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Integral Operators on Local Orlicz-Morrey Spaces

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Abstract. We establish a general principle on the boundedness of operators on local Orlicz-Morrey spaces. As applications of this principle, we obtain the boundedness of the Calderón-Zygmund operators, the nonlinear commutators of the Calderón-Zygmund operators, the oscillatory singular integral operators, the singular integral operators with rough kernels and the Marcinkiewicz integrals on the local Orlicz-Morrey spaces.

1. Introduction

This paper aims to study the boundedness of operators on local Orlicz-Morrey spaces.

The Morrey spaces were introduced in [35] for the studies of quasi-linear elliptic partial differential equations. Since then, it had been developed to be one of the major topics in the theory of function spaces and extensions of Morrey spaces had been introduced by a number of researchers in harmonic analysis and theory of functions spaces. One of the extensions is the local Morrey spaces. The boundedness of the singular integral operator, the Hardy-Littlewood maximal function and the fractional integral operator had been extended to local Morrey spaces in [2–8, 43, 44].

Another important generalizations of Morrey spaces is the Orlicz-Morrey spaces [39]. The boundedness of the singular integral operator, the Hardy-Littlewood maximal function and the fractional integral operator had been extended to the Orlicz-Morrey spaces in [10–13, 17, 19, 20, 22, 23, 29, 37–40, 50].

As motivated by the preceding mentioned results on local Morrey spaces and Orlicz-Morrey spaces, we study the boundedness of operators on the local Orlicz-Morrey spaces. Our main result gives a principle for the boundedness of operators on the local Orlicz-Morrey spaces. This principle is obtained by refining the extrapolation theory introduced by Rubio de Francia in [47–49] by using the ideas from [27]. Our main result does not only apply to linear operators, it can also be used to obtain boundedness for nonlinear operators. As applications of our main result, we establish the boundedness of the Calderón-Zygmund operators, the nonlinear commutators of the Calderón-Zygmund operators, the oscillatory singular integral operators, the singular integral operators with rough kernels and the Marcinkiewicz integrals on the local Orlicz-Morrey spaces.

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This paper is organized as follows. The definition of the local Orlicz-Morrey space is given in Section 2. We study a pre-dual of the local Orlicz-Morrey space, namely, the local Orlicz-block spaces in Section 3. The main result of this paper and the boundedness of the Calderón-Zygmund operators, the nonlinear commutators of the Calderón-Zygmund operators, the oscillatory singular integral operators, the singular integral operators with rough kernels and the Marcinkiewicz integral on the local Orlicz-Morrey spaces are presented in Section 4.

2. Local Orlicz-Morrey spaces

This section gives the definition of the Young's function, the Orlicz spaces and the local Orlicz-Morrey spaces.

Let $B(z,r) = \{x \in \mathbb{R}^n : |x - z| < r\}$ denote the open ball with center $z \in \mathbb{R}^n$ and radius $r > 0$. Let $\mathbb{B} = \{B(z,r) : z \in \mathbb{R}^n,\ r > 0\}$.

A function $\Phi : [0,+\infty] \to [0,+\infty]$ is a Young's function if there exists an increasing and left-continuous function ϕ satisfying $\phi(0) = 0$ and that ϕ is neither identically zero nor identically infinite such that

$$\Phi(s) = \int_0^s \phi(u)du, \quad s \geq 0.$$

A Young's function Φ is said to satisfy the $\triangle_2$-condition if there exists a constant $K > 1$ such that

$$\Phi(2t) \leq K\Phi(t), \quad t > 0.$$

We write $\Phi \in \triangle_2$ if it satisfies the $\triangle_2$-condition.

Let Φ be a Young's function associated with ϕ. Let

$$\psi(v) = \inf\{u \geq 0 : \phi(u) \geq v\}, \quad 0 \leq v \leq \infty.$$

The function Ψ defined by

$$\Psi(t) = \int_0^t \psi(v)dv, \quad 0 \leq t \leq \infty$$

is called the conjugate (complementary) function of Φ [1, Chapter 4, Definition 8.11].

We write $\Phi \in \triangledown_2$ if there exists a constant $K > 1$ such that

$$2K\Phi(t) \leq \Phi(Kt).$$

The Orlicz space L_Φ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi(|f|/\lambda)dx \leq 1\right\} < \infty.$$

For any Lebesgue measurable set E with $|E| < \infty$, we have $\|\chi_E\|_{L_\Phi} = \frac{1}{\Phi^{-1}(|E|^{-1})}$ where Φ^{-1} denotes the right-continuous inverse of Φ given by

$$\Phi^{-1}(t) = \sup_{s\geq 0} \Phi(s) \leq t, \quad 0 \leq t < \infty.$$

We have the Hölder inequality for Orlicz spaces,

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq C\|f\|_{L_\Phi}\|g\|_{L_\Psi}$$

for some $C > 0$ where L_Ψ denotes the Orlicz space generated by Ψ, see [42, Section 6.7.14.8].

For any $r > 0$ and Lebesgue measurable function $\Phi : (0, \infty) \to (0, \infty)$, define $\Phi_r(t) = \Phi(t^r)$. Whenever Φ and Φ_r are Young's functions, we find that

$$\begin{aligned} \| |f|^r \|_{L_\Phi} &= \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi(|f|^r/\lambda) dx \le 1 \right\} \\ &= \inf \left\{ \lambda^r > 0 : \int_{\mathbb{R}^n} \Phi((|f|/\lambda)^r) dx \le 1 \right\} = \|f\|^r_{L_{\Phi_r}}. \end{aligned} \tag{1}$$

Definition 2.1. *Let Φ be a Young's function and $v : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. The local Orlicz-Morrey space $LM_{\Phi,v}$ consists of all Lebesgue measurable functions f satisfying*

$$\|f\|_{LM_{\Phi,v}} = \sup_{r>0} \frac{1}{v(r)} \|\chi_{B(0,r)} f\|_{L_\Phi} < \infty.$$

In particular, when $v \equiv 1$, the local Orlicz-Morrey space $LM_{\Phi,v}$ becomes the Orlicz space L_Φ. When $\Phi(t) = t^p$, $p \in [1, \infty)$, the local Orlicz-Morrey space reduces to the local Morrey space.

