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An Existence Results for a Fractional Differential Equation with ϕ-Fractional Derivative

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Abstract. In this article, we establish certain sufficient conditions to show the existence of solutions of a fractional differential equation with the ϕ-Riemann-Liouville and ϕ-Caputo fractional derivative in a special Banach space. Our approach is based on fixed point theorems for Meir-Keeler condensing operators via measure of non-compactness. Also an example is given to illustrate our approach..

1. Introduction

The theoretical study of fractional differential equations and inclusions has recently acquired great importance in applied mathematics and the modeling of many phenomena in various sciences. Let us quote for example [6, 12, 13, 17–19, 22]. The monographs [15, 20, 21, 24, 27] contain basic concepts and theory in fractional differential equations and fractional calculus.

Recently, excellent works has been done to study fractional differential equations with various conditions which resides in the existence and uniqueness theorem by involving various fractional derivatives [1, 2, 8, 9, 11, 14, 23, 25].

In [26] there are new concepts of the fractional integral and the fractional derivative. Many fractional differential equations solved over Banach spaces using these new concepts and certain basic tools from functional analysis, we mention for example [7, 16].

We consider, in this work, the following fractional differential equation

$$(\mathbf{P}) \quad \begin{cases} {}^{rl}\mathcal{D}_{0^+}^{\alpha,\phi}({}^{c}\mathcal{D}^{\beta,\phi})y(r) = f(r, y(r), D^{\beta}y(r)), \quad r \in (0, L], \\ \Im_{0^+}^{(1-\alpha),\phi}({}^{c}\mathcal{D}^{\beta,\phi})y(0^+) = a, \\ y(0) = b, \end{cases}$$

where ${}^{rl}\mathcal{D}^{\alpha,\phi}$(resp.${}^{c}\mathcal{D}^{\beta,\phi}$) denotes the left-sided ϕ-Riemann-Liouville (resp. ϕ-Caputo) fractional derivative , $0 < \alpha < 1$, $0 < \beta < 1$ with $\alpha + \beta > 1$. The operator $\Im_{0^+}^{(1-\alpha),\phi}$ denotes the left-sided ϕ- Riemann-Liouville

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fractional integral, E is a Banach space with the norm $\|.\|$, $a, b \in E$, $f : (0, L] \times E \times E \to E$ a function satisfying some specified conditions (see, section 3) and $\phi \in C^1([0, L], \mathbb{R}^+)$ satisfies $\phi'(r) > 0$, for all $r \in [0, L]$.

The present work is organized as follows: to make the reader understand our problem, we give in section 2 some definitions, lemmas and basic results. Next, in the section 3, we present our main results using a new method to show the existence of solutions to the problem (**P**). Finally an example to reinforce our work in the section 4

2. Background and basic results

In this section, we introduce some definitions and results that are very useful in this work.

Let $C([0, L], E)$ be the space of E-valued continuous functions on $[0, L]$ endowed with the uniform norm topology

$$\|x\|_\infty = \sup\{\|x(r)\|,\ r \in [0, L]\}.$$

Let $C_{\alpha,\phi}([0, L], E)$ and $C^\beta_{\alpha,\phi}([0, L], E)$ two Banach spaces of functions defined as follows:

$$C_{\alpha,\phi}([0, L], E) = \{w \in C((0, L], E) : \lim_{r\to 0^+}(w(r) - w(0))^{1-\alpha} w(r) \text{ exists and finite}\},$$

with the norm

$$\|w\|_{C_{\alpha,\phi}} = \sup_{r\in[0,L]} (\phi(r) - \phi(0))^{1-\alpha}\|w(r)\|,$$

and

$$C^\beta_{\alpha,\phi}([0, L], E) = \{w : [0, L] \to E : w \in C([0, L], E) \text{ and } {}^c\mathcal{D}^{\beta,\phi} w \in C_{\alpha,\phi}([0, L], E)\},$$

with the norm

$$\|w\|_{C^\beta_{\alpha,\phi}} = \sup_{r\in[0,L]} \|w(r)\| + \sup_{r\in[0,L]} (\phi(r) - \phi(0))^{1-\alpha}\|{}^c\mathcal{D}^{\beta,\phi} w(r)\|.$$

We begin with some definitions from the theory of fractional calculus. For all $\eta > -1$ and $s, r \in [0, L]$ with $r \geq s$, we pose $\psi_\eta(r, s) = (\phi(r) - \phi(s))^\eta$.

Definition 2.1. *[15]. Let Γ be the gamma function, ξ a non-negative real number and $\delta \in C^1((0, L], E)$.*

(i) We recall that the ϕ-Riemann- Liouville fractional integral of order $\xi > 0$ of δ is given by

$$\Im^{\xi,\phi}_{0^+}\delta(r) = \frac{1}{\Gamma(\xi)}\int_0^r \phi'(s)\psi_{\xi-1}(r, s)\delta(s)ds,$$

(ii) Let $0 < \alpha < 1$. The ϕ-Riemann- Liouville fractional derivative of order ξ of the function δ is given as

$${}^{rl}\mathcal{D}^{\xi,\phi}_{0^+}\delta(r) = \frac{1}{\phi'(r)\Gamma(1 - \xi)}\frac{d}{dr}\left(\int_0^r \phi'(s)\psi_{-\xi}(r, s)\delta(s)ds\right),$$

