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Further Study on Induced L-Convex Spaces

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Abstract. In this paper, the relationships of induced L-convex spaces with L-hull operators, product spaces, and quotient spaces are discussed. It is shown that the quotient L-convex structure of induced L-convex structure is exactly the induced L-convex structure by quotient convex structure. Moreover, sub-S_1, sub-S_2, S_2 and S_3 separation axioms are introduced in L-convex spaces and induced L-convex spaces. Some properties and relationships of them are investigated.

1. Introduction

Axiomatic convexity theory (also called abstract convexity theory in [27]) plays an important role in mathematics. For different mathematical objects, there are so many collections of sets that can form convex structures, such as convexities in lattices [26], convexities in graphs [24], convexities in real vector spaces [25]. Also, convex structures appeared naturally in topology, especially in the theory of supercompact spaces [9].

With the development of fuzzy mathematics, axiomatic convex structures have been endowed with fuzzy set theory. Weiss [30] considered a convex fuzzy set in a vector space over real or complex number. Maruyama [8] and Rosa [18] independently introduced the concept of fuzzy convex structures, which is called L-convex structures nowadays. As a topology-like structure, convex structures possess some similar characters of topologies and hence are also discussed in the fuzzy case. In the framework of L-convex spaces, Pang and Xiu [14] firstly proposed the axiomatic approach to bases and subbases in the framework of L-convex spaces. Furthermore, Pang et al. [15, 19] provided several characterizations of L-convex spaces. Later, Based on L-concave prefilters, Xiu [32] introduced L-convergence structures in the framework of L-concave spaces.

In a different way, Shi and Xiu [22] provided a new fuzzification method to convex structures from a logical viewpoint. In this way, the new resulting concept is called M-fuzzifying convex structures. In this framework, Shi et al. [21, 36] introduced the concepts of M-fuzzifying restricted hull operators and M-fuzzifying interval operators, and established their relationships with M-fuzzifying convex structures. In [35], Xiu and Pang investigated the mutual relationships among M-fuzzifying closure systems, M-fuzzifying convex structures and M-fuzzifying cotopologies from a categorical aspect. Later, Pang and Xiu

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[13] proposed the notion of lattice-valued interval operators and investigated its categorical relationships with L-fuzzifying convex structures and L-convex structures, respectively. Relevant results of M-fuzzifying convex spaces can be seen in [7, 34, 37, 38].

Furthermore, Shi and Xiu [23] proposed the notion of broader convex spaces, which is called (L, M)-fuzzy convex spaces. It contains L-convex spaces and M-fuzzifying convex spaces as special cases. In this framework, Xiu [33] equipped L-CP and L-CC mappings with some degrees. Li [6] and Wu [31] respectively provided a categorical approach to (L, M)-fuzzy convex structures. Recently, Pang [10] proposed the concept of (L, M)-fuzzy hull operators, (L, M)-fuzzy restricted hull operators and (L, M)-fuzzy interval operators, and established the relationships of them with (L, M)-fuzzy convex structures from categorical aspect.

As we all know, separation axioms and induced L-convex spaces are important concepts in (fuzzy) convex spaces. In the framework of M-fuzzifying convex structures, Liang et al. [3, 5] firstly proposed a degree approach to S_0, S_1, S_2, S_3 and S_4 separation axioms to characterize the degree to which an M-fuzzifying convex space fulfills the separation properties. Later, Liang et al. [4] introduced the degrees of S_0, S_1 and S_2 separation axioms, L-CP and L-CC mappings in (L, M)-fuzzy convex spaces. In the framework of L-convex structures, Pang and Shi [11] proposed induced L-convex spaces, and introduced several subcategories of L-convex spaces. Zhou and Shi [39] introduced S_{-1}, sub-S_0, S_0, S_1 and S_2 separation axioms in L-convex spaces. However, sub-S_1, sub-S_2 and S_3 separation axioms have not been discussed in L-convex spaces. Motivated by this, one purpose of this paper is to introduce sub-S_1, sub-S_2, S_3 and a new kind of S_2 separation axioms in L-convex spaces and induced L-convex spaces. Moreover, we will focus on the relationships of induced L-convex spaces with L-hull operators, product spaces, and quotient spaces.

This article is arranged as follows. In Section 2, we review some preliminaries that are needed in the subsequent sections. In Section 3, we consider the relationships of induced L-convex spaces with L-hull operators, product spaces and quotient spaces. In Section 4, we introduce sub-S_1 and sub-S_2 separation axioms in L-convex spaces. Then we study the relationships of them in L-convex spaces and induced L-convex spaces. In Section 5, we define S_2 and S_3 separation axioms in L-convex spaces, and discuss the relationships of them and S_1 separation axiom in L-convex spaces and induced L-convex spaces.

2. Preliminaries

Throughout this paper, L denote completely distributive De Morgan algebra, i.e., a completely distributive lattice with an order-reversing involution $'$. The smallest element and the largest element in L are denoted by $\perp_L$ and $\top_L$, respectively. For $a, b \in L$, we say that a is wedge below b in L [16], in symbols $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geqslant b$ implies $d \geqslant a$ for some $d \in D$. The set $\{a \in L \mid a \prec b\}$ denoted by $\beta(b)$ is called the greatest minimal family of b in the sense of [29]. A complete lattice L is completely distributive if and only if $b = \bigvee\{a \in L \mid a \prec b\}$ for each $b \in L$. An element a in L is called co-prime if $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$. The set of non-zero co-prime elements in L is denoted by $J(L)$.

For a nonempty set X, 2^X denotes the powerset of X. For any nonempty subset $A \in 2^X$, let χ_A denote the characteristic function of A. L^X is the set of all L-subsets on X. L^X is also a completely distributive De Morgan algebra when it inherits the structure of the lattice L in a natural way, by defining $\vee$, $\wedge$, $\leqslant$ and $'$ pointwisely. It is easy to see that the set $J(L^X)$ of non-zero coprimes in L^X is $\{x_\lambda \mid \lambda \in J(L)\}$. The smallest element and the largest element in L^X are denoted by $\perp_{L^X}$ and $\top_{L^X}$, respectively. For each $a \in L$, $\underline{a}$ denotes the constant mapping $X \to L$, $x \mapsto a$, which is called constant L-subset.

For each $A \in L^X$, the support set of A is provided by $\mathrm{Supp}A = \{x \in X \mid A(x) \neq \perp_L\}$. We say $\{A_j\}_{j\in J}$ is a directed subset of L^X, in symbols $\{A_j\}_{j\in J} \overset{dir}{\subseteq} L^X$, if for each $A_{j_1}, A_{j_2} \in \{A_j\}_{j\in J}$, there exists $A_{j_s} \in \{A_j\}_{j\in J}$ such that $A_{j_1}, A_{j_2} \leqslant A_{j_s}$. For a directed subset $D \subseteq L$, we usually use $\bigvee^{\uparrow} D$ to denote its supremum.

Let $f : X \to Y$ be a mapping. The forward L-power operator $f_L^{\to} : L^X \to L^Y$ and the backward L-powerset operator $f_L^{\leftarrow} : L^Y \to L^X$ induced by f [17] are defined by $f_L^{\to}(A)(y) = \bigvee_{f(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $f_L^{\leftarrow}(B) = B \circ f$ for $B \in L^Y$, respectively.

For $a \in L$ and $U \in L^X$, we use the following notations [20]:

(1) $U_{[a]} = \{x \in X \mid U(x) \geqslant a\}$;

(2) $U_{(a)} = \{x \in X \mid a \in \beta(U(x))\}$;
(3) $U^{(a)} = \{x \in X \mid U(x) \nleqslant a\}$.

Lemma 2.1. ([20]) *Let $U \in L^X$ and $a \in L$, then $U^{(a)} = \bigcup\limits_{b \nleqslant a} U_{[b]}$.*

Definition 2.2. ([8]) A subset $\mathscr{C}$ of L^X is called an L-convex structure on X if it satisfies:
(LC1) $\perp_{L^X}, \top_{L^X} \in \mathscr{C}$;
(LC2) $\{U_i\}_{i\in I} \subseteq \mathscr{C}$ implies $\bigwedge_{i\in I} U_i \in \mathscr{C}$;
(LC3) If $\{U_i\}_{i\in I} \subseteq \mathscr{C}$ is totally ordered, then $\bigvee_{i\in I} U_i \in \mathscr{C}$.
The pair $(X, \mathscr{C})$ is called an L-convex space. The members of $\mathscr{C}$ are called L-convex sets.