A similar function space, called as the central Morrey-Orlicz spaces, are introduced and studied in [33, 34].

Let $q > 0$. We find that

$$\begin{aligned} \| |f|^q \|_{LM_{\Phi,v}} &= \sup_{r>0} \frac{1}{v(r)} \|\chi_{B(0,r)} |f|^q \|_{L_\Phi} = \sup_{r>0} \frac{1}{v(r)} \|f\|^q_{L_{\Phi_q}} \\ &= \left(\sup_{r>0} \frac{1}{v(r)^{1/q}} \|f\|_{L_{\Phi_q}} \right)^q = \|f\|^q_{LM_{\Phi_q, v^{1/q}}}. \end{aligned} \tag{2}$$

The following results give conditions that guarantee $\chi_B \in LM_{\Phi,v}$, $B \in \mathbb{B}$. Consequently, it also guarantees that $\chi_E \in LM_{\Phi,v}$ whenever E is a Lebesgue measurable set with $|E| < \infty$.

Proposition 2.2. *Let Φ be a Young's function and $v : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. If $\Phi \in \nabla_2$ and there is a constant $C > 0$ such that v satisfies*

$$C \le v(r), \quad \forall r \ge 1, \tag{3}$$

$$\frac{1}{\Phi^{-1}(|B(0, r)|^{-1})} \le C v(r), \quad \forall r \le 1, \tag{4}$$

then for any $B \in \mathbb{B}$, $\chi_B \in LM_{\Phi,v}$.

Proof: Let $s > 0$. When $r \ge 1$, according to (3), we have

$$\frac{1}{v(r)} \|\chi_{B(0,s)} \chi_{B(0,r)}\|_{L_\Phi} \le C \|\chi_{B(0,s)}\|_{L_\Phi} \tag{5}$$

for some $C > 0$. When $r < 1$, (4) yields

$$\begin{aligned} \frac{1}{v(r)} \|\chi_{B(0,s)} \chi_{B(0,r)}\|_{L_\Phi} &\le \frac{1}{v(r)} \|\chi_{B(0,r)}\|_{L_\Phi} \\ &= \frac{1}{v(r)} \frac{1}{\Phi^{-1}(|B(0, r)|^{-1})} \le C. \end{aligned} \tag{6}$$

Consequently, (5) and (6) assure that

$$\|\chi_{B(0,s)}\|_{LM_{\Phi,v}} = \sup_{r>0} \frac{1}{v(r)} \|\chi_{B(0,s)} \chi_{B(0,r)}\|_{L_\Phi} < C + C \|\chi_{B(0,s)}\|_{L_\Phi}.$$

Thus, $\chi_{B(0,s)} \in LM_{\Phi,v}$. For any $B(x, r) \in \mathbb{B}$, we have a $s > 0$ such that $B(x, r) \subseteq B(0, s)$. Therefore, we have $\chi_{B(x,r)} \in LM_{\Phi,v}$. ■

3. Local Orlicz-block spaces

This section studies the local Orlicz-block space, that is a pre-dual of the local Orlicz-Morrey space. The local Orlicz-block space is used to obtain the main result of this paper in the next section.

The results presented in this section are the analogues of the results for the block type spaces obtained in [25]. For completeness, we also give the details for the result presented in this section.

We begin with the definition of the local Orlicz-block spaces.

Definition 3.1. *Let Φ be a Young's function and $v : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. For any Lebesgue measurable function b , we write $b \in \mathfrak{b}_{\Phi,v}$ if $\operatorname{supp} b \subseteq B(0,r)$ for some $r > 0$ and*

$$\|b\|_{L_\Phi} \leq \frac{1}{v(r)}. \tag{7}$$

The local Orlicz-block space $\mathfrak{LB}_{\Phi,v}$ is defined as

$$\mathfrak{LB}_{\Phi,v} = \Big\{ \sum_{k=1}^{\infty} \lambda_k b_k \ : \ \sum_{k=1}^{\infty} |\lambda_k| < \infty \text{ and } b_k \in \mathfrak{b}_{\Phi,v} \Big\}. \tag{8}$$

The local Orlicz-block space $\mathfrak{LB}_{\Phi,v}$ is endowed with the norm

$$\|f\|_{\mathfrak{LB}_{\Phi,v}} = \inf \Big\{ \sum_{k=1}^{\infty} |\lambda_k| \text{ such that } f = \sum_{k=1}^{\infty} \lambda_k b_k \text{ a.e.} \Big\}. \tag{9}$$

For any $B(x,r) \in \mathbb{B}$, we have $\chi_{B(x,r)} \in \mathfrak{LB}_{\Phi,v}$ with $\|\chi_{B(x,r)}\|_{\mathfrak{LB}_{\Phi,v}} \leq \|\chi_B\|_{L_\Phi} v(|x|+r)$.

We now show that the dual space of the local Orlicz-block space is the local Orlicz-Morrey space.

Theorem 3.2. *Let Φ be a Young's function and $v : (0,\infty) \to (0,\infty)$ be Lebesgue measurable function. We have*

$$(\mathfrak{LB}_{\Psi,v})^* = LM_{\Phi,v}$$

where Ψ is the conjugate function of Φ and $(\mathfrak{LB}_{\Psi,v})^*$ denotes the dual space of $\mathfrak{LB}_{\Psi,v}$.