(iii) Let $0 < \alpha < 1$. The ϕ-Caputo fractional derivative of order ξ of the function δ is given as

$${}^{c}\mathcal{D}^{\xi,\phi}_{0^+}\delta(r) = \frac{1}{\Gamma(1 - \xi)}\left(\int_0^r \psi_{-\xi}(r, s)\delta'(s)ds\right).$$

Lemma 2.2. *[15, 26] Let $\xi, \zeta \in \mathbb{R}^*_+$. Then we have:*

1. $\Im^{\xi,\phi}_{0^+}\psi_{\zeta-1}(r, 0) = \frac{\Gamma(\zeta)}{\Gamma(\xi+\zeta)}\psi_{\xi+\zeta-1}(r, 0)$.
2. *If $0 < \xi < 1$, then ${}^{rl}\mathcal{D}^{\xi,\phi}_{0^+}\psi_{\xi-1}(r, 0) = 0$.*
3. *If $\zeta > 1$, then ${}^{c}\mathcal{D}^{\xi,\phi}_{0^+}\psi_{\zeta-1}(r, 0) = \frac{\Gamma(\zeta)}{\Gamma(\zeta-\xi)}\psi_{\zeta-\xi-1}(r, 0)$.*

Lemma 2.3. *[9] Let $\zeta, \xi, \omega > 0$. Thus*

$$\int_0^r (r-s)^{\xi-1} s^{\zeta-1} e^{-\omega s} ds \leq C r^{\xi-1},$$

where

$$C = \left(\frac{(1+\zeta(1+\zeta)/\xi)\omega^{-\zeta}}{\Gamma(\zeta)} \right) \max\{1, 2^{1-\xi}\}.$$

Remark 2.4. *Under the data of the previous lemma, the inequality that follows is also valid*

$$\int_0^r \phi'(s)\psi_{\xi-1}(r,s)\psi_{\zeta-1}(r,0) e^{-\omega\psi_1(s,0)} ds \leq C\psi_{\xi-1}(r,0).$$

Let us now give the definition of the measure of non-compactness in the sense of Kuratowski and its properties. For all $G \subseteq E$, we denote by $S_b(G)$ the set of all bounded subsets of G.

Definition 2.5. *[4, 5] Let $D \in S_b(E)$. The Kuratowski measure of non-compactness ϑ of the subset D is defined as follows:*

$$\vartheta(D) = \inf\{e > 0 : \Omega \text{ admits a finite cover by sets of diameter} \leq e\}.$$

Lemma 2.6. *[4, 5] Let $A, B \in S_b(E)$. The following properties hold:*

- *(i_1) $\vartheta(A) = 0$ if and only if A is relatively compact,*
- *(i_2) $\vartheta(A) = \vartheta(\overline{A})$, where $\overline{A}$ denotes the closure of A,*
- *(i_3) $\vartheta(A + B) \leq \vartheta(A) + \vartheta(B)$,*
- *(i_4) $A \subset B$ implies $\gamma(A) \leq \gamma(B)$,*
- *(i_5) $\vartheta(a.A) = \|a\|.\vartheta(A)$ for all $a \in E$,*
- *(i_6) $\vartheta(\{a\} \cup A) = \vartheta(A)$ for all $a \in E$,*
- *(i_7) $\vartheta(A) = \vartheta(Conv(A))$, where $Conv(A)$ is the smallest convex that contains A.*

Lemma 2.7. *[7] Let $D \in S_b(E)$ and $\varepsilon > 0$. Then, there is a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset D$, such that*

$$\vartheta(D) \leq 2\vartheta(\{\mu_n, n \in \mathbb{N}\}) + \varepsilon.$$

Lemma 2.8. *[10] If D is a equicontinuous and bounded subset of $C([a,b],E)$, then $\vartheta(D(.)) \in C([a,b],\mathbb{R}^+)$*

$$\vartheta_C(D) = \max_{r\in[a,b]} \vartheta(D(r)), \ \vartheta\left(\left\{ \int_a^b w(r)dr : w \in D \right\} \right) \leq \int_a^b \vartheta(D(r))dr,$$

where $D(r) = \{w(r) : w \in D\}$ and ϑ_C is the non-compactness measure on the space $C([a,b],E)$.

Meir-Keeler has been introduced since 1969 the notion of Meir-Keeler contraction mapping in a metric space. Most recently in 2015, the authors introduced the following definition and fixed point theorem.

Definition 2.9. *[3] Let κ be an arbitrary measure of non-compactness on E and G be a nonempty subset of E . Let Δ be an operator from G to G. Δ is said Meir-Keeler condensing operator if*

$$\forall\varepsilon > 0, \ \exists k(\varepsilon) > 0, \forall D \in S_b(G) : \varepsilon \leq \kappa(D) < \varepsilon + k \Longrightarrow \kappa(\Lambda D) < \varepsilon.$$

Theorem 2.10. *[3] Let κ be an arbitrary measure of non-compactness on* E *and G a closed, bounded and convex subset of* E*. Let Δ be an operator from G to G, assume that Δ is a Meir-Keeler condensing operator and continuous, then the set $\{w \in G : \Delta(w) = w\}$ is nonempty and compact.*