Definition 2.3. ([39]) Let $(X, \mathscr{C})$ be an L-convex space. For any $H \in L^X$, H is called an L-biconvex set if H and H' are L-convex sets.

Definition 2.4. ([11]) An L-convex structure $\mathscr{C}$ on X is called stratified if it satisfies:

$$\forall a \in L, \underline{a} \in \mathscr{C}.$$

For a stratified L-convex structure on X, the pair $(X, \mathscr{C})$ is called a stratified L-convex space.

Proposition 2.5. ([12]) *Let $(X, \mathscr{C})$ be an L-convex space and define $co^{\mathscr{C}} : L^X \longrightarrow L^X$ by*

$$\forall A \in L^X, co^{\mathscr{C}}(A) = \wedge\{B \in L^X \mid A \leqslant B \in \mathscr{C}\}.$$

Then $co^{\mathscr{C}}$ is an L-hull operator on X.

Definition 2.6. ([11]) Let $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ be a mapping between two L-convex spaces.
(1) f is called L-convex preserving (L-CP, for short) mapping provided that for any $V \in \mathscr{D}$ implies $f_L^{\leftarrow}(V) \in \mathscr{C}$;
(2) f is called L-convex to convex (L-CC, for short) mapping provided that for any $A \in \mathscr{C}$ implies $f_L^{\rightarrow}(A) \in \mathscr{D}$;
(3) f is called an L-isomorphism provided that f is an L-CP and L-CC bijection.

Proposition 2.7. ([39]) *If $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ is an L-CP mapping between two L-convex spaces and H is an L-biconvex set of Y, then $f_L^{\leftarrow}(H)$ is an L-biconvex set of X.*

Definition 2.8. ([23]) Let $(X, \mathscr{C})$ be an L-convex space and $\emptyset \neq Y \subseteq X$. Then $\mathscr{C}|_Y = \{U|_Y \mid U \in \mathscr{C}\}$ is an L-convex structure on Y. $(Y, \mathscr{C}|_Y)$ is called the L-convex subspace of $(X, \mathscr{C})$.

Definition 2.9. ([14]) Let $(X, \mathscr{C})$ be an L-convex space and $\mathscr{B} \subseteq \mathscr{C}$. Then $\mathscr{B}$ is called a base of $(X, \mathscr{C})$ (or $\mathscr{C}$) provided that for each $C \in \mathscr{C}$, there is a directed family $\mathscr{B}_C \subseteq \mathscr{B}$ such that $C = \bigvee^{\uparrow} \mathscr{B}_C$.

Definition 2.10. ([14]) Let $(X, \mathscr{C})$ be an L-convex space and $\mathscr{U} \subseteq \mathscr{C}$. Then $\mathscr{U}$ is called a subbase of $(X, \mathscr{C})$ (or $\mathscr{C}$) provided that $\mathscr{B}_{\mathscr{U}}$ is a base of $(X, \mathscr{C})$, where $\mathscr{B}_{\mathscr{U}} = \{\bigwedge_{i\in I} A_i \mid \{A_i\}_{i\in I} \subseteq \mathscr{U} \text{ and } I \neq \emptyset\}$.

Definition 2.11. ([14]) Let $\{(X_i, \mathscr{C}_i)\}_{i\in I}$ be a family of L-convex spaces, and X be the product of $\{X_i\}_{i\in I}$. For each $i \in I$, $P_i : X \to X_i$ denote the projection. X can be equipped with the L-convex structure $\prod\limits_{i\in I} \mathscr{C}_i$ generated by the family $\{(P_i)_L^{\leftarrow}(V) \mid V \in \mathscr{C}_i, i \in I\}$ as a subbase. Then $\prod\limits_{i\in I} \mathscr{C}_i$ is called the product L-convex structure for X and $(X, \prod\limits_{i\in I} \mathscr{C}_i)$ is called the product L-convex space.

Theorem 2.12. ([14]) *Let $\{(X_i, \mathscr{C}_i)\}_{i\in I}$ be a family of L-convex spaces and $(X, \prod\limits_{i\in I} \mathscr{C}_i)$ be the product space of $\{(X_i, \mathscr{C}_i)\}_{i\in I}$. Then $\prod\limits_{i\in I} \mathscr{C}_i$ is the coarsest L-convex structure on X which guarantees that all projection mappings are L-CP mappings.*

Proposition 2.13. ([14]) *Let $\{(X_i, \mathscr{C}_i)\}_{i\in I}$ be a family of stratified L-convex spaces, $X = \prod_{i\in I} X_i$, and let $\{P_i : X \to X_i\}_{i\in I}$ be the family of projection mappings. Then $P_i : (X, \prod_{j\in I} \mathscr{C}_j) \to (X_i, \mathscr{C}_i)$ is an L-CC mapping for each $i \in I$.*

Proposition 2.14. ([39]) *Let $(X, \mathscr{C})$ be an L-convex space and $(Y, \mathscr{C}|_Y)$ a subspace of $(X, \mathscr{C})$. If H is an L-biconvex set of X, then $H|_Y$ is an L-biconvex set of Y.*

Definition 2.15. ([2, 27]) Let $(X, \mathcal{C})$ be a convex space.

(1) $(X, \mathcal{C})$ is said to be S_1 separated if all singletons in X are convex;

(2) $(X, \mathcal{C})$ is said to be S_2 separated if for all $x, y \in X$ with $x \neq y$, then there exists biconvex set H of X with $x \in H, y \notin H$;

(3) $(X, \mathcal{C})$ is said to be S_3 separated if for any $C \in \mathcal{C}$ and $x \notin C$, there exists biconvex set H of X such that $C \subseteq H, x \notin H$.

Definition 2.16. ([39]) An L-convex space $(X, \mathscr{C})$ is said to be S_1, if for any $x_\lambda, y_\mu \in J(L^X)$ with $x_\lambda \nleqslant y_\mu$, there exists $U \in \mathscr{C}$ such that $x_\lambda \nleqslant U$, $y_\mu \leqslant U$.

Proposition 2.17. ([39]) *Let $(X, \mathscr{C})$ be an L-convex space, then $(X, \mathscr{C})$ is S_1 if and only if for any $y_\mu \in J(L^X)$, $co^{\mathscr{C}}(y_\mu) = y_\mu$.*

Definition 2.18. ([28, 29]) For any $A \in L^X$, A is said to be pseudocrisp if there exists $a \in L$ with $a \neq \perp_L$ such that $A(x) \geqslant a$ if and only if $A(x) \neq \perp_L$, $\forall x \in X$.

Remark 2.19. ([28, 29]) The characteristic function χ_A is a pseudocrisp set.

Proposition 2.20. ([11]) *Let $(X, \mathcal{C})$ be a convex space and define $\omega(\mathcal{C})$ as follows:*

$$\omega(\mathcal{C}) = \{U \in L^X \mid \forall a \in L, U_{[a]} \in \mathcal{C}\}.$$

Then $\omega(\mathcal{C})$ is a stratified L-convex structure on X and $(X, \omega(\mathcal{C}))$ is a stratified L-convex space. $(X, \omega(\mathcal{C}))$ is called an induced L-convex space by $(X, \mathcal{C})$.

3. Induced L-convex spaces, L-hull operators, product spaces and quotient spaces

In this section, we discuss the relationships of L-hull operator, product space, quotient space with induced L-convex space. It is shown that the quotient L-convex structure of induced L-convex structure is exactly the induced L-convex structure by quotient convex structure.

Firstly, we present characterization of L-hull operators in induced L-convex spaces.