Proof: Let $f \in LM_{\Phi,v}$ and $b \in \mathfrak{b}_{\Psi,v}$ with $\operatorname{supp} b \in B = B(0,r)$, $r > 0$. The Hölder inequality yields

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)b(x)|dx &= \int_{B(0,r)} |f(x)b(x)|dx \\ &\leq C\|\chi_{B(0,r)} f\|_{L_\Phi} \|\chi_{B(0,r)} b\|_{L_\Psi} \\ &\leq C\frac{1}{v(r)} \|\chi_{B(0,r)} f\|_{L_\Phi}\end{aligned}$$

for some $C > 0$. Thus,

$$\int_{\mathbb{R}^n} |f(x)b(x)|dx \leq C\frac{1}{v(r)} \|\chi_{B(0,r)} f\|_{L_\Phi} \leq C\|f\|_{LM_{\Phi,v}}.$$

For any $g = \sum_{k\in\mathbb{N}} \lambda_k b_k \in \mathfrak{LB}_{\Psi,v}$, we have

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \sum_{k=1}^{\infty} |\lambda_k| \int_{\mathbb{R}^n} |f(x)b_k(x)|dx \leq C\|g\|_{\mathfrak{LB}_{\Psi,v}} \|f\|_{LM_{\Phi,v}} \tag{10}$$

for some $C > 0$. Thus, $LM_{\Phi,v} \hookrightarrow (\mathfrak{LB}_{\Psi,v})^*$.

We prove the reverse embedding. For any $r > 0$ and $L \in (\mathfrak{LB}_{\Psi,v})^*$, define $X = \{g\chi_{B(0,r)} : g \in L_\Phi\}$. It is easy to see that X is a subspace of L_Φ. Define the linear functional $l : X \to \mathbb{C}$ by $l(h) = L(\chi_{B(0,r)} g)$ where $h = \chi_{B(0,r)} g \in X$ and $g \in L_\Phi$.

For any $r > 0$,

$$G = \frac{1}{\|g\chi_{B(0,r)}\|_{L_\Psi} v(r)} g\chi_{B(0,r)}$$

belongs to $\mathfrak{b}_{\Psi,v}$. According to (9), we get $\|G\|_{\mathfrak{LB}_{\Psi,v}} \leq 1$. That is,

$$\|g\chi_{B(0,r)}\|_{\mathfrak{LB}_{\Psi,v}} \leq \|g\chi_{B(0,r)}\|_{L_\Psi} v(r). \tag{11}$$

Since $L \in (\mathfrak{LB}_{\Psi,v})^*$, (11) guarantees that

$$|l(h)| = |L(g\chi_{B(0,r)})| \leq C\|g\chi_{B(0,r)}\|_{\mathfrak{LB}_{\Psi,v}} \leq K\|g\chi_{B(0,r)}\|_{L_\Psi} = K\|h\|_{L_\Psi}$$

for some $K > 0$ independent of h. Therefore, l is a bounded functional on X. The Hahn-Banach theorem assures that l can be extended to be a member of $(L_\Psi)^*$. As $(L_\Psi)^* = L_\Phi$, there exists a $f_r \in L_\Phi$ such that

$$l(g) = \int_{\mathbb{R}^n} f_r(x)g(x)dx, \quad \forall g \in L_\Psi$$

and without loss of generality, we can assume that $\operatorname{supp} f_r \subseteq B(0,r)$.

Let $r > s > 0$. For any Lebesgue measurable set E with $E \subset B(0,s)$, we find that $\int_E f_r(x)dx = l(\chi_E) = \int_E f_s(x)dx$. That is, $f_r = f_s$ almost everywhere on $B(0,r) \cap B(0,s)$. It guarantees that there exists an unique Lebesgue measurable function f such that $f(x) = f_r(x)$ on $B(0,r)$ for all r.

Next, we show that $f \in LM_{\Phi,v}$. For any $B(0,r) \in \mathbb{B}$ and Lebesgue measurable function h with $\|h\|_{L_\Psi} = 1$, $H = \frac{\chi_{B(0,r)}h}{v(r)}$ belongs to $\mathfrak{b}_{\Psi,v}$. In addition, we have $\|H\|_{\mathfrak{LB}_{\Psi,v}} \leq 1$. That is, $\|\chi_{B(x,r)}h\|_{\mathfrak{LB}_{\Psi,v}} \leq v(r)$.

As $H \in \mathfrak{b}_{\Psi,v}$, we get

$$\begin{aligned}
\frac{1}{v(r)}\|\chi_{B(0,r)}f\|_{L_\Phi} &= \frac{1}{v(r)} \sup_{\|h\|_{L_\Psi}=1} \left| \int_{B(0,r)} f(y)h(y)dy \right| \\
&\leq \sup_{\|h\|_{L_\Psi}=1} \left| \int_{B(0,r)} f_r(x) \frac{\chi_{B(0,r)}(x)h(x)}{v(r)} dx \right| \\
&\leq \|L\|_{(\mathfrak{LB}_{\Psi,v})^*} \sup_{\|h\|_{L_\Psi}=1} \left\| \frac{h\chi_{B(0,r)}}{v(r)} \right\|_{\mathfrak{LB}_{\Psi,v}} \leq \|L\|_{(\mathfrak{LB}_{\Psi,v})^*}.
\end{aligned}$$

By taking supremum over $B(0,r) \in \mathbb{B}$ on both sides of the above inequalities, we find that $f \in LM_{\Phi,v}$ and $\|f\|_{LM_{\Phi,v}} \leq \|L\|_{(\mathfrak{LB}_{\Psi,v})^*}$. Moreover, the functional $L_f(g) = \int_{\mathbb{R}^n} f(x)g(x)dx$ and L are identical on the set $\mathfrak{b}_{\Psi,v}$. In view of Definition 3.1, the set of finite linear combinations of functions in $\mathfrak{b}_{\Psi,v}$ is dense in $\mathfrak{LB}_{\Psi,v}$, therefore $L_f = L$ and $(\mathfrak{LB}_{\Psi,v})^* \hookrightarrow LM_{\Phi,v}$. ∎

For any locally integrable function f, the Hardy-Littlewood maximal operator $\mathrm{M}f$ is defined as

$$\mathrm{M}f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|dy$$

where the supremum is taken over all $B \in \mathbb{B}$ containing x.

We obtain some preliminary results for the boundedness of the Hardy-Littlewood maximal function on $\mathfrak{LB}_{\Phi,v}$.