Lemma 2.11. *Let $\gamma \in C([0,L],E)$ and $y \in C^{\beta}_{\alpha,\phi}([0,L],E)$, then y is a solution of the problem*

$$^{rl}\mathcal{D}^{\alpha,\phi}_{0^+}(^{c}\mathcal{D}^{\beta,\phi})y(r) = \gamma(r), \quad r \in (0,L], \tag{1}$$

$$\Im^{(1-\alpha),\phi}_{0^+}(^{c}\mathcal{D}^{\beta,\phi})y(0^+) = a, \tag{2}$$

$$y(0) = b, \tag{3}$$

if y satisfies the following integral equation,

$$y(r) = b + \frac{a\psi_{\beta+\alpha-1}(r,0)}{\Gamma(\beta+\alpha)} + \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_0^r\int_0^s \phi'(s)\phi'(\tau)\psi_{\beta-1}(r,s)\psi_{\alpha-1}(s,\tau)\gamma(\tau)d\tau ds. \tag{4}$$

Proof. Let $y \in C^{\beta}_{\alpha,\phi}([0,L],E)$ be a solution of (4), we have $y(0) = b$. Next, by applying $^{c}\mathcal{D}^{\beta,\phi}$ to both sides of (4), we get

$$^{c}\mathcal{D}^{\beta,\phi}y(r) = \frac{a\psi_{\alpha-1}(r,0)}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^r \phi'(s)\psi_{\alpha-1}(r,s)\gamma(s)ds. \tag{5}$$

Applying $\Im^{1-\alpha}_{0^+}$ to both sides of (5) and utilizing Lemma 2.2, we get

$$\Im^{(1-\alpha),\phi}_{0^+}(^{c}\mathcal{D}^{\beta,\phi})y(r) = a + \Im^{1,\phi}_{0^+}\gamma(r)$$

By taking r tends to 0, we get (2) □

3. Existence of the solutions

We first put the following hypotheses:

(H$_1$) The function $f : (0,L] \times E \times E \to E$ is continuous and for all $x, y, u, v \in E$ and $r \in (0,L]$:

$$\|f(r,x,y) - f(r,u,v)\| \le A\|x-u\| + \psi_{1-\alpha}(r,0)B\|y-v\|,$$

where $A,\ B \in \mathbb{R}^+$.

(H$_2$) For $r \in (0,L]$ and $x, y \in E$,

$$\|f(r,x,y)\| \le a(r)\|x\| + \psi_{\lambda}(r,0)e^{-\sigma\psi_1(r,0)}b(r)\|y\|,$$

where $\sigma > 0, \lambda \ge 1-\alpha$ and $a(.),\ b(.) : [0,\infty) \to \mathbb{R}^+$ are continuous functions,

(H$_3$) There exists two functions $\imath, \jmath \in C([0,L],\mathbb{R}^+)$ such that for each nonempty, bounded set $\Omega \subset C^{\beta}_{\alpha,\phi}([0,L],E)$,

$$\vartheta(f(r,\Omega(r),^{c}\mathcal{D}^{\beta,\phi}\Omega(r))) \le \imath(r)\vartheta(\Omega(r)) + \psi_{1-\alpha}(r,0)\jmath(r)\vartheta(^{c}\mathcal{D}^{\beta,\phi}\Omega(r)), \quad \text{for all } r \in (0,L],$$

and

$$4\left(F(L) + \psi_{1-\alpha}(L,0)^{c}\mathcal{D}^{\beta,\phi}F(L)\right) < 1,$$

where $\Omega(r) = \{y(r) : y \in \Omega\}$, $^{c}\mathcal{D}^{\beta,\phi}\Omega(r) = \{^{c}\mathcal{D}^{\beta,\phi}y(r) : y \in \Omega\}$ and

$$F(r) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^r\int_0^s \phi'(s)\phi'(\tau)\psi_{\beta-1}(r,s)\psi_{\alpha-1}(s,\tau)\left[\imath(\tau) + \jmath(\tau)\right]d\tau ds, \;\; r \in (0,L].$$

(H₄)

$$\frac{\Gamma(\alpha+\beta+1)\left[b^*\alpha C + a^*\psi_1(L,0)\right] + \Gamma(\alpha+1)\left[a^*\psi_{\beta+\alpha}(L,0) + b^*(\alpha+\beta)C\psi_{\alpha+\beta-1}(L,0)\right]}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)} < 1,$$

where $a^* = \sup_{[0,L]} a(r)$ and $b^* = \sup_{[0,L]} b(r)$.

We define the operator $N : C^{\beta}_{\alpha,\phi}([0,L],E) \to C^{\beta}_{\alpha,\phi}([0,L],E)$ by

$$\begin{aligned} Ny(r) &= b + \frac{a\psi_{\beta+\alpha-1}(r,0)}{\Gamma(\beta+\alpha)} \\ &\quad + \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_0^r\int_0^s \phi'(s)\phi'(\tau)\psi_{\beta-1}(r,s)\psi_{\alpha-1}(s,\tau)f(\tau,y(\tau),{}^c\mathcal{D}^{\beta,\phi}y(\tau))d\tau ds, \end{aligned}$$

and the operator ${}^c\mathcal{D}^{\beta,\phi}N : C_{\alpha,\phi}([0,L],E) \to C_{\alpha,\phi}([0,L],E)$ by

$$({}^c\mathcal{D}^{\beta,\phi}N)y(r) = \frac{a\psi_{\alpha-1}(r,0)}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^r \phi'(s)\psi_{\alpha-1}(r,s)f(s,y(s),{}^c\mathcal{D}^{\beta,\phi}y(s))ds.$$

The theorem below is the main result. Let

$$B = \{y \in C^{\beta}_{\alpha,\phi}([0,L],E) : \|y\|_{C^{\beta}_{\alpha,\phi}} \leq R\},$$

where R is a strictly positive real number.