Theorem 3.1. *Let $(X, \mathcal{C})$ be a convex space, and $(X, \omega(\mathcal{C}))$ be the induced L-convex space by $(X, \mathcal{C})$. Then*

$$\forall A \in L^X, co^{\omega(\mathcal{C})}(A) = \bigvee_{\lambda \in L} (\underline{\lambda} \wedge \chi_{co(A_{(\lambda)})}) = \bigvee_{\lambda \in L} (\underline{\lambda} \wedge \chi_{co(A_{[\lambda]})}).$$

Proof. Suppose that $B = \bigvee_{\lambda \in L} (\underline{\lambda} \wedge \chi_{co(A_{(\lambda)})})$, $C = \bigvee_{\lambda \in L} (\underline{\lambda} \wedge \chi_{co(A_{[\lambda]})})$. Since $(co^{\omega(\mathcal{C})}(A))_{[\lambda]} \in \mathcal{C}$ and $A_{(\lambda)} \subseteq A_{[\lambda]} \subseteq (co^{\omega(\mathcal{C})}(A))_{[\lambda]}$, we have $co(A_{(\lambda)}) \subseteq co(A_{[\lambda]}) \subseteq (co^{\omega(\mathcal{C})}(A))_{[\lambda]}$. Moreover, since $A = \bigvee_{\lambda \in L} (\underline{\lambda} \wedge \chi_{A_{(\lambda)}})$, it follows that

$$\begin{aligned} A &\leqslant \bigvee_{\lambda \in L} (\underline{\lambda} \wedge \chi_{co(A_{(\lambda)})}) \\ &\leqslant \bigvee_{\lambda \in L} (\underline{\lambda} \wedge \chi_{co(A_{[\lambda]})}) \\ &\leqslant \bigvee_{\lambda \in L} (\underline{\lambda} \wedge \chi_{(co^{\omega(\mathcal{C})}(A))_{[\lambda]}}) \\ &= co^{\omega(\mathcal{C})}(A). \end{aligned}$$

This implies

$$A \leqslant B \leqslant C \leqslant co^{\omega(\mathcal{C})}(A). \tag{1}$$

Now we prove that for any $\lambda \in L$, $B_{[\lambda]} = \bigcap_{a \prec \lambda} co(A_{(a)})$. On one hand, take any $x \in \bigcap_{a \prec \lambda} co(A_{(a)})$. This implies for any $a \prec \lambda$, $x \in co(A_{(a)})$. Then

$$\begin{aligned} B(x) &= \bigvee_{a \in L} (\underline{a} \wedge \chi_{co(A_{(a)})})(x) \\ &\geqslant \bigvee_{a \prec \lambda} (\underline{a} \wedge \chi_{co(A_{(a)})})(x) \\ &= \bigvee_{a \prec \lambda} a = \lambda. \end{aligned}$$

This means $x \in B_{[\lambda]}$. By the arbitrariness of x, we obtain $\bigcap_{a \prec \lambda} co(A_{(a)}) \subseteq B_{[\lambda]}$. On the other hand, take any $x \in B_{[\lambda]}$. Then

$$\lambda \leqslant B(x) = \bigvee_{\lambda \in L} (\underline{\lambda} \wedge \chi_{co(A_{(\lambda)})})(x).$$

For any $a \prec \lambda$, there exists $\mu \in L$ such that $a \prec (\underline{\mu} \wedge \chi_{co(A_{(\mu)})})(x)$. This implies $a \leqslant \chi_{co(A_{(\mu)})}(x)$ and $a \prec \mu$. Since $x \in co(A_{(\mu)}) \subseteq co(A_{(a)})$, it follows that $x \in \bigcap_{a \prec \lambda} co(A_{(a)})$. Hence, $B_{[\lambda]} \subseteq \bigcap_{a \prec \lambda} co(A_{(a)})$. Therefore, for any $\lambda \in L$, $B_{[\lambda]} = \bigcap_{a \prec \lambda} co(A_{(a)})$. This means $B_{[\lambda]} \in \mathcal{C}$. Thus $B \in \omega(\mathcal{C})$. By (1), we know that

$$co^{\omega(\mathcal{C})}(A) \leqslant co^{\omega(\mathcal{C})}(B) = B \leqslant C \leqslant co^{\omega(\mathcal{C})}(A).$$

This shows that $co^{\omega(\mathcal{C})}(A) = B = C$, as desired. □

Next, we discuss the relationship between product space and induced L-convex space.

Theorem 3.2. *Let $\{(X_t, \mathcal{C}_t)\}_{t \in T}$ be a collection convex spaces. Then $\prod_{t \in T} \omega(\mathcal{C}_t) \subseteq \omega(\prod_{t \in T} \mathcal{C}_t)$.*

Proof. First we check $P_t^{-1}(\omega(\mathcal{C}_t)) \subseteq \omega(P_t^{-1}(\mathcal{C}_t))$. For any $A \in P_t^{-1}(\omega(\mathcal{C}_t))$, there exists $B \in \omega(\mathcal{C}_t)$ such that $A = P_t^{\leftarrow}(B)$. For any $\lambda \in L$,

$$\begin{aligned} A_{[\lambda]} &= \{x \in X \mid A(x) \geqslant \lambda\} \\ &= \{x \in X \mid P_t^{\leftarrow}(B)(x) \geqslant \lambda\} \\ &= \{x \in X \mid B(P_t(x)) \geqslant \lambda\} \\ &= \{x \in X \mid P_t(x) \in B_{[\lambda]}\} \\ &= P_t^{-1}(B_{[\lambda]}). \end{aligned}$$

Since $B_{[\lambda]} \in \mathcal{C}_t$, this implies $A_{[\lambda]} = P_t^{-1}(B_{[\lambda]}) \in P_t^{-1}(\mathcal{C}_t)$. Thus $A \in \omega(P_t^{-1}(\mathcal{C}_t))$. Then it follows that $P_t^{-1}(\omega(\mathcal{C}_t)) \subseteq \omega(P_t^{-1}(\mathcal{C}_t))$.

Next we check $\prod_{t \in T} \omega(\mathcal{C}_t) \subseteq \omega(\prod_{t \in T} \mathcal{C}_t)$. Since the product convex structure $\prod_{t \in T} \mathcal{C}_t$ is generated by the subbase $\{P_t^{-1}(C_t) \mid C_t \in \mathcal{C}_t, t \in T\}$, we can obtain that for any $t \in T$, $P_t^{-1}(\mathcal{C}_t) \subseteq \prod_{t \in T} \mathcal{C}_t$. This implies $\omega(P_t^{-1}(\mathcal{C}_t)) \subseteq \omega(\prod_{t \in T} \mathcal{C}_t)$. Hence

$$P_t^{-1}(\omega(\mathcal{C}_t)) \subseteq \omega(P_t^{-1}(\mathcal{C}_t)) \subseteq \omega(\prod_{t \in T} \mathcal{C}_t).$$

Then it follows that $\prod_{t \in T} \omega(\mathcal{C}_t) \subseteq \omega(\prod_{t \in T} \mathcal{C}_t)$, as $\bigcup_{t \in T} P_t^{-1}(\omega(\mathcal{C}_t))$ is a subbase of $\prod_{t \in T} \omega(\mathcal{C}_t)$. □

In order to discuss the relationship between quotient space and induced L-convex space, let us recall the concepts of quotient convex structure and quotient L-convex structure.

Definition 3.3. ([27]) Let $(X, \mathcal{C})$ be a convex space and $f : X \to Y$ be a surjective mapping. Define a convex structure $\mathcal{C}/f = \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{C}\}$. Then $(Y, \mathcal{C}/f)$ is called a quotient space of X and the convex structure $\mathcal{C}/f$ a quotient convex structure.

Definition 3.4. ([23]) Let $(X, \mathscr{C})$ be an L-convex space and $f : X \to Y$ be a surjective mapping. Define an L-convex structure $\mathscr{C}/f = \{B \in L^Y \mid f_L^{\leftarrow}(B) \in \mathscr{C}\}$. Then $(Y, \mathscr{C}/f)$ is an L-convex space and we call $\mathscr{C}/f$ a quotient L-convex structure of X with respect to f and $\mathscr{C}$.

Now, we discuss the relationship between quotient space and induced L-convex space.

Theorem 3.5. *Let (X, C) be a convex space and $f : X \to Y$ a surjective mapping. Then $\omega(C/f) = \omega(C)/f$.*

Proof. First we prove $\omega(C/f) \subseteq \omega(C)/f$. Take any $B \in \omega(C/f)$. For any $a \in L$, we have $B_{[a]} \in C/f$, then $f^{-1}(B_{[a]}) \in C$. Since

$$\begin{aligned} f_L^{\leftarrow}(B)_{[a]} &= \{x \in X \mid f_L^{\leftarrow}(B)(x) \geqslant a\} \\ &= \{x \in X \mid B(f(x)) \geqslant a\} \\ &= \{x \in X \mid f(x) \in B_{[a]}\} \\ &= f^{-1}(B_{[a]}), \end{aligned}$$

it follows that $f_L^{\leftarrow}(B)_{[a]} \in C$. This implies $f_L^{\leftarrow}(B) \in \omega(C)$. This means $B \in \omega(C)/f$. Hence $\omega(C/f) \subseteq \omega(C)/f$.