Proposition 3.3. *Let Φ be a Young's function, $v : (0,\infty) \to (0,\infty)$ be Lebesgue measurable function and $f \in \mathfrak{LB}_{\Phi,v}$. If $g \in \mathcal{M}$ satisfies $|g| \leq |f|$, then $g \in \mathfrak{LB}_{\Phi,v}$.*

Proof: As $f \in \mathfrak{LB}_{\Phi,v}$, (8) and (9) assure that for any $\epsilon > 0$, there exists a family of $\{b_i\}_{i=1}^{\infty} \subset \mathfrak{b}_{\Phi,v}$ and a family of scalars $\{\lambda_i\}_{i=1}^{\infty}$ such that $f = \sum_{i=1}^{\infty} \lambda_i b_i$ and $\sum_{i=1}^{\infty} |\lambda_i| \leq (1+\epsilon)\|f\|_{\mathfrak{LB}_{\Psi,v}}$. We find that $g = \sum_{i=1}^{\infty} \lambda_i c_i$ where

$$c_i(x) = \begin{cases} \dfrac{g(x)}{f(x)} b_i(x), & f(x) \neq 0, \\ 0, & f(x) = 0. \end{cases}$$

As $|g| \leq |f|$, $\{c_i\}_{i=1}^{\infty} \subset \mathfrak{b}_{\Phi,v}$ and $g \in \mathfrak{LB}_{\Phi,v}$. Moreover, since ϵ is arbitrary, we have $\|g\|_{\mathfrak{LB}_{\Phi,v}} \leq \|f\|_{\mathfrak{LB}_{\Phi,v}}$. ■

Theorem 3.4. *Let Φ be a Young's function and $v : (0,\infty) \to (0,\infty)$ be Lebesgue measurable function. If v fulfills (3) and*

$$\frac{1}{\Psi^{-1}(|B(0,r)|^{-1})} \leq Cv(r), \quad \forall r \leq 1, \tag{12}$$

where Ψ is the conjugate function of Φ, then $\mathfrak{LB}_{\Phi,v} \subset L^1_{\text{loc}}$ and $\mathfrak{LB}_{\Phi,v}$ is a Banach space.

Proof: Proposition 2.2 asserts that $\chi_B \in LM_{\Psi,v}$, $\forall B \in \mathbb{B}$. According to Theorem 3.2, we have $\chi_B \in (\mathfrak{LB}_{\Phi,v})^*$. For any $f \in \mathfrak{LB}_{\Phi,v}$, (10) gives

$$\int_B |f(x)|dx \leq C\|\chi_B\|_{LM_{\Psi,v}} \|f\|_{\mathfrak{LB}_{\Phi,v}}. \tag{13}$$

As a result of this inequality, we find that $\mathfrak{LB}_{\Phi,v} \hookrightarrow L^1_{\text{loc}}$.

We are going to show that $\mathfrak{LB}_{\Phi,v}$ is a Banach space. Let $\{f_i\}_{i=1}^{\infty} \subset \mathfrak{LB}_{\Phi,v}$ satisfy $\sum_{i=1}^{\infty} \|f_i\|_{\mathfrak{LB}_{\Phi,v}} < \infty$.

For any $B \in \mathbb{B}$, (13) yields $\int_B \sum_{i=1}^{\infty} |f_i(x)|dx \leq C\|\chi_B\|_{LM_{\Psi,v}} (\sum_{i=1}^{\infty} \|f_i\|_{\mathfrak{LB}_{\Phi,v}})$. Consequently, $f = \sum_{i=1}^{\infty} f_i$ is a well defined Lebesgue measurable function and $f \in L^1_{\text{loc}}$.

For any $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that,

$$\sum_{i=N+1}^{\infty} \|f_i\|_{\mathfrak{LB}_{\Phi,v}} < \epsilon. \tag{14}$$

The definition of $\mathfrak{LB}_{\Phi,v}$ asserts that for any $\epsilon > 0$, $f_i = \sum_{k=1}^{\infty} \lambda_{k,i} b_{k,i}$ where $\{b_{k,i}\}_{i,k\in\mathbb{N}} \subset \mathfrak{b}_{\Phi,v}$ and $\sum_{k=1}^{\infty} |\lambda_{k,i}| \leq (1+\epsilon)\|f_i\|_{\mathfrak{LB}_{\Phi,v}}$.

For any $1 \leq i \leq n$, there exists a $N_i \in \mathbb{N}$ such that

$$\left\| f_i - \sum_{k=1}^{N_i} \lambda_{k,i} b_{k,i} \right\|_{\mathfrak{LB}_{\Phi,v}} \leq \sum_{k=N_i+1}^{\infty} |\lambda_{k,i}| < 2^{-i}\epsilon. \tag{15}$$

For any $B \in \mathbb{B}$,

$$\begin{aligned} &\int_B \left| f(x) - \sum_{i=1}^{N} \sum_{k=1}^{N_i} \lambda_{k,i} b_{k,i}(x) \right| dx \\ &\leq \int_B \left| f(x) - \sum_{i=1}^{N} f_i(x) \right| dx + \int_B \left| \sum_{i=1}^{N} f_i(x) - \sum_{i=1}^{N} \sum_{k=1}^{N_i} \lambda_{k,i} b_{k,i}(x) \right| dx \\ &\leq \int_B \sum_{i=N+1}^{\infty} |f_i(x)|dx + \sum_{i=1}^{N} \int_B \left| f_i(x) - \sum_{k=1}^{N_i} \lambda_{k,i} b_{k,i}(x) \right| dx. \end{aligned}$$

According to (13), (14) and (15), we get

$$\int_B \left| f(x) - \sum_{i=1}^{N} \sum_{k=1}^{N_i} \lambda_{k,i} b_{k,i}(x) \right| dx$$

$$\leq C \|\chi_B\|_{LM_{\Psi,v}} \left(\sum_{i=N+1}^{\infty} \|f_i\|_{\mathfrak{LB}_{\Phi,v}} + \sum_{i=1}^{N} \left\| f_i - \sum_{k=1}^{N_i} \lambda_{k,i} b_{k,i} \right\|_{\mathfrak{LB}_{\Phi,v}} \right)$$

$$\leq C \|\chi_B\|_{LM_{\Psi,v}} \left(\epsilon + \sum_{i=1}^{N} 2^{-i} \epsilon \right) < 2C \|\chi_B\|_{LM_{\Psi,v}} \epsilon.$$