Theorem 3.1. *Suppose that the conditions* **(H₁)** − **(H₄)** *are valid. If*

(H₅)

$$\begin{aligned} \frac{1}{R} &< \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+1)\left[\|b\|\Gamma(\alpha+1)+\alpha\|a\|\right] + \Gamma(\alpha+1)\|a\|(\alpha+\beta)\psi_{\beta+\alpha-1}(L,0)} \\ &\quad - \frac{\Gamma(\alpha+\beta+1)\left[b^*\alpha C + a^*\psi_1(L,0)\right] + \Gamma(\alpha+1)\left[a^*\psi_{\beta+\alpha}(L,0) + b^*(\alpha+\beta)C\psi_{\alpha+\beta-1}(L,0)\right]}{\Gamma(\alpha+\beta+1)\left[\|b\|\Gamma(\alpha+1)+\alpha\|a\|\right] + \Gamma(\alpha+1)\|a\|(\alpha+\beta)\psi_{\beta+\alpha-1}(L,0)}. \end{aligned}$$

is valid. Then the problem **(P)** *has at least one solution.*

Proof. From the definition of the operator N and Lemma 2.11, we see that the fixed points of N are solutions of problem **(P)**. For this reason, it suffices to verify the axioms of Theorem 2.10, it is done in four steps.

First step. We start to prove that N is bounded. Let $y \in C^{\beta}_{\alpha,\phi}([0,L],E)$, from **(H₂)** it is easy to deduce that $Ny \in C^{\beta}_{\alpha,\phi}([0,L],E)$. Using **(H₂)** and Lemma 2.2, for all $y \in B_{\varrho} = \{y \in C^{\beta}_{\alpha,\phi}([0,L],E) : \|y\|_{C^{\beta}_{\alpha,\phi}} < \varrho\}$ and $r \in (0,L]$ we get

$$\begin{aligned} \|Ny(r)\| &\leq \|b\| + \frac{\|a\|\psi_{\beta+\alpha-1}(r,0)}{\Gamma(\beta+\alpha)} \\ &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^r\int_0^s \phi'(s)\phi'(\tau)\psi_{\beta-1}(r,s)\psi_{\alpha-1}(s,\tau)\|f(\tau,y(\tau),{}^c\mathcal{D}^{\beta,\phi}y(\tau))\|d\tau ds \\ &\leq \|b\| + \frac{\|a\|\psi_{\beta+\alpha-1}(L,0)}{\Gamma(\beta+\alpha)} + \frac{a^*\rho}{\Gamma(\alpha)\Gamma(\beta)}\int_0^r\int_0^s \phi'(s)\phi'(\tau)\psi_{\beta-1}(r,s)\psi_{\alpha-1}(s,\tau)d\tau ds \\ &\quad + \frac{b^*\rho}{\Gamma(\alpha)\Gamma(\beta)}\int_0^r\int_0^s \phi'(s)\phi'(\tau)\psi_{\beta-1}(r,s)\psi_{\alpha-1}(s,\tau)\psi_{\lambda+\alpha-1}(\tau,0)e^{-\sigma\psi_1(\tau,0)}d\tau ds. \end{aligned}$$

So,

$$\|Ny\|_\infty \le \|b\| + \frac{\|a\|\psi_{\beta+\alpha-1}(L,0)}{\Gamma(\alpha+\beta)} + \frac{a^*\rho\psi_{\beta+\alpha}(L,0)}{\Gamma(\alpha+\beta+1)} + \frac{b^*\rho C\psi_{\alpha+\beta-1}(L,0)}{\Gamma(\alpha+\beta)} = M_1. \tag{6}$$

And we have also

$$\begin{aligned}\|\psi_{1-\alpha}(r,0)({}^c\mathcal{D}^{\beta,\phi}N)y(r)\| &\le \frac{\|a\|}{\Gamma(\alpha)} + \frac{\psi_{1-\alpha}(r,0)}{\Gamma(\alpha)}\int_0^r \phi'(s)\psi_{\alpha-1}(r,s)\|f(s,y(s),c\mathcal{D}^{\beta,\phi}y(s))\|ds\\ &\le \frac{\|a\|}{\Gamma(\alpha)} + \frac{a^*\rho\psi_{1-\alpha}(r,0)}{\Gamma(\alpha)}\int_0^r \phi'(s)\psi_{\alpha-1}(r,s)ds\\ &+ \frac{b^*\rho\psi_{1-\alpha}(r,0)}{\Gamma(\alpha)}\int_0^r \phi'(s)\psi_{\alpha-1}(r,s)\psi_{\lambda+\alpha-1}(s,0)e^{-\sigma\psi_1(s,0}ds.\end{aligned}$$