Next, we prove $\omega(C)/f \subseteq \omega(C/f)$. Take any $B \in \omega(C)/f$. Since $f_L^{\leftarrow}(B) \in \omega(C)$, we can obtain that for any $a \in L$, $f_L^{\leftarrow}(B)_{[a]} \in C$. Since $f^{-1}(B_{[a]}) = f_L^{\leftarrow}(B)_{[a]} \in C$, we have $B_{[a]} \in C/f$, then $B \in \omega(C/f)$. This implies $\omega(C)/f \subseteq \omega(C/f)$. Therefore $\omega(C/f) = \omega(C)/f$, as desired. □

Let $C(X)$ be the set of all convex structures on X, $\Delta(X)$ be the set of all L-convex structures on X. The previous theorem shows that the following diagram commutes.

$$\begin{array}{ccc} C(X) & \xrightarrow{1/f} & C(Y) \\ \omega\downarrow & & \omega\downarrow \\ \Delta(X) & \xrightarrow{1/f} & \Delta(Y) \end{array}$$

4. Sub-S_1 and sub-S_2 separation axioms in L-convex spaces and induced L-convex spaces

In this section, we introduce sub-S_1 and sub-S_2 separation axioms in L-convex spaces and induced L-convex spaces. Moreover, we discuss the relationships among them and S_1 separation axioms.

Definition 4.1. Let $(X, \mathscr{C})$ be an L-convex space.

(1) $(X, \mathscr{C})$ is said to be sub-S_1 separated if for any $x, y \in X$ with $x \neq y$, there exist $\lambda \in J(L)$ and $U \in \mathscr{C}$ such that $x_\lambda \nleqslant U$, $y_\lambda \leqslant U$;

(2) $(X, \mathscr{C})$ is said to be sub-S_2 separated if for any $x, y \in X$ with $x \neq y$, there exist $\lambda \in J(L)$ and L-biconvex set H such that $x_\lambda \leqslant H$, $y_\lambda \nleqslant H$.

Next we discuss the hereditary property of sub-S_1 and sub-S_2 separation axioms in L-convex spaces.

Proposition 4.2. *If $(X, \mathscr{C})$ is a sub-S_1(resp. sub-S_2) L-convex space and $(Y, \mathscr{C}|_Y)$ is its subspace, then $(Y, \mathscr{C}|_Y)$ is sub-S_1(resp. sub-S_2).*

Proof. (1) Let $(X, \mathscr{C})$ be a sub-S_1 L-convex space. Take any $x, y \in Y$ with $x \neq y$. Since $Y \subseteq X$, there exist $\lambda \in J(L)$ and $U \in \mathscr{C}$ such that $x_\lambda \nleqslant U$, $y_\lambda \leqslant U$. By Definition 2.8, we know that $U|_Y \in \mathscr{C}|_Y$.
Since

$$\lambda \nleqslant U(x) = U|_Y(x),$$

and

$$\lambda \leqslant U(y) = U|_Y(y),$$

we have $x_\lambda \nleqslant U|_Y, y_\lambda \leqslant U|_Y$. Therefore $(Y, \mathscr{C}|_Y)$ is sub-S_1.

(2) Let $(X, \mathscr{C})$ be a sub-S_2 L-convex space. Take any $x, y \in Y$ with $x \neq y$. Since $Y \subseteq X$, there exist $\lambda \in J(L)$ and L-biconvex set H such that $x_\lambda \leqslant H$, $y_\lambda \nleqslant H$. By proposition 2.14, we know that $H|_Y$ is an L-biconvex set of Y.
Since

$$\lambda \leqslant H(x) = H|_Y(x),$$

and

$$\lambda \nleqslant H(y) = H|_Y(y),$$

we have $x_\lambda \leqslant H|_Y$, $y_\lambda \nleqslant H|_Y$. Therefore $(Y, \mathscr{C}|_Y)$ is sub-S_2. □

Now let us discuss the relationships among L-CP mapping, L-CC mapping and separation axioms in L-convex spaces.

Proposition 4.3. *Let $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ be a bijective and L-CP mapping between two L-convex spaces. If $(Y, \mathscr{D})$ is sub-S_1(resp. sub-S_2), then $(X, \mathscr{C})$ is sub-S_1(resp. sub-S_2).*

Proof. (1) Let $(Y, \mathscr{D})$ be a sub-S_1 L-convex space. Since f is a bijective mapping, for any $x, y \in X$ with $x \neq y$, we have $f(x), f(y) \in Y$ with $f(x) \neq f(y)$. Then there exist $\lambda \in J(L)$ and $U \in \mathscr{D}$ such that $f_L^{\rightarrow}(x_\lambda) = f(x)_\lambda \nleqslant U$, $f_L^{\rightarrow}(y_\lambda) = f(y)_\lambda \leqslant U$. Hence we have $x_\lambda \nleqslant f_L^{\leftarrow}(U)$ and $y_\lambda \leqslant f_L^{\leftarrow}(U)$. Since $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ is an L-CP mapping, we know that $f_L^{\leftarrow}(U)$ is an L-convex set of X. Therefore, $(X, \mathscr{C})$ is sub-S_1.

(2) Let $(Y, \mathscr{D})$ be a sub-S_2 L-convex space. Since f is a bijective mapping, for any $x, y \in X$ with $x \neq y$, we have $f(x), f(y) \in Y$ with $f(x) \neq f(y)$. Then there exist $\lambda \in J(L)$ and L-biconvex set H such that $f_L^{\rightarrow}(x_\lambda) = f(x)_\lambda \leqslant H$, $f_L^{\rightarrow}(y_\lambda) = f(y)_\lambda \nleqslant H$. Hence we have $x_\lambda \leqslant f_L^{\leftarrow}(H)$ and $y_\lambda \nleqslant f_L^{\leftarrow}(H)$. Since $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ is an L-CP mapping, we know that $f_L^{\leftarrow}(H)$ is an L-biconvex set of X. Thus $(X, \mathscr{C})$ is sub-S_2. □

Proposition 4.4. *Let $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ be a bijective and L-CC mapping between two L-convex space. If $(X, \mathscr{C})$ is sub-S_1(resp. sub-S_2), then $(Y, \mathscr{D})$ is sub-S_1(resp. sub-S_2).*

Proof. (1) Let $(X, \mathscr{C})$ be a sub-S_1 L-convex space. Since f is a bijective mapping, for any $x, y \in Y$ with $x \neq y$, we have $f^{-1}(x), f^{-1}(y) \in X$ with $f^{-1}(x) \neq f^{-1}(y)$. Then there exist $\lambda \in J(L)$ and $U \in \mathscr{C}$ such that $f_L^{\leftarrow}(x_\lambda) = f^{-1}(x)_\lambda \nleqslant U$, $f_L^{\leftarrow}(y_\lambda) = f^{-1}(y)_\lambda \leqslant U$. Hence we have $x_\lambda \nleqslant f_L^{\rightarrow}(U)$ and $y_\lambda \leqslant f_L^{\rightarrow}(U)$. Since $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ is an L-CC mapping, we know that $f_L^{\rightarrow}(U)$ is L-convex set of X. Therefore, $(Y, \mathscr{D})$ is sub-S_1.

(2) Let $(X, \mathscr{C})$ be a sub-S_2 L-convex space. Since f is a bijective mapping, for any $x, y \in Y$ with $x \neq y$, we have $f^{-1}(x), f^{-1}(y) \in X$ with $f^{-1}(x) \neq f^{-1}(y)$. Then there exist $\lambda \in J(L)$ and L-biconvex set H such that $f_L^{\leftarrow}(x_\lambda) = f^{-1}(x)_\lambda \leqslant H$, $f_L^{\leftarrow}(y_\lambda) = f^{-1}(y)_\lambda \nleqslant H$. Hence we have $x_\lambda \leqslant f_L^{\rightarrow}(H)$ and $y_\lambda \nleqslant f_L^{\rightarrow}(H)$. Since $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ is an L-CC mapping, we know that $f_L^{\rightarrow}(H)$ is L-biconvex set of X. Therefore, $(Y, \mathscr{D})$ is sub-S_2. □

By Proposition 4.3 and Proposition 4.4, we can easily obtain the following result.