Consequently, we find that $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \lambda_{k,i} b_{k,i}$ converges to f in $L^1_{\rm loc}$ and $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \lambda_{k,i} b_{k,i}$ converges to f locally in measure. A subsequence of $\{\sum_{i=1}^{N} \sum_{k=1}^{M} \lambda_{k,i} b_{k,i}\}_{N,M}$ converges to f a.e. We find that $\{\lambda_{k,i}\}_{i,k\in\mathbb{N}}$, satisfies $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |\lambda_{k,i}| \leq (1+\epsilon) \sum_{i=1}^{\infty} \|f_i\|_{\mathfrak{LB}_{\Phi,v}} < \infty$. That is, $\sum_{i=1}^{\infty} f_i$ converges to f in $\mathfrak{LB}_{\Phi,v}$. Since $\epsilon > 0$ is arbitrary, we have $\left\| \sum_{i=1}^{\infty} f_i \right\|_{\mathfrak{LB}_{\Phi,v}} \leq \sum_{i=1}^{\infty} \|f_i\|_{\mathfrak{LB}_{\Phi,v}}$. Therefore, $\mathfrak{LB}_{\Phi,v}$ is a Banach space. ∎

We now establish the boundedness of the Hardy-Littlewood maximal operator on $\mathfrak{LB}_{\Phi,v}$.

Theorem 3.5. *Let Φ be a Young's function and $v : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If $\Phi \in \triangle_2 \cap \nabla_2$, v satisfies (3), (12) and for any $r > 0$*

$$v(2r) \leq Cv(r), \tag{16}$$

$$\sum_{j=0}^{\infty} \frac{\Psi^{-1}(|B(0,2^{j+1}r)|^{-1})}{\Psi^{-1}(|B(0,r)|^{-1})} v(2^{j+1}r) \leq Cv(r) \tag{17}$$

for some $C > 0$ where Ψ is the conjugate function of Φ, then the Hardy-Littlewood maximal operator M *is bounded on $\mathfrak{LB}_{\Phi,v}$.*

Proof: Since v satisfies (3) and (12), Theorem 3.4 ensures that $\mathfrak{LB}_{\Phi,v} \subset L^1_{loc}$, therefore the Hardy-Littlewood maximal operator is well defined on $\mathfrak{LB}_{\Phi,v}$.

Let $b \in \mathfrak{b}_{\Phi,v}$ with support $B(0,r)$, $r > 0$. For any $k \in \mathbb{N}$, write $B_k = B(0,2^k r)$. Define $m_k = \chi_{B_{k+1}\backslash B_k} \mathrm{M}(b)$, $k \in \mathbb{N}\backslash\{0\}$ and $m_0 = \chi_{B(0,2r)} \mathrm{M}(b)$. We have $\operatorname{supp} m_0 \subseteq B(0,2r)$, $\operatorname{supp} m_k \subseteq B_{k+1}\backslash B_k$ and $\mathrm{M}(b) = \sum_{k=0}^{\infty} m_k$.

Since $\Phi \in \triangle_2 \cap \nabla_2$, the Hardy-Littlewood maximal operator M is bounded on the Orlicz space L_Φ. Consequently, $\|m_0\|_{L_\Phi} \leq \|\mathrm{M}\, b\|_{L_\Phi} \leq C\|b\|_{L_\Phi}$ for some $C > 0$. According to (7) and (16), we find that $\|m_0\|_{L_\Phi} \leq C\|b\|_{L_\Phi} \leq C\frac{1}{v(r)} \leq C\frac{1}{v(2r)}$ for some $C > 0$ independent of $r > 0$ and b. Consequently, $m_0/C \in \mathfrak{b}_{\Phi,v}$.

The Hölder inequality yields

$$m_k = \chi_{B_{k+1}\backslash B_k} |\mathrm{M}(b)| \leq \frac{\chi_{B_{k+1}\backslash B_k}}{2^{kn} r^n} \int_{B(0,r)} |b(y)| dy$$

$$\leq C\chi_{B_{k+1}\backslash B_k} \frac{1}{2^{kn} r^n} \|b\|_{L_\Phi} \|\chi_{B(0,r)}\|_{L_\Psi}$$

for some $C > 0$ independent of k.

Consequently, in view of [23, (2.1)], we obtain

$$\|m_k\|_{L_\Phi} \leq C \frac{\|\chi_{B_{k+1}\backslash B_k}\|_{L_\Phi}}{2^{kn} r^n} \|b\|_{L_\Phi} \|\chi_{B(0,r)}\|_{L_\Psi}$$

$$\leq C \frac{\|\chi_{B(0,r)}\|_{L_\Psi}}{\|\chi_{B_{k+1}}\|_{L_\Psi}} \frac{v(2^{k+1}r)}{v(r)} \frac{1}{v(2^{k+1}r)}.$$

Define $m_k = \sigma_k b_k$ where

$$\sigma_k = \frac{\|\chi_{B(0,r)}\|_{L_\Psi}}{\|\chi_{B_{k+1}}\|_{L_\Psi}} \frac{v(2^{k+1}r)}{v(r)}.$$

We find that $b_k/C \in \mathfrak{b}_{\Phi,v}$ for some $C > 0$ independent of k. Since v satisfies (17), we find that

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(0,r)}\|_{L_\Psi}}{\|\chi_{B(0,2^{j+1}r)}\|_{L_\Psi}} v(2^{j+1}r) \le Cv(r).$$

That is, $\sum_{k=0}^{\infty} \sigma_k < C$ for some $C > 0$. In view of Definition 3.1, $\mathrm{M}(b) \in \mathfrak{LB}_{\Psi,v}$. In addition, there exists a constant $C_0 > 0$ so that for any $b \in \mathfrak{b}_{\Phi,v}$, $\|\mathrm{M}(b)\|_{\mathfrak{LB}_{\Phi,v}} < C_0$.