So,

$$\|{}^c\mathcal{D}^{\beta,\phi}N)y\|_{C_{\alpha,\phi}} \le \frac{\|a\|}{\Gamma(\alpha)} + \frac{a^*\rho\psi_1(L,0)}{\Gamma(\alpha+1)} + \frac{b^*\rho C}{\Gamma(\alpha)} = M_2, \tag{7}$$

where $C = \max\{1, 2^{1-\alpha}\}\Gamma(\alpha+\lambda)[1 + (\alpha+\lambda)(\alpha+\lambda+1)/\alpha]\sigma^{-(\alpha+\lambda)}$.
From (6) and (7), we get

$$\|(Ny)\|_{C^\beta_{\alpha,\phi}} \le M = M_1 + M_2.$$

Second step. We prove that N is continuous. Let $\{y_n\}_{n\in\mathbb{N}}$ converges to y in $C^\beta_{\alpha,\phi}([0,L],E)$ and $\epsilon > 0$. Hypothesis ($\mathbf{H_1}$) assume that there exists $m \in \mathbb{N}$ such that, for all $n \ge m$ and $r \in (0,L]$, we have

$$\|f(r,y_n(r),{}^c\mathcal{D}^{\beta,\phi}y_n(r)) - f(r,y(r),{}^c\mathcal{D}^{\beta,\phi}y(r))\| < \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+1)\psi_1(L,0) + \Gamma(\alpha+1)\psi_{\alpha+\beta}(L,0)}\epsilon. \tag{8}$$

We have then

$$\begin{aligned}&\|Ny_n(r) - Ny(r)\| + \psi_{1-\alpha}(r,0)\|({}^c\mathcal{D}^{\beta,\phi}N)y_n(r) - ({}^c\mathcal{D}^{\beta,\phi}N)y(r)\| \le \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\\ &\times\int_0^r\int_0^s \phi'(s)\phi'(\tau)\psi_{\beta-1}(r,s)\psi_{\alpha-1}(s,\tau)\|f(\tau,y_n(\tau),{}^c\mathcal{D}^{\beta,\phi}y_n(\tau)) - f(\tau,y(\tau),{}^c\mathcal{D}^{\beta,\phi}y(\tau))\|d\tau ds\\ &+ \frac{\psi_{1-\alpha}(r,0)}{\Gamma(\alpha)}\int_0^r \phi'(s)\psi_{\alpha-1}(r,s)\|f(s,y_n(s),{}^c\mathcal{D}^{\beta,\phi}y_n(s)) - f(s,y(s),{}^c\mathcal{D}^{\beta,\phi}y(s))\|ds.\end{aligned}$$

From (8), we conclude that

$$\exists m \in \mathbb{N}, \forall n \ge m : \|Ny_n - Ny\|_{C^\beta_{\alpha,\phi}} < \epsilon.$$

Third step. We prove that NB is equicontinuous for all bounded subset B of $C^\beta_{\alpha,\phi}([0,L],E)$, let B_ρ be the subset which was previously defined. It suffices to prove that NB_ρ and $({}^c\mathcal{D}^{\beta,\phi}N)B_\rho$ are equicontinuous respectively in $C([0,L],E)$ and in $C_{\alpha,\phi}([0,L],E)$. Let $y \in B_\rho$ and $r_1, r_2 \in (0,L]$ with $r_1 < r_2$. First of all, we have

$$\begin{aligned}&\|Ny(r_2) - Ny(r_1)\| \le \frac{\|a\|\left(\psi_{\alpha+\beta-1}(r_2,0) - \psi_{\alpha+\beta-1}(r_1,0)\right)}{\Gamma(\alpha+\beta)}\\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{r_1}\int_0^s \phi'(s)\phi'(\tau)[\psi_{\beta-1}(r_1,s) - \psi_{\beta-1}(r_2,s)]\psi_{\alpha-1}(s,\tau)\|f(\tau,y(\tau),{}^c\mathcal{D}^{\beta,\phi}y(\tau))\|d\tau ds\\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{r_1}^{r_2}\int_0^s \phi'(s)\phi'(\tau)\psi_{\beta-1}(r_2,s)\psi_{\alpha-1}(s,\tau)\|f(\tau,y(\tau),{}^c\mathcal{D}^{\beta,\phi}y(\tau))\|d\tau ds\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\|a\|\left(\psi_{\alpha+\beta-1}(r_2,0)-\psi_{\alpha+\beta-1}(r_1,0)\right)}{\Gamma(\alpha+\beta)}+\frac{a^*\rho}{\Gamma(\alpha)\Gamma(\beta)}\\
&\times\int_0^{r_1}\int_0^{s}\phi'(s)\phi'(\tau)[\psi_{\beta-1}(r_1,s)-\psi_{\beta-1}(r_2,s)]\psi_{\alpha-1}(s,\tau)d\tau ds+\frac{b^*\rho}{\Gamma(\alpha)\Gamma(\beta)}\\
&\times\int_0^{r_1}\int_0^{s}\phi'(s)[\psi_{\beta-1}(r_1,s)-\psi_{\beta-1}(r_2,s)]\psi_{\alpha-1}(s,\tau)\psi_{\lambda+\alpha-1}(\tau,0)e^{-\sigma\psi_1(\tau,0)}d\tau ds\\
&+\frac{a^*\rho}{\Gamma(\alpha)\Gamma(\beta)}\int_{r_1}^{r_2}\int_0^{s}\phi'(s)\phi'(\tau)\psi_{\beta-1}(r_2,s)\psi_{\alpha-1}(s,\tau)d\tau ds\\
&+\frac{b^*\rho}{\Gamma(\alpha)\Gamma(\beta)}\int_{r_1}^{r_2}\int_0^{s}\phi'(s)\psi_{\alpha-1}(r_2,s)\psi_{\beta-1}(s,\tau)\psi_{\lambda+\alpha-1}(\tau,0)e^{-\sigma\psi_1(\tau,0)}d\tau ds\\
&\leq \frac{\|a\|\left(\psi_{\alpha+\beta-1}(r_2,0)-\psi_{\alpha+\beta-1}(r_1,0)\right)}{\Gamma(\alpha+\beta)}+\frac{a^*\rho\psi_{\alpha}(L,0)+b^*\rho\psi_{2\alpha+\lambda-1}(L,0)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\\
&\times[2\psi_{\beta}(r_2,r_1)+\psi_{\beta}(r_1,0)-\psi_{\beta}(r_1,0)].
\end{aligned}
$$