Proposition 4.5. *Suppose two L-convex spaces $(X, \mathscr{C})$ and $(Y, \mathscr{D})$ be L-isomorphism. If $(X, \mathscr{C})$ is sub-S_1(resp. sub-S_2), then so is $(Y, \mathscr{D})$.*

Proof. It is straightforward and omitted. □

By the above propositions, we can obtain the productive property of sub-S_1 and sub-S_2 separation axioms in L-convex spaces.

Proposition 4.6. *Let $\{(X_t, \mathscr{C}_t)\}_{t \in T}$ be a family of L-convex spaces, and $(X, \mathscr{C})$ be the product space of $\{(X_t, \mathscr{C}_t)\}_{t \in T}$. If for each $t \in T$, $(X_t, \mathscr{C}_t)$ is sub-S_1(resp. sub-S_2), then so is $(X, \mathscr{C})$. Conversely, if $(X, \mathscr{C})$ is sub-S_1(resp. sub-S_2) and $(X_t, \mathscr{C}_t)$ is stratified for some $t \in T$, then $(X_t, \mathscr{C}_t)$ is sub-S_1(resp. sub-S_2).*

Proof. (1) Suppose that for each $t \in T$, $(X_t, \mathscr{C}_t)$ is sub-S_1. Take any $x, y \in X$ with $x \neq y$, where $x = \{x^t\}_{t\in T}$, $y = \{y^t\}_{t\in T}$. Then there exists $r \in T$ such that $x^r \neq y^r$. Since $(X_r, \mathscr{C}_r)$ is sub-S_1, there exist $\lambda \in J(L)$ and $U_r \in \mathscr{C}_r$ such that $(x^r)_\lambda \nleqslant U_r$, $(y^r)_\lambda \leqslant U_r$. By $P_r : (X, \mathscr{C}) \to (X_r, \mathscr{C}_r)$ is an L-CP mapping, we know that $P_r^{\leftarrow}(U_r)$ is L-convex set of X.
Since

$$P_r^{\rightarrow}(x_\lambda) = P_r^{\rightarrow}(x)_\lambda = (x^r)_\lambda \nleqslant U_r,$$

and

$$P_r^{\rightarrow}(y_\lambda) = P_r^{\rightarrow}(y)_\lambda = (y^r)_\lambda \leqslant U_r,$$

we have $x_\lambda \nleqslant P_r^{\leftarrow}(U_r)$ and $y_\lambda \leqslant P_r^{\leftarrow}(U_r)$. Therefore, $(X, \mathscr{C})$ is sub-S_1.

(2) Suppose that for each $t \in T$, $(X_t, \mathscr{C}_t)$ is sub-S_2. Take any $x, y \in X$ with $x \neq y$, where $x = \{x^t\}_{t\in T}$, $y = \{y^t\}_{t\in T}$. Then there exists $r \in T$ such that $x^r \neq y^r$. Since $(X_r, \mathscr{C}_r)$ is sub-S_2, there exist $\lambda \in J(L)$ and L-biconvex set H_r such that $(x^r)_\lambda \leqslant H_r$, $(y^r)_\lambda \nleqslant H_r$. By $P_r : (X, \mathscr{C}) \to (X_r, \mathscr{C}_r)$ is an L-CP mapping, we know that $P_r^{\leftarrow}(H_r)$ is L-biconvex set of X.
Since

$$P_r^{\rightarrow}(x_\lambda) = P_r^{\rightarrow}(x)_\lambda = (x^r)_\lambda \leqslant H_r,$$

and

$$P_r^{\rightarrow}(y_\lambda) = P_r^{\rightarrow}(y)_\lambda = (y^r)_\lambda \nleqslant H_r,$$

we have $x_\lambda \leqslant P_r^{\leftarrow}(H_r)$ and $y_\lambda \nleqslant P_r^{\leftarrow}(H_r)$. Therefore, $(X, \mathscr{C})$ is sub-S_2.

(3) Conversely, suppose that $(X, \mathscr{C})$ is sub-S_1(resp. sub-S_2) and $(X_t, \mathscr{C}_t)$ is stratified. By Theorem 2.12 and Proposition 2.13, we know $(X_t, \mathscr{C}_t)$ is L-isomorphic to a subspace $(\widetilde{X}_t, \mathscr{C}\mid_{\widetilde{X}_t})$ of $(X, \mathscr{C})$, where $\widetilde{X}_t$ is a subset of X parallelling to X_t through $x = (x_t)_{t\in T}$. By Propositions 4.2 and 4.5 , we can obtain $(X_t, \mathscr{C}_t)$ is sub-S_1(resp. sub-S_2). □

Proposition 4.7. *An S_1 L-convex space is sub-S_1.*

Proof. Let $(X, \mathscr{C})$ be a S_1 L-convex space. Take any $x, y \in X$ with $x \neq y$, then for any $\lambda \in J(L)$, we have $x_\lambda \nleqslant y_\lambda$. Therefore there exists $U \in \mathscr{C}$ such that $x_\lambda \nleqslant U$, $y_\lambda \leqslant U$. Thus we complete the proof. □

The following example shows that the converse of Proposition 4.7 is not true.

Example 4.8. Let $X = \{x, y, z\}$ and $L = [0, 1]$. We define $\mathscr{C} = \{\underline{0}, \underline{1}, U_1, U_2\}$, where

$$U_1(x) = \frac{1}{4}, U_1(y) = \frac{1}{2}, U_1(z) = 0;$$

$$U_2(x) = \frac{1}{4}, U_2(y) = \frac{1}{2}, U_2(z) = \frac{2}{3}.$$

It is easy to verify that $(X, \mathscr{C})$ is a sub-S_1 L-convex space. However, for any $x, y \in X$, take any $\lambda, \mu \in (0, \frac{1}{4})$ with $\mu < \lambda$. Then there is no $U \in \mathscr{C}$ which satisfying $x_\lambda \nleqslant U$, $y_\mu \leqslant U$. Hence, $(X, \mathscr{C})$ is not S_1.

Proposition 4.9. *An sub-S_2 L-convex space is sub-S_1.*

Proof. It is straightforward and omitted. □

The following example shows that the converse of Proposition 4.9 is not true.

Example 4.10. Consider the L-convex space $(X, \mathscr{C})$ in Example 4.8. It is sub-S_1 but not sub-S_2. In fact, there are only two L-biconvex sets in the L-convex space $(X, \mathscr{C})$, i.e. $\underline{0}$ and $\underline{1}$. Therefore, for any $x, y \in X$ with $x \neq y$, there is no $\lambda \in J(L)$ and L-biconvex set H which satisfying $x_\lambda \leqslant H$, $y_\lambda \nleqslant H$. Hence $(X, \mathscr{C})$ is not sub-S_2.

Next, we discuss the relationships of S_1, sub-S_1 and S_2, sub-S_2 separation axioms between a convex space and the induced L-convex space.

Lemma 4.11. *A convex space $(X, \mathcal{C})$ is S_1 if and only if for any $x, y \in X$ with $x \neq y$, there exists $F \in \mathcal{C}$ such that $x \in F, y \notin F$.*

Proof. Necessity. For any $x, y \in X$ with $x \neq y$, since $(X, \mathcal{C})$ is S_1, it follows that $\{x\} \in \mathcal{C}$ and $x \in \{x\}$, $y \notin \{x\}$.

Sufficiency. Take any $x, y \in X$ with $x \neq y$, there exists $F_y \in \mathcal{C}$ such that $x \in F_y$, $y \notin F_y$. On one hand, it is easy to check that $\{x\} \subseteq \bigcap_{y \neq x} F_y$. On the other hand, for any $z \notin \{x\}$, this means $z \neq x$, then there exists $F_z \in \mathcal{C}$ such that $z \notin F_z$, thus $z \notin \bigcap_{y \neq x} F_y$. This implies $\{x\} \supseteq \bigcap_{y \neq x} F_y$. Hence, we have $\{x\} = \bigcap_{y \neq x} F_y$. Since $F_y \in \mathcal{C}$, we have $\{x\} = \bigcap_{y \neq x} F_y \in \mathcal{C}$. Therefore, $(X, \mathcal{C})$ is S_1. □

Lemma 4.12. *Let $(X, \mathcal{C})$ be a convex space, $(X, \omega(\mathcal{C}))$ be the induced L-convex space. If $\lambda \in J(L)$ and H is an L-biconvex set of $(X, \omega(\mathcal{C}))$, then $H_{[\lambda]}$ is a biconvex set of $(X, \mathcal{C})$.*