Finally, let $f \in \mathfrak{LB}_{\Phi,v}$. There is a $\{c_k\}_{k=1}^{\infty} \subset \mathfrak{b}^{u}_{\Phi,v}$ and a sequence $\Lambda = \{\lambda_k\}_{k=1}^{\infty} \in l^1$ such that $f = \sum_{k=1}^{\infty} \lambda_k c_k$ with $\|\Lambda\|_{l^1} \le 2\|f\|_{\mathfrak{LB}_{\Phi,v}}$. As $\mathfrak{LB}_{\Phi,v}$ is a Banach space, we have

$$\begin{aligned}\left\|\sum_{k=1}^{\infty} \lambda_k \mathrm{M}(c_k)\right\|_{\mathfrak{LB}_{\Phi,v}} &\le \sum_{k=1}^{\infty} |\lambda_k| \|\mathrm{M}(c_k)\|_{\mathfrak{LB}_{\Phi,v}} \\ &\le C_0 \sum_{k=1}^{\infty} |\lambda_k| \le 2C_0 \|f\|_{\mathfrak{LB}_{\Phi,v}}.\end{aligned}$$

As $\mathrm{M} f \le \sum_{k=1}^{\infty} |\lambda_k| \mathrm{M}(c_k)$, Proposition 3.3 asserts that $\mathrm{M} f \in \mathfrak{LB}_{\Phi,v}$ and $\|\mathrm{M} f\|_{\mathfrak{LB}_{\Phi,v}} \le C\|f\|_{\mathfrak{LB}_{\Phi,v}}$ for some $C > 0$. ∎

4. Main results

The main result is obtained in this section. It relies on the refined extrapolation theory for the local Orlicz-Morrey space. It is obtained by refining the extrapolation theory from Rubio de Francia [47–49] by using the ideas given in [26, 27] for studying the extrapolation theory of the Orlicz-slice spaces and Morrey-Banach spaces.

We begin with the definition of the well-known Muckenhoupt weight functions.

Definition 4.1. *For $1 < p < \infty$, a locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ is said to be an A_p weight if*

$$\sup_{B \in \mathbb{B}} \left(\frac{1}{|B|} \int_B \omega(x) dx\right) \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{p'}{p}} dx\right)^{\frac{p}{p'}} < \infty$$

where $p' = \frac{p}{p-1}$. A locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ is said to be an A_1 weight if

$$\frac{1}{|B|} \int_B \omega(y) dy \le C\omega(x), \quad a.e.\, x \in B \tag{18}$$

for some constant $C > 0$ independent of the balls B. Define $A_\infty = \cup_{p \ge 1} A_p$.

It is a well known fact that for any $p \in (1, \infty)$, the Hardy-Littlewood maximal operator is bounded on the weighted Lebesgue space $L^p(\omega)$ if and only if $\omega \in A_p$.

For any locally integrable function h, define

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{\mathrm{M}^k h(x)}{2^k \|\mathrm{M}^k\|_{\mathfrak{LB}_{\Psi,v}}}.$$

If $\mathrm{M} : \mathfrak{LB}_{\Psi,v} \to \mathfrak{LB}_{\Psi,v}$ is bounded, then $\mathcal{R}$ is well defined. Consequently, $\mathcal{R}$ satisfies

$$h(x) \le \mathcal{R}h(x), \tag{19}$$

$$\|\mathcal{R}h\|_{\mathfrak{LB}_{\Psi,v}} \le 2\|h\|_{\mathfrak{LB}_{\Psi,v}}, \tag{20}$$

$$[\mathcal{R}h]_{A_1} \le 2\|\mathrm{M}\|_{\mathfrak{LB}_{\Psi,v}}. \tag{21}$$

We see that (19) follows from the definition of $\mathcal{R}$, (20) follows from the boundedness of the Hardy-Littlewood maximal function on $\mathfrak{LB}_{\Psi,v}$ and (21) is a consequence of the boundedness of Hardy-Littlewood maximal function on $\mathfrak{LB}_{\Psi,v}$ and (18).

The following theorem is the main result of this paper.

Theorem 4.2. *Let $q > 1$, Φ be a Young's function and $v : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. If $\Phi_{1/q}$ is a Young's function, $\Phi_{1/q} \in \triangle_2 \cap \nabla_2$, v satisfies (3), (4), (16) and*

$$\sum_{j=0}^{\infty} \frac{(\Phi_{1/q})^{-1}(|B(0, 2^{j+1}r)|^{-1})}{(\Phi_{1/q})^{-1}(|B(0, r)|^{-1})} v^q(2^{j+1}r) \le C v^q(r) \tag{22}$$

for some $C > 0$. Suppose that for every

$$\omega \in \left\{\mathcal{R}h : h \in \mathfrak{LB}_{\Psi^q, v^q}, \|h\|_{\mathfrak{LB}_{\Psi^q, v^q}} \le 1\right\} \tag{23}$$

where Ψ^q is the conjugate function of $\Phi_{1/q}$, the mapping $T : L^q(\omega) \to L^q(\omega)$ satisfies

$$\int_{\mathbb{R}^n} |T(f)(x)|^q \omega(x) dx \le C \int_{\mathbb{R}^n} |f(x)|^q \omega(x) dx \tag{24}$$

for some $C > 0$ independent of f. Then, we have

$$\|Tf\|_{LM_{\Phi,v}} \le C\|f\|_{LM_{\Phi,v}}, \quad \forall f \in LM_{\Phi,v}. \tag{25}$$

Proof: The conjugate function of Ψ^q is $\Phi_{1/q}$. In addition, we have $\Phi_{1/q} \in \triangle_2 \cap \nabla_2$ if and only if $\Psi^q \in \triangle_2 \cap \nabla_2$. Therefore, Theorem 3.5 guarantees that the Hardy-Littlewood maximal operator M is bounded on $\mathfrak{LB}_{\Psi^q, v^q}$ and Theorem 3.4 assures that $\mathfrak{LB}_{\Psi^q, v^q} \subset L^1_{loc}$. Consequently, $\mathcal{R}$ is well defined on $\mathfrak{LB}_{\Psi^q, v^q}$ and satisfies (19)–(21).