Taking r_2 tends towards r_1, we get that, the last formula tends to zero. Then NB_ρ is equicontinuous in $C([0,L],E)$.
And, we also have

$$
\begin{aligned}
&\|\psi_{1-\alpha}(r_2,0)({}^c\mathcal{D}^{\beta,\phi}N)y(r_2)-\psi_{1-\alpha}(r_1,0)({}^c\mathcal{D}^{\beta,\phi}N)(y)(r_1)\|\\
&\leq\frac{\psi_{1-\alpha}(r_1,0)}{\Gamma(\alpha)}\int_0^{r_1}\phi'(s)[\psi_{\alpha-1}(r_1,s)-\psi_{\alpha-1}(r_2,s)]\|f(s,y(s),{}^c\mathcal{D}^{\beta,\phi}y(s))\|ds\\
&+\frac{[\psi_{1-\alpha}(r_2,0)-\psi_{1-\alpha}(r_1,0)]}{\Gamma(\alpha)}\int_0^{r_1}\phi'(s)\psi_{\alpha-1}(r_2,s)\|f(s,y(s),{}^c\mathcal{D}^{\beta,\phi}y(s))\|ds\\
&+\frac{\psi_{1-\alpha}(r_2,0)}{\Gamma(\alpha)}\int_{r_1}^{r_2}\phi'(s)\psi_{\alpha-1}(r_2,s)\|f(s,y(s),{}^c\mathcal{D}^{\beta,\phi}y(s))\|ds\\
&\leq\frac{a^*\rho\psi_{1-\alpha}(r_1,0)}{\Gamma(\alpha)}\int_0^{r_1}\phi'(s)[\psi_{\alpha-1}(r_1,s)-\psi_{\alpha-1}(r_2,s)]ds\\
&+\frac{b^*\rho\psi_{1-\alpha}(r_1,0)}{\Gamma(\alpha)}\int_0^{r_1}\phi'(s)[\psi_{\alpha-1}(r_1,s)-\psi_{\alpha-1}(r_2,s)]\psi_{\alpha+\lambda-1}(s,0)ds\\
&+\frac{a^*\rho[\psi_{1-\alpha}(r_2,0)-\psi_{1-\alpha}(r_1,0)]}{\Gamma(\alpha)}\int_0^{r_1}\phi'(s)\psi_{\alpha-1}(r_2,s)ds\\
&+\frac{b^*\rho[\psi_{1-\alpha}(r_2,0)-\psi_{1-\alpha}(r_1,0)]}{\Gamma(\alpha)}\int_0^{r_1}\phi'(s)\psi_{\alpha-1}(r_2,s)\psi_{\alpha+\lambda-1}(s,0)ds\\
&+\frac{a^*\rho\psi_{1-\alpha}(r_2,0)}{\Gamma(\alpha)}\int_{r_1}^{r_2}\phi'(s)\psi_{\alpha-1}(r_2,s)ds\\
&+\frac{b^*\rho\psi_{1-\alpha}(r_2,0)}{\Gamma(\alpha)}\int_{r_1}^{r_2}\phi'(s)\psi_{\alpha-1}(r_2,s)\psi_{\alpha+\lambda-1}(s,0)ds\\
&\leq\frac{a^*\rho\psi_{1-\alpha}(L,0)+b^*\rho\psi_{\lambda}(L,0)}{\Gamma(\alpha+1)}\left(\psi_{\alpha}(r_1,0)-\psi_{\alpha}(r_2,0)+\psi_{\alpha}(r_2,r_1)\right)\\
&+\left(\frac{a^*\rho\psi_{\alpha}(L,0)+b^*\rho\psi_{2\alpha+\lambda-1}(L,0)}{\Gamma(\alpha+1)}\right)\left(\psi_{1-\alpha}(r_2,0)-\psi_{1-\alpha}(r_1,0)\right)\\
&+\frac{a^*\rho\psi_{1-\alpha}(L,0)+b^*\rho\psi_{\lambda}(L,0)}{\Gamma(\alpha+1)}\psi_{\alpha}(r_2,r_1).
\end{aligned}
$$

Taking r_2 tends towards r_1, we get that, the last inequality tends to zero. Then $({}^c\mathcal{D}^{\beta,\phi}N)B_\rho$ is equicontinuous in $C_{\alpha,\phi}([0,L],E)$.