Proof. Since H is an L-biconvex set of $(X, \omega(\mathcal{C}))$, then for any $\lambda \in L$, we have $H_{[\lambda]} \in \mathcal{C}$ and $(H')_{[\lambda]} \in \mathcal{C}$. By Lemma 2.1, we have

$$(H_{[\lambda]})' = (H')^{(\lambda')} = \bigcup_{a \not\leqslant \lambda'} (H')_{[a]}.$$

For any $(H')_{[a]}, (H')_{[b]}$, let $c = a \wedge b$. Then $(H')_{[a]} \subseteq (H')_{[c]}$ and $(H')_{[b]} \subseteq (H')_{[c]}$. This means $\{(H')_{[a]} \mid a \not\leqslant \lambda', a \in L\}$ is directed. Hence $(H_{[\lambda]})' \in \mathcal{C}$. Since $H_{[\lambda]} \in \mathcal{C}$, we know that $H_{[\lambda]}$ is biconvex set of $(X, \mathcal{C})$, as desired. □

Theorem 4.13. *Let $(X, \mathcal{C})$ be a convex space, and $(X, \omega(\mathcal{C}))$ be the induced L-convex space by $(X, \mathcal{C})$. $(X, \omega(\mathcal{C}))$ is sub-S_1(resp. sub-S_2) if and only if $(X, \mathcal{C})$ is S_1(resp. S_2).*

Proof. (1) Necessity. Let $(X, \omega(\mathcal{C}))$ be a sub-S_1 L-convex space. Take any $x, y \in X$ with $x \neq y$. Since $(X, \omega(\mathcal{C}))$ is sub-S_1, there exist $\lambda \in J(L)$ and $U \in \omega(\mathcal{C})$ such that $x_\lambda \not\leqslant U$ and $y_\lambda \leqslant U$. This means $\lambda \not\leqslant U(x)$, $\lambda \leqslant U(y)$. Hence $x \notin U_{[\lambda]}$, $y \in U_{[\lambda]}$. Obviously, $U_{[\lambda]} \in \mathcal{C}$. Therefore $(X, \mathcal{C})$ is S_1.

Sufficiency. Take any $x, y \in X$ with $x \neq y$. Since $(X, \mathcal{C})$ is S_1, there exists $F \in \mathcal{C}$ such that $x \in F, y \notin F$. For any $\lambda \in J(L)$, $(\chi_F)_{[\lambda]} = \{z \in X \mid (\chi_F)(z) \geqslant \lambda\} = F$. This means $\chi_F \in \omega(\mathcal{C})$. Hence $x_\lambda \leqslant \chi_F$, $y_\lambda \not\leqslant \chi_F$. By the arbitrariness of x and y, we know that $(X, \omega(\mathcal{C}))$ is sub-S_1.

(2) Necessity. Let $(X, \omega(\mathcal{C}))$ be a sub-S_2 L-convex space. Take any $x, y \in X$ with $x \neq y$. Since $(X, \omega(\mathcal{C}))$ is sub-S_2, there exist $\lambda \in J(L)$ and an L-biconvex set H such that $x_\lambda \leqslant H$, $y_\lambda \not\leqslant H$. This means $\lambda \leqslant H(x)$, $\lambda \not\leqslant H(y)$. Hence $x \in H_{[\lambda]}$ and $y \notin H_{[\lambda]}$. By Lemma 4.12, we know that $H_{[\lambda]}$ is biconvex set of X. Therefore, $(X, \mathcal{C})$ is S_2.

Sufficiency. Take any $x, y \in X$ with $x \neq y$. Since $(X, \mathcal{C})$ is S_2, there exists a biconvex set D of X such that $x \in D, y \notin D$. Hence $x_\lambda \leqslant \chi_D$, $y_\lambda \not\leqslant \chi_D$. For any $\lambda \in J(L)$, then $(\chi_D)_{[\lambda]} = D$ and $(\chi_{D'})_{[\lambda]} = D'$. This means $\chi_D, \chi_{D'} \in \omega(\mathcal{C})$. Since $\chi_{D'} = (\chi_D)'$, we know that χ_D is L-biconvex set of X. Therefore, $(X, \omega(\mathcal{C}))$ is sub-S_2. □

5. S_2 and S_3 separation axioms in L-convex spaces and induced L-convex spaces

In this section, we introduce S_2 and S_3 separation axioms in L-convex spaces and induced L-convex spaces. In particular, we discuss the relationships among them and S_1.

Definition 5.1. *ȷm Let $(X, \mathscr{C})$ be an L-convex space.*

(1) $(X, \mathscr{C})$ is said to be S_2 separated if for any $x_\lambda, y_\mu \in J(L^X)$ with $x_\lambda \not\leqslant y_\mu$, there exists an L-biconvex set H such that $x_\lambda \not\leqslant H$, $y_\mu \leqslant H$;

(2) $(X, \mathscr{C})$ is said to be S_3 separated if for any $x_\lambda \in J(L^X)$ and pseudocrisp set $C \in \mathscr{C}$ with $x \notin SuppC$, there exists an L-biconvex set H such that $x_\lambda \not\leqslant H$, $C \leqslant H$.

Remark 5.2. If L is replaced by $\{0, 1\}$, then Definition 5.1 reduced to the definition of S_2 and S_3 separation axioms in convex space. So we can see that the Definition 5.1 is reasonable generalization of S_2 and S_3 separation axioms.

Now we discuss the hereditary property of S_2 and S_3 separation axioms in L-convex spaces.

Proposition 5.3. *If $(X, \mathscr{C})$ is a S_2(resp. S_3) L-convex space and $(Y, \mathscr{C}|_Y)$ is its subspace, then $(Y, \mathscr{C}|_Y)$ is S_2(resp. S_3).*

Proof. (1) Suppose that $(X, \mathscr{C})$ be S_2. For any $x_\lambda, y_\mu \in J(L^Y)$ with $x_\lambda \nleqslant y_\mu$, we have $x_\lambda, y_\mu \in J(L^X)$. Then there exists L-biconvex set $H \in L^X$ such that $x_\lambda \nleqslant H$, $y_\mu \leqslant H$. It follows that

$$x_\lambda = x_\lambda|_Y \nleqslant H|_Y,$$

and

$$y_\mu = y_\mu|_Y \leqslant H|_Y.$$

By Proposition 2.14, we have $H|_Y$ is L-biconvex set of Y. Therefore, $(Y, \mathscr{C}|_Y)$ is S_2.

(2) Suppose that $(X, \mathscr{C})$ be S_3. For any $x_\lambda \in J(L^Y)$ and L-convex set $C|_Y \in \mathscr{C}|_Y$, where $C|_Y$ is pseudocrisp set and $x \notin \mathrm{Supp}(C|_Y)$. Then we know that $C \in \mathscr{C}$ is pseudocrisp set in X with $x \notin \mathrm{Supp}C$. Since $(X, \mathscr{C})$ is S_3, there exists an L-biconvex set $H \in L^X$ such that $x_\lambda \nleqslant H$, $C \leqslant H$. Hence we have $x_\lambda \nleqslant H|_Y, C|_Y \leqslant H|_Y$. By Proposition 2.14, we know that $H|_Y$ is L-biconvex set of Y. Therefore, $(Y, \mathscr{C}|_Y)$ is S_3. □

Proposition 5.4. *Let $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ be an L-CP and bijective mapping between two L-convex spaces. If $(Y, \mathscr{D})$ is S_2(resp. S_3), then $(X, \mathscr{C})$ is S_2(resp. S_3).*

Proof. (1) Suppose that $(Y, \mathscr{D})$ is S_2. Since f is a bijective and L-CP mapping, for any $x_\lambda, y_\mu \in J(L^X)$ with $x_\lambda \nleqslant y_\mu$, then we have $f(x)_\lambda, f(y)_\mu \in J(L^Y)$ with $f(x)_\lambda \nleqslant f(y)_\mu$. Then there exists L-biconvex set H such that $f_L^{\rightarrow}(x_\lambda) = f(x)_\lambda \nleqslant H$, $f_L^{\rightarrow}(y_\mu) = f(y)_\mu \leqslant H$. Hence, we have $x_\lambda \nleqslant f_L^{\leftarrow}(H)$ and $y_\mu \leqslant f_L^{\leftarrow}(H)$. By Proposition 2.7, we know that $f_L^{\leftarrow}(H)$ is L-biconvex set of X. Therefore, $(X, \mathscr{C})$ is S_2.