Let $f \in LM_{\Phi,v}$. For any $h \in \mathfrak{LB}_{\Psi^q, v^q}$ with $\|h\|_{\mathfrak{LB}_{\Psi^q, v^q}} \le 1$, (2) and (20) give

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)|^q \mathcal{R}h(x) dx &\le \||f|^q\|_{LM_{\Phi_{1/q}, v^q}} \|\mathcal{R}h\|_{\mathfrak{LB}_{\Psi^q, v^q}} \\ &\le C\|f\|^q_{LM_{\Phi,v}} \|h\|_{\mathfrak{LB}_{\Psi^q, v^q}} < \infty.\end{aligned}$$

The above inequality shows that

$$LM_{\Phi,v} \hookrightarrow \bigcap_{h \in \mathfrak{LB}_{\Psi^q, v^q}, \|h\|_{\mathfrak{LB}_{\Psi^q, v^q}} \le 1} L^q(\mathcal{R}h). \tag{26}$$

For any

$$\omega \in \left\{\mathcal{R}h : h \in \mathfrak{LB}_{\Psi^q, v^q}, \|h\|_{\mathfrak{LB}_{\Psi^q, v^q}} \le 1\right\},$$

(26) guarantees that $LM_{\Phi,v} \hookrightarrow L^q(\omega)$. Therefore, (24) is valid for $f \in LM_{\Phi,v}$

Let $f \in LM_{\Phi,v}$. For any $h \in \mathfrak{LB}_{\Psi^q, v^q}$, (19) and the boundedness of T on $L^q(\mathcal{R}h)$ yield

$$\int_{\mathbb{R}^n} |Tf(x)|^q |h(x)| dx \le \int_{\mathbb{R}^n} |Tf(x)|^q \mathcal{R}h(x) dx \le \int_{\mathbb{R}^n} |f(x)|^q \mathcal{R}h(x) dx.$$

Consequently, Theorem 3.2 and (21) give

$$\begin{aligned}\int_{\mathbb{R}^n} |Tf(x)|^q |h(x)| dx &\le C\||f|^q\|_{LM_{\Phi_{1/q}, v^q}} \|\mathcal{R}h\|_{\mathfrak{LB}_{\Psi^q, v^q}} \\ &\le C\||f|^q\|_{LM_{\Phi_{1/q}, v^q}} \|h\|_{\mathfrak{LB}_{\Psi^q, v^q}}.\end{aligned}$$

By taking supremum over $h \in \mathfrak{LB}_{\Psi^q,v^q}$ with $\|h\|_{\mathfrak{LB}_{\Psi^q,v^q}} \leq 1$ on both sides of the above inequality, (2) and Theorem 3.2 guarantee that

$$\begin{aligned}\|Tf\|_{LM_{\Phi,v}} &= \||Tf|^q\|^{1/q}_{LM_{\Phi_{1/q},v^q}} \\ &= \left(\sup\left\{\int_{\mathbb{R}^n}|Tf(x)|^q|h(x)|dx : \|h\|_{\mathfrak{LB}_{\Psi^q,v^q}} \leq 1\right\}\right)^{1/q} \\ &\leq C\||f|^q\|^{1/q}_{LM_{\Phi_{1/q},v^q}} = C\|f\|_{LM_{\Phi,v}}\end{aligned}$$

which is (25). ■

Notice that the classical extrapolation theory only yields the result for some subset of $LM_{\Phi,v}$ only, such as the class of bounded functions with compact support of the class of Schwartz functions. The preceding theorem gives result on the entire local Orlicz-Morrey space. Thanks to the embedding (26), we can obtain the result for the entire local Orlicz-Morrey space.

We first apply Theorem 4.2 to the Calderón-Zygmund operators. Let $\mathcal{D}'$ be the space of distributions on $\mathbb{R}^n$. A linear operator $T : C_0^\infty \to \mathcal{D}'$ is a Calderón-Zygmund operator, if T is bounded on L^2 and there exists a kernel $C, \delta > 0$ and $K(x, y) : \mathbb{R}^n \times \mathbb{R}^n\backslash\{(x, x) : x \in \mathbb{R}^n\} \to \mathbb{R}$ such that for any $f \in C_0^\infty$ and $x \notin \operatorname{supp} f$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy,$$

where K satisfies

$$\begin{aligned}|K(x, y)| &\leq C|x - y|^{-n}, \quad x \neq y, \\ |K(x, y) - K(z, y)| &\leq C|x - z|^\delta|x - y|^{-n-\delta}, \quad |x - z| \leq |x - y|/2, \\ |K(x, y) - K(x, z)| &\leq C|y - z|^\delta|x - y|^{-n-\delta}, \quad |y - z| \leq |x - y|/2.\end{aligned}$$

One of the celebrated result for the Calderón-Zygmund operators is the following weighted norm inequality.

Theorem 4.3. *Let $p \in (1, \infty)$ and $\omega \in A_p$. If T is a Calderón-Zygmund operator, then T is bounded on $L^p(\omega)$.*

As (21) asserts that $\{\mathcal{R}h : h \in \mathfrak{LB}_{\Psi^q,v^q}, \|h\|_{\mathfrak{LB}_{\Psi^q,v^q}} \leq 1\} \subseteq A_1$, Theorems 4.2 and 4.3 yield the boundedness of Calderón-Zygmund operators on $LM_{\Phi,v}$.

Theorem 4.4. *Let $q > 1$, T be a Calderón-Zygmund operator, Φ be a Young's function and $v : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. If $\Phi_{1/q}$ is a Young's function, $\Phi_{1/q} \in \Delta_2 \cap \nabla_2$, v satisfies (3), (4), (16) and (22), then T is bounded on $LM_{\Phi,v}$.*

The boundedness of the Calderón-Zygmund operators on the central Morrey-Orlicz spaces and the Orlicz-Morrey spaces were given in [34] and [39, 40], respectively.

We consider a number of important operators related with the Calderón-Zygmund operators. We begin with the nonlinear commutators generated by the Calderón-Zygmund operators. Let T be a Calderón-Zygmund operator. We define

$$Nf = T(f \log |f|) - Tf \log |Tf|. \tag{27}$$

The operator N is introduced by Rochberg and Weiss in [46]. The nonlinear commutators generated by the Calderón-Zygmund operators have applications on the estimates of the Jacobian [18] and the weak minima of variational integrals [30].