Final step. We verify that N satisfies the assumptions of theorem 2.10.
First, we now show that N is defined from B to B, Indeed, for any $y \in B$, by above condition $(\mathbf{H_2})$ and by according to a little calculation, we have

$$\begin{aligned}
&\|Ny(r)\| + \|\psi_{1-\alpha}(r,0)({}^c\mathcal{D}^{\beta,\phi}N)y(r)\| \\
&\leq \frac{\Gamma(\alpha+\beta)\left[\|b\|\Gamma(\alpha+1)+\alpha\|a\|\right]+\Gamma(\alpha+1)\|a\|\psi_{\beta+\alpha-1}(L,0)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)} \\
&+ \frac{\Gamma(\alpha+\beta+1)\left[b^*\alpha C + a^*\psi_1(L,0)\right]+\Gamma(\alpha+1)\left[a^*\psi_{\beta+\alpha}(L,0)+b^*(\alpha+\beta)C\psi_{\alpha+\beta-1}(L,0)\right]}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}R.
\end{aligned}$$

From $(\mathbf{H_4})$ and $(\mathbf{H_5})$, we obtain

$$\forall y \in B : \|Ny\|_{C^\beta_{\alpha,\phi}} < R.$$

Thus, N is defined from B to B. We put $D = \overline{conv}(NB)$, where $\overline{conv}(NB)$ is the closure of the convex hull of NB. Since $ND \subset NB \subset D$, then D is a subset closed, bounded and convex of B and N remains defined from D to D.
We denote by $\vartheta_{(\alpha,\phi)}$ the Kuratowski measure of non-compactness defined on any bounded subset of $C^\beta_{\alpha,\phi}([0,L],E)$. We can easily show the following inequality

$$\vartheta_{(\alpha,\phi)}(D) \leq \sup_{r\in[0,L]} \vartheta(D(r)) + \sup_{r\in(0,L]} \vartheta(\psi_{1-\alpha}(r,0){}^c\mathcal{D}^{\beta,\phi}D(r)) \leq 2\vartheta_{(\alpha,\phi)}(D), \tag{9}$$

$D(r) = \{y(r) : y \in D\}$ and ${}^c\mathcal{D}^{\beta,\phi}D(r) = \{{}^c\mathcal{D}^{\beta,\phi}y(r) : y \in D\}$.
Next, it remains to prove that N is a Meir-Keeler condensing operator via the measure of non-compactness $\vartheta_{(\alpha,\phi)}$, this is equivalent to demonstrating the following implication

$$\forall \epsilon > 0, \exists \varrho(\epsilon) > 0 : \epsilon \leq \vartheta_{(\alpha,\phi)}(V) < \epsilon + \varrho \Longrightarrow \vartheta_{(\alpha,\phi)}(NV) < \epsilon, \text{ for any } V \subset D. \tag{10}$$

Let ϵ be a strictly positive real, $V \subset D$. From Lemmas 2.6,2.7,2.8, $(\mathbf{H_3})$, the inequality (9) and the previous steps, we have that there exists a sequence $\{\mu_n\}_{n=0}^\infty \subset V$ such that

$$\begin{aligned}
&\vartheta(NV(r)) + \vartheta(\psi_{1-\alpha}(r,0)({}^c\mathcal{D}^{\beta,\phi}N)V(r)) \leq \frac{\epsilon}{2} \\
&+ \frac{2}{\Gamma(\beta)\Gamma(\alpha)}\vartheta\left\{\int_0^r\int_0^s \phi'(s)\phi'(\tau)\psi_{\beta-1}(r,s)\psi_{\alpha-1}(s,\tau)f(\tau,\mu_n(\tau),{}^c\mathcal{D}^{\beta,\phi}\mu_n(\tau))d\tau ds,\ n\in\mathbb{N}\right\} \\
&+ \frac{2\psi_{1-\alpha}(r,0)}{\Gamma(\alpha)}\vartheta\left\{\int_0^r \phi'(s)\psi_{\alpha-1}(r,s)f(s,\mu_n(s),{}^c\mathcal{D}^{\beta,\phi}\mu_n(s))ds,\ n\in\mathbb{N}\right\} \\
&\leq \frac{\epsilon}{2} + \frac{2}{\Gamma(\beta)\Gamma(\alpha)}\int_0^r\int_0^s \phi'(s)\phi'(\tau)\psi_{\beta-1}(r,s)\psi_{\alpha-1}(s,\tau)\vartheta\left\{f(\tau,\mu_n(\tau),{}^c\mathcal{D}^{\beta,\phi}\mu_n(\tau)),\ n\in\mathbb{N}\right\}d\tau ds \\
&+ \frac{2\psi_{1-\alpha}(r,0)}{\Gamma(\alpha)}\int_0^r \phi'(s)\psi_{\alpha-1}(r,s)\vartheta\left\{f(s,\mu_n(s),{}^c\mathcal{D}^{\beta,\phi}\mu_n(s)),\ n\in\mathbb{N}\right\}ds \\
&\leq \frac{\epsilon}{2} + 4\vartheta_{(\alpha,\phi)}(V)\left\{F(L) + \psi_{1-\alpha}(L,0){}^c\mathcal{D}^{\beta,\phi}F(L)\right\}.
\end{aligned}$$

From (9), we know that

$$\vartheta_{(\alpha,\phi)}(NV) \leq \frac{\epsilon}{2} + 4\vartheta_{(\alpha,\phi)}(V)\left\{F(L) + \psi_{1-\alpha}(L,0){}^c\mathcal{D}^{\beta,\phi}F(L)\right\}.$$