(2) Suppose that $(Y, \mathscr{D})$ is S_3. Take any $x_\lambda \in J(L^X)$ and pseudocrisp set $C \in \mathscr{C}$ with $x \notin \mathrm{Supp}C$. Since f is a bijective and L-CP mapping, we know that $f(x)_\lambda \in J(L^Y)$ and $f_L^{\rightarrow}(C) \in \mathscr{D}$ is pseudocrisp set in Y with $f(x) \notin \mathrm{Supp}(f_L^{\rightarrow}(C))$. Since $(Y, \mathscr{D})$ is S_3, there exists L-biconvex set H such that $f_L^{\rightarrow}(x_\lambda) = f(x)_\lambda \nleqslant H$, $f_L^{\rightarrow}(C) \leqslant H$. Hence, we have $x_\lambda \nleqslant f_L^{\leftarrow}(H)$ and $C \leqslant f_L^{\leftarrow}(H)$. By Proposition 2.7, we know that $f_L^{\leftarrow}(H)$ is L-biconvex set of X. Therefore, $(X, \mathscr{C})$ is S_3. □

Lemma 5.5. *If $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ is a bijection and L-CC mapping between two L-convex spaces, H is an L-biconvex set of X, then $f_L^{\rightarrow}(H)$ is an L-biconvex set of Y.*

Proof. Since f is an L-CC mapping and H is an L-biconvex set of X, we have $f_L^{\rightarrow}(H)$ and $f_L^{\rightarrow}(H')$ are L-convex sets of Y. For any $y \in Y$, there exists a unique $x \in X$ such that $f(x) = y$. We know that

$$f_L^{\rightarrow}(H')(y) = \bigvee_{f(x)=y} H'(x) = H'(x),$$

and

$$(f_L^{\rightarrow}(H))'(y) = (f_L^{\rightarrow}(H)(y))' = (\bigvee_{f(x)=y} H(x))' = \bigwedge_{f(x)=y} H'(x) = H'(x).$$

This means $(f_L^{\rightarrow}(H))' = f_L^{\rightarrow}(H')$. Hence $f_L^{\rightarrow}(H)$ is an L-biconvex set of Y. □

Proposition 5.6. *Let $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ be a bijective and L-CC mapping between two L-convex spaces. If $(X, \mathscr{C})$ is S_2(resp. S_3), then $(Y, \mathscr{D})$ is S_2(resp. S_3).*

Proof. (1) Suppose that $(X, \mathscr{C})$ is S_2. Since f is a bijective and L-CC mapping, for any $x_\lambda, y_\mu \in J(L^Y)$ with $x_\lambda \nleqslant y_\mu$, we have $f^{\leftarrow}(x)_\lambda, f^{\leftarrow}(y)_\mu \in J(L^X)$ with $f^{\leftarrow}(x)_\lambda \nleqslant f^{\leftarrow}(y)_\mu$. Since $(X, \mathscr{C})$ is S_2, there exists L-biconvex set H such that $f_L^{\leftarrow}(x_\lambda) = f^{\leftarrow}(x)_\lambda \nleqslant H$, $f_L^{\leftarrow}(y_\mu) = f^{\leftarrow}(y)_\mu \leqslant H$. Hence, we have $x_\lambda \nleqslant f_L^{\rightarrow}(H)$ and $y_\mu \leqslant f_L^{\rightarrow}(H)$. By Proposition 5.5, we know that $f_L^{\rightarrow}(H)$ is L-biconvex set of X. Therefore, $(Y, \mathscr{D})$ is S_2.

(2) Suppose that $(X, \mathscr{C})$ is S_3. Take any $y_\lambda \in J(L^Y)$ and pseudocrisp set $C \in \mathscr{D}$ with $y \notin \mathrm{Supp}C$. Since f is a bijective and L-CC mapping, then we have $f^{\leftarrow}(y)_\lambda \in J(L^X)$ and $f_L^{\leftarrow}(C) \in \mathscr{C}$ is pseudocrisp set with $f^{\leftarrow}(y) \notin \mathrm{Supp}(f_L^{\leftarrow}(C))$. Since $(X, \mathscr{C})$ is S_3, there exists L-biconvex set H such that $f_L^{\leftarrow}(y_\lambda) = f^{\leftarrow}(y)_\lambda \nleqslant H$, $f_L^{\leftarrow}(C) \leqslant H$. Hence, we have $y_\lambda \nleqslant f_L^{\rightarrow}(H)$ and $C \leqslant f_L^{\rightarrow}(H)$. By Proposition 5.5, we know that $f_L^{\rightarrow}(H)$ is L-biconvex set of X. Therefore, $(Y, \mathscr{D})$ is S_3. □

By Proposition 5.4 and Proposition 5.6, we can easily obtain the following result.

Proposition 5.7. *Let two L-convex spaces $(X,\mathscr{C})$ and $(Y,\mathscr{D})$ be L-isomorphism. If $(X,\mathscr{C})$ is S_2(resp. S_3), then so is $(Y,\mathscr{D})$.*

Proof. It is straightforward and omitted. □

Next we discuss the productive property of S_2 separation axiom in L-convex spaces.

Proposition 5.8. *Let $\{(X_t,\mathscr{C}_t)\}_{t\in T}$ be a family of L-convex spaces, and $(X,\mathscr{C})$ be the product space of $\{(X_t,\mathscr{C}_t)\}_{t\in T}$. If for each $t \in T$, $(X_t,\mathscr{C}_t)$ is S_2, then so is $(X,\mathscr{C})$. Conversely, if $(X,\mathscr{C})$ is S_2 and $(X_t,\mathscr{C}_t)$ is stratified for some $t \in T$, then $(X_t,\mathscr{C}_t)$ is S_2.*

Proof. Take any $x_\lambda, y_\mu \in J(L^X)$ with $x_\lambda \nleqslant y_\mu$, where $x = \{x^t\}_{t\in T}$, $y = \{y^t\}_{t\in T}$. Then there exists $r \in T$ such that $(x^r)_\lambda \nleqslant (y^r)_\mu$. Since $(X_r,\mathscr{C}_r)$ is S_2, there exists L-biconvex set H_r such that $(x^r)_\lambda \nleqslant H_r$, $(y^r)_\mu \leqslant H_r$. By $P_r : (X,\mathscr{C}) \to (X_r,\mathscr{C}_r)$ is an L-CP mapping, we know that $P_r^{\leftarrow}(H_r)$ is an L-biconvex set of X. It follows that

$$P_r^{\rightarrow}(x_\lambda) = P_r^{\rightarrow}(x)_\lambda = (x^r)_\lambda \nleqslant H_r,$$

and

$$P_r^{\rightarrow}(y_\mu) = P_r^{\rightarrow}(y)_\mu = (y^r)_\mu \leqslant H_r.$$

Hence we have $x_\lambda \nleqslant P_r^{\leftarrow}(H_r)$ and $y_\mu \leqslant P_r^{\leftarrow}(H_r)$. Therefore, $(X,\mathscr{C})$ is S_2.

Conversely, suppose that $(X,\mathscr{C})$ is S_2 and $(X_t,\mathscr{C}_t)$ is stratified. By Theorem 2.12 and Proposition 2.13, we know $(X_t,\mathscr{C}_t)$ is L-isomorphic to a subspace $(\widetilde{X}_t,\mathscr{C}\mid_{\widetilde{X}_t})$ of $(X,\mathscr{C})$, where $\widetilde{X}_t$ is a subset of X parallelling to X_t through $x = (x_t)_{t\in T}$. By Propositions 5.3 and 5.7, we can obtain $(X_t,\mathscr{C}_t)$ is S_2. □

Proposition 5.9. *An S_2 L-convex space is S_1.*

Proof. The proof is easy and omitted. □

The following example shows that the converse of Proposition 5.9 is not true.

Example 5.10. Let $X = \{x, y\}$ and $L = \{\bot, a, b, \top\}$, where $a' = b, b' = a, \bot' = \top, \top' = \bot$. We define $\mathscr{C} = \{\underline{\bot}, \underline{a}, \underline{b}, U_1, U_2, U_3, \underline{\top}\}$, where

$$U_1(x) = a, U_1(y) = \top;$$

$$U_2(x) = \bot, U_2(y) = b;$$

$$U_3(x) = \bot, U_3(y) = a.$$

It is easy to verify that $(X,\mathscr{C})$ is an L-convex space. For any $x_\lambda, y_\mu \in J(L^X)$ with $x_\lambda \nleqslant y_\mu$, we can find $U \in \mathscr{C}$ which satisfying $x_\lambda \nleqslant U$, $y_\mu \leqslant U$. Then $(X,\mathscr{C})$ is S_1. For the L-convex space $(X,\mathscr{C})$, $x_a, y_a \in J(L^X)$ and $x_a \nleqslant y_a$, there is no L-biconvex set H which satisfying $x_a \nleqslant H$, $y_a \leqslant H$. Hence $(X,\mathscr{C})$ is not S_2.

