ \leq& M\left(\frac{\lambda_n^\beta(x)}{\eta_1 x^2 + \eta_2 x}\right)^{\mu/2}. \end{aligned}$$

This concludes the proof. □

Remark 5.5. *One of the crucial remark is that we have investigated the local approximation results between Theorem 5.1 to Theorem 5.4. Therefore, we indicate that λ_n^β and δ_n^β tend to zero when taking $\beta = \beta_n$ $0 \leq \beta_n < 1$, $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} n\beta_n = 0$. Otherwise, these theorems just stand an inequality or an estimate.*

6. A Voronovskaya type theorem

Now, we will analyze the asymptotic behavior of given operators $D_n^{\beta_n}$ with Voronovskaya type theorem.

Theorem 6.1. *Let* $0 \leq \beta_n < 1$, *such that* $\lim_{n\to\infty} \beta_n = 0$ *and* $\lim_{n\to\infty} n\beta_n = 0$. *Also let* f *be a bounded integrable function on* $[0, \infty)$ *and* f', f'' *exists at a point* $x \in [0, \infty)$ *then*

$$\lim_{n\to\infty} n(D_n^{\beta_n}(f;x) - f) = f'(x) + 3xf''(x).$$

Proof. By the Taylor's expansion, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \sigma(t,x)(t-x)^2.$$

where $\sigma(t,x)$ belongs to $C_B[0,\infty)$. Taking into account the linearity of the operators, we apply the operators $D_n^{\beta_n}$ to both sides of the above Taylor's expansions to get

$$n(D_n^{\beta_n}(f;x) - f) = nD_n^{\beta_n}(t-x;x)f'(x) + \frac{n}{2}D_n^{\beta_n}((t-x)^2;x)f''(x) + nD_n^{\beta_n}(\sigma(t,x)(t-x)^2;x). \tag{14}$$

After applying the Cauchy-Schwarz inequality to the third term of the right hand side of (14), we find

$$n\left|D_n^{\beta_n}(\sigma(t,x)(t-x)^2;x)\right| \leq \left(n^2B_n^{\beta_n}((t-x)^4;x)\right)^{1/2}\left(D_n^{\beta_n}(\sigma^2(t,x);x)\right)^{1/2},$$

which yields,

$$\begin{aligned}\lim_{n\to\infty} n\left(D_n^{\beta_n}(f;x) - f(x)\right) \leq{}& \lim_{n\to\infty} nD_n^{\beta_n}(t-x;x)f'(x) + \lim_{n\to\infty}\frac{n}{2}D_n^{\beta_n}((t-x)^2;x)f''(x)\\ &+ \lim_{n\to\infty}\left(n^2D_n^{\beta_n}((t-x)^4;x)\right)^{1/2}\lim_{n\to\infty}\left(D_n^{\beta_n}(\sigma^2(t,x);x)\right)^{1/2}.\end{aligned}$$

Hence, from Lemma 2.3,

$$\lim_{n\to\infty} n\left(D_n^{\beta_n}(f;x) - f(x)\right) \leq f'(x) + 3xf''(x) + 27x^2\lim_{n\to\infty}\left(D_n^{\beta_n}(\sigma^2(t,x);x)\right)^{1/2}.$$

Let us take $\xi(t,x) := \sigma^2(t,x)$. Then, we observe that $\xi(x,x) = 0$ and $\xi(t,x) \in C_B[0,\infty)$. Thus, we get $\lim_{n\to\infty}\left(D_n^{\beta_n}(\xi(t,x);x)\right)^{1/2} = 0$. Here, as $\lim_{n\to\infty}\left(D_n^{\beta_n}(\xi(t,x);x)\right)^{1/2} = 0$, we have to notice that:

"Let $f \in C_B[0,\infty)$ and $\beta = \{\beta_n\}_{n\geq1}$ be a sequence such that $0 \leq \beta_n < 1$ and $\lim_{n\to\infty}\beta_n = 0$, then $\lim_{n\to\infty} D_n^{\beta}(f;x) = f(x)$ uniformly compact subset on $[0,\infty)$ where $C_B[0,\infty)$ be the space of all real valued continuous bounded functions f defined on $[0,\infty)$."

Finally,

$$\lim_{n\to\infty} n(D_n^{\beta_n}(f;x) - f) = f'(x) + 3xf''(x),$$

we get the assertion of theorem. □

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