If

$$\vartheta_{(\alpha,\phi)}(NV) \leq \frac{\epsilon}{2} + 4\vartheta_{(\alpha,\phi)}(V)\left\{F(L) + \psi_{1-\alpha}(L,0)^{c}\mathcal{D}^{\beta,\phi}F(L)\right\} < \epsilon,$$

this implies that

$$\vartheta_{(\alpha,\phi)}(V) < \frac{1}{8\left(F(L) + \psi_{1-\alpha}(L,0)^{c}\mathcal{D}^{\beta,\phi}F(L)\right)}\epsilon,$$

so that implication (10) is fulfilled, we take

$$\varrho = \frac{1 - 8\left(F(L) + \psi_{1-\alpha}(L,0)^{c}\mathcal{D}^{\beta,\phi}F(L)\right)}{8\left(F(L) + \psi_{1-\alpha}(L,0)^{c}\mathcal{D}^{\beta,\phi}F(L)\right)}\epsilon.$$

So, N is a Meir-Keeler condensing operator via $\vartheta_{(\alpha,\phi)}$, finally all the hypotheses of the theorem 2.10 are fulfilled. Then, the problem (**P**) admits at least one solution. □

4. Example

As an example of the use of the main result , we shall now consider the following fractional differential equation

$$^{rl}\mathcal{D}^{\frac{1}{2},\phi}({}^{c}\mathcal{D}^{\frac{3}{4},\phi})y(r) = \frac{1}{40(1+r^2)}\left(y_n(r) + \frac{\sqrt{\arctan r}}{e^{5\sqrt{2}\arctan r}}({}^{c}\mathcal{D}^{\frac{3}{4},\phi})y_n(r)\right)_{n=1}^{\infty},\ r \in (0,1], \tag{11}$$

$$\mathfrak{I}_{0^+}^{\frac{1}{2}}({}^{c}\mathcal{D}^{\frac{3}{4},\phi})y(0^+) = (1,0,\ldots,0,\ldots) \tag{12}$$

$$y(0) = (1,0,\ldots,0,\ldots). \tag{13}$$

where $\phi(r) = \frac{4}{\pi}\arctan r$. Let

$$E = \{(y_1, y_2, \ldots, y_n, \ldots) : \sup_n |y_n| < \infty\},$$

with the norm $\|y\| = \sup_n |y_n|$, then $(E, \|.\|)$ consists a Banach space, by comparing with the (**P**), we notice that

$$\alpha = \lambda = \frac{1}{2}, \beta = \frac{3}{4} \text{ and } f(r, y(r), {}^{c}\mathcal{D}^{\frac{3}{4},\phi}y(r)) = (f(r, y_1(r), {}^{c}\mathcal{D}^{\frac{3}{4},\phi}y_1(r)), \ldots, f(r, y_n(r), {}^{c}\mathcal{D}^{\frac{3}{4},\phi}y_n(r)), \ldots),$$

where

$$f(r, y_n(r), {}^{c}\mathcal{D}^{\frac{3}{4},\phi}y_n(r)) = \frac{1}{40(1+r^2)}y_n(r) + \frac{\sqrt{\arctan r}}{40(1+r^2)e^{5\sqrt{2}\arctan r}}{}^{c}\mathcal{D}^{\frac{3}{4},\phi}y_n(r), n \in \mathbb{N}^*.$$

Clear that $f : (0,1] \times E \times E \to E$ is continuous and

$$\|f(r, y(r), {}^{c}\mathcal{D}^{\frac{3}{4},\phi}y(r))\| \leq \frac{1}{40(1+r^2)}\|y(r)\| + \frac{\sqrt{\arctan r}}{40(1+r^2)e^{5\sqrt{2}\arctan r}}\|{}^{c}\mathcal{D}^{\frac{3}{4},\phi}y(r)\|.$$

Hence, (**H₁**) and (**H₂**) are satisfied. Next, For any bounded set $B \subset C^{\beta}_{\alpha,\phi}([0,1], E)$, we have

$$f(r, B(r), {}^{c}\mathcal{D}^{\frac{3}{4},\phi}B(r)) = \frac{1}{40(1+r^2)}B(r) + \frac{\sqrt{\arctan r}}{40(1+r^2)e^{5\sqrt{2}\arctan r}}{}^{c}\mathcal{D}^{\frac{3}{4},\phi}B(r).$$

Then

$$\vartheta(f(r, B(r), {}^{c}\mathcal{D}^{\frac{3}{4},\phi}B(r)) \leq \frac{1}{40(1+r^2)}\vartheta(B(r)) + \frac{1}{40(1+r^2)}\vartheta(\sqrt{\arctan r} \times {}^{c}\mathcal{D}^{\frac{3}{4},\phi}B(r)),$$

since $4\left(F(1) + \phi_{1-\alpha}(1,0)^{c}\mathcal{D}^{\frac{3}{4},\phi}F(1)\right) \leq 0.8043 < 1$. So, (**H₃**) holds. Finally, we check (**H₄**), we have $C = 1$, thus

$$\frac{1}{40}\frac{\Gamma(2.25)\,[0.5+1] + \Gamma(1.5)\,[1+1.25]}{\Gamma(1.5)\Gamma(2.25)} \simeq \frac{0.1839}{2} < 1.$$

So, (**H₄**) holds. Therefore, Theorem 2 ensures that problem (11)-(13) has a solution.

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