The main result in [46] is the boundedness of N on L^p, $p \in (1, \infty)$. The weighted norm inequalities for N is obtained in [41, Theorem 1.3].

Theorem 4.5. *Let $p \in (1, \infty)$ and $\omega \in A_p$. There is a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} |Nf(x)|^p \omega(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx.$$

The reader is referred to [41, Theorem 1.3] for the proof of the preceding theorem. The reader is referred to [28] for the boundedness of N on Morrey type spaces.

Theorem 4.6. *Let $q > 1$, Φ be a Young's function and $v : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. If $\Phi_{1/q}$ is a Young's function, $\Phi_{1/q} \in \triangle_2 \cap \nabla_2$, v satisfies (3), (4), (16) and (22), then N is bounded on $LM_{\Phi,v}$.*

The above result shows that Theorem 4.2 is also applied to nonlinear operators. Next, we study the oscillatory singular integral operators. Let $K(x, y)$ satisfy

$$|K(x, y)| \leq C|x - y|^{-n}, \quad x \neq y, \tag{28}$$

$$|\nabla_x K(x, y)| + |\nabla_y K(x, y)| \leq C|x - y|^{-n-1}, \quad x \neq y. \tag{29}$$

Let $P(x, y)$ be a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$. The oscillatory singular integral operator $T_{K,P}$ associated with K and P is defined as

$$T_{K,P}f(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} K(x, y)f(y)dy.$$

The studies of the oscillatory singular integral operators were started from [45]. The weighted norm inequalities for the oscillatory singular integral operators were obtained in [32].

Theorem 4.7. *Let $p \in (1, \infty)$, $P(x, y)$ be real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$ and $K(x, y)$ satisfies (28)–(29). If the Calderón-Zygmund operator*

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy$$

is bounded on L^2, then for any $\omega \in A_p$, $T_{K,P}$ is bounded on $L^p(\omega)$.

In view of the preceding theorem and Theorem 4.2, we obtain the boundedness of the oscillatory singular integral operators on $LM_{\Phi,v}$.

Theorem 4.8. *Let $q > 1$, $P(x, y)$ be real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$ and $K(x, y)$ satisfies (28)–(29), Φ be a Young's function and $v : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. If T is bounded on L^2, $\Phi_{1/q}$ is a Young's function, $\Phi_{1/q} \in \triangle_2 \cap \nabla_2$, v satisfies (3), (4), (16) and (22), then $T_{K,P}$ is bounded on $LM_{\Phi,v}$.*

We now turn to the singular integrals with rough kernels. The studies of the singular integrals with rough kernels can be chased back to Calderón and Zygmund [9] where they introduced the celebrated method of rotation to obtain the boundedness of the singular integrals with odd kernels.

Let $n \geq 2$ and Ω be a Lebesgue measurable function. We say that Ω is a homogeneous function of degree zero if $\Omega(\lambda x) = \Omega(x)$, for any $\lambda > 0$ and $x \in \mathbb{R}^n$. We say that Ω satisfies the vanishing moment condition if

$$\int_{\mathbb{S}^{n-1}} \Omega(z)dz = 0. \tag{30}$$

Let $\Omega \in L^1(\mathbb{S}^{n-1})$ be a homogeneous function satisfying the vanishing moment condition. The singular integral with rough kernel Ω is defined as

$$T_\Omega f(x) = p.v. \int \frac{\Omega(x - y)}{|x - y|^n} f(y)dy.$$

We have the following weighted norm inequality for the singular integrals with rough kernels, see [15, 53].

Theorem 4.9. *Let $\theta \in (1,\infty)$ and $\Omega \in L^\theta(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero that satisfies the vanishing moment condition. If $p \in (\theta',\infty)$ and $\omega \in A_{p/\theta'}$, then T_Ω is bounded on $L^p(\omega)$.*

The preceding result and Theorem 4.2 yield the boundedness of T_Ω on $LM_{\Phi,v}$.

Theorem 4.10. *Let $\theta \in (1,\infty)$ and $\Omega \in L^\theta(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero that satisfies (30). Let $q > \theta'$, Φ be a Young's function and $v : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If $\Phi_{1/q}$ is a Young's function, $\Phi_{1/q} \in \triangle_2 \cap \nabla_2$, v satisfies (3), (4), (16) and (22), then T_Ω is bounded on $LM_{\Phi,v}$.*

The preceding theorem gives a complementary result for the boundedness of singular integral operators with rough kernels on Orlicz-Morrey spaces. For the boundedness of the singular integral operators with rough kernels on Morrey-Banach spaces, in particular, the Orlicz-Morrey spaces, the reader is referred to [24].

Finally, we study the boundedness of the Marcinkiewicz integral on $LM_{\Phi,v}$. Let $\Omega \in L^1(\mathbb{S}^{n-1})$. The Marcinkiewicz integral $\mathcal{M}_\Omega$ is defined as

$$\mathcal{M}_\Omega f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}.$$

The Marcinkiewicz integral was introduced by Stein in [51] as a generalization of Littlewood-Paley function. The weighted norm inequalities for the Marcinkiewicz integral are given in the following theorem.

Theorem 4.11. *Let $\theta \in (1,\infty)$ and $\Omega \in L^\theta(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero satisfying (30). If $p \in (\theta',\infty)$ and $\omega \in A_{p/\theta'}$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\omega)$.*

For the proof of the above theorem, the reader is referred to [14, 16].

By applying Theorem 4.2 to the above weighted norm inequality for $\mathcal{M}_\Omega$, we obtain the boundedness of $\mathcal{M}_\Omega$ on $LM_{\Phi,v}$ in the following theorem.

Theorem 4.12. *Let $\theta \in (1,\infty)$ and $\Omega \in L^\theta(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero satisfying (30). Let $q > \theta'$, Φ be a Young's function and $v : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If $\Phi_{1/q}$ is a Young's function, $\Phi_{1/q} \in \triangle_2 \cap \nabla_2$, v satisfies (3), (4), (16) and (22), then $\mathcal{M}_\Omega$ is bounded on $LM_{\Phi,v}$.*

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