Proposition 5.11. *An S_2 L-convex space is sub-S_2.*

Proof. The proof is easy and omitted. □

The following example shows that the converse of Proposition 5.11 is not true.

Example 5.12. Let $X = \{x, y, z\}$ and $L = [0, 1]$. We define $\mathscr{C} = \{\underline{0}, \underline{1}, H_1, H_2, H_3, H_4\}$, where

$$H_1(x) = \frac{1}{2}, H_1(y) = \frac{1}{4}, H_1(z) = \frac{1}{3};$$

$$H_2(x) = \frac{1}{2}, H_2(y) = \frac{3}{4}, H_2(z) = \frac{2}{3};$$

$$H_3(x) = \frac{1}{2}, H_3(y) = \frac{1}{4}, H_3(z) = \frac{2}{3};$$

$$H_4(x) = \frac{1}{2}, H_4(y) = \frac{3}{4}, H_4(z) = \frac{1}{3}.$$

Obviously, $(X, \mathscr{C})$ is an L-convex space. For any $H_i \in \mathscr{C}$, then $H_i' \in \mathscr{C}$ $(i = 1, 2, 3, 4)$. This means that all elements in $\mathscr{C}$ are L-biconvex sets. Then $(X, \mathscr{C})$ is a sub-S_2 L-convex space. Let $x \in X$ and $\lambda, \mu \in (0, \frac{1}{4})$ with $\mu < \lambda$. Then there is no L-biconvex set H which satisfying $x_\lambda \nleqslant H$, $y_\mu \leqslant H$. Hence, $(X, \mathscr{C})$ is not S_2.

Theorem 5.13. *If an L-convex space $(X, \mathscr{C})$ is S_3 and S_1, then $(X, \mathscr{C})$ is S_2.*

Proof. Since $(X, \mathscr{C})$ is S_1, for any $x_\lambda, y_\mu \in J(L^X)$ with $x_\lambda \nleqslant y_\mu$, by Proposition 2.17, we have $y_\mu = co^{\mathscr{C}}(y_\mu) \in \mathscr{C}$. Obviously, $y_\mu \in J(L^X)$ is pseudocrisp set. Since $x \notin \mathrm{supp}(y_\mu)$, then there exists L-biconvex set H such that $x_\lambda \nleqslant H$, $y_\mu \leqslant H$. Therefore, $(X, \mathscr{C})$ is S_2. □

Next we discuss the relationships of S_2 and S_3 separation axioms between a convex space and the induced L-convex space.

Theorem 5.14. *Let $(X, \mathcal{C})$ be a convex space, and $(X, \omega(\mathcal{C}))$ be the induced L-convex space by $(X, \mathcal{C})$. $(X, \omega(\mathcal{C}))$ is S_2(resp. S_3) if and only if $(X, \mathcal{C})$ is S_2(resp. S_3).*

Proof. (1) Necessity. Let $(X, \omega(\mathcal{C}))$ be a S_2 L-convex space. Take any $x, y \in X$ with $x \neq y$ and $\lambda \in J(L)$. Since $(X, \omega(\mathcal{C}))$ is S_2, there exists L-biconvex set H such that $x_\lambda \nleqslant H$, $y_\lambda \leqslant H$. Hence $\lambda \nleqslant H(x)$ and $\lambda \leqslant H(y)$. This means $x \notin H_{[\lambda]}$ and $y \in H_{[\lambda]}$. By Lemma 4.12, we know that $H_{[\lambda]}$ is a biconvex set of $(X, \mathcal{C})$. By the arbitrariness of x, y, we know that $(X, \mathcal{C})$ is S_2.

Sufficiency. Take any $x_\lambda, y_\mu \in J(L^X)$ with $x_\lambda \nleqslant y_\mu$. Then $x \neq y$ or $x = y$, $\lambda \nleqslant \mu$.
Case 1: If $x \neq y$, since $(X, \mathcal{C})$ is S_2, then there exists a biconvex set P such that $x \notin P$, $y \in P$. Hence $x_\lambda \nleqslant \chi_P$, $y_\mu \leqslant \chi_P$. For any $a \in L$, $(\chi_P)_{[a]} = P$ or X, and $(\chi_{P'})_{[a]} = P'$ or X. Hence $\chi_P, \chi_{P'} \in \omega(\mathcal{C})$. By $\chi_{P'} = (\chi_P)'$, we know that χ_P is L-biconvex set of $(X, \omega(\mathcal{C}))$. This means $(X, \omega(\mathcal{C}))$ is S_2.
Case 2: If $x = y$, $\lambda \nleqslant \mu$, since $(X, \omega(\mathcal{C}))$ is a stratified L-convex space, it follows that $H = \underline{\mu}$ is an L-biconvex set of $(X, \omega(\mathcal{C}))$. Hence $x_\lambda \nleqslant \underline{\mu}$, $y_\mu = x_\mu \leqslant \underline{\mu}$. Therefore, $(X, \omega(\mathcal{C}))$ is S_2.

(2) Necessity. Let $(X, \omega(\mathcal{C}))$ be a S_3 L-convex space. Take any $C \in \mathcal{C}$, $x \in X$ with $x \notin C$. For any $\lambda \in J(L)$, $(\chi_C)_{[\lambda]} = C$. This means $\chi_C \in \omega(\mathcal{C})$. Since $x \notin C$, we have $\chi_C(x) = \bot_L$. Hence $x \notin \mathrm{Supp}\chi_C$. Since χ_C is pseudocrisp set and $(X, \omega(\mathcal{C}))$ is S_3, there exists L-biconvex set H such that $x_\lambda \nleqslant H$, $\chi_C \leqslant H$. This means for any $y \in C$, $\chi_C(y) = \top_L \leqslant H(y)$. Then $H(y) = \top_L$. This implies $y \in H_{[\lambda]}$. Therefore, $C \subseteq H_{[\lambda]}$. By $x_\lambda \nleqslant H$, we know that $\lambda \nleqslant H(x)$. Hence $x \notin H_{[\lambda]}$. By Lemma 4.12, we have $H_{[\lambda]}$ is a biconvex set of $(X, \mathcal{C})$. Therefore, $(X, \mathcal{C})$ is S_3.

Sufficiency. Take any $x_\lambda \in J(L^X)$ and pseudocrisp set $C \in \omega(\mathcal{C})$ with $x \notin \mathrm{Supp}C$. Since C is pseudocrisp set, there exists $a \in L$ with $a \neq \bot_L$ such that for any $y \in X$, $C(y) \neq \bot_L$ if and only if $C(y) \geqslant a$. This means $C_{[a]} = \{y \mid C(y) \neq \bot_L\} = \mathrm{Supp}C$. By $x \notin \mathrm{Supp}C$, we know that $x \notin C_{[a]}$. Since $(X, \mathcal{C})$ is S_3, there exists biconvex set H of $(X, \mathcal{C})$ such that $C_{[a]} \subseteq H$, $x \notin H$. Then for any $y \notin H$, we have $y \notin C_{[a]}$. Since $C_{[a]} = \mathrm{Supp}C$, we have $C(y) = \bot_L$. Hence $C \leqslant \chi_H$. Since $x \notin H$ and $\lambda \neq \bot_L$, then we have $x_\lambda \nleqslant \chi_H$. For any $\lambda \in J(L)$, $(\chi_H)_{[\lambda]} = H$ and $(\chi_{H'})_{[\lambda]} = H'$. Since H is biconvex set, hence $\chi_H, \chi_{H'} \in \omega(\mathcal{C})$. By $\chi_{H'} = (\chi_H)'$, we know that χ_H is L-biconvex set of X. Therefore, $(X, \omega(\mathcal{C}))$ is S_3. □

6. Conclusions

In this paper, we established the relationships of induced L-convex spaces with L-hull operators, product spaces, and quotient spaces. Moreover, we introduced sub-S_1, sub-S_2, S_2 and S_3 separation axioms in L-convex space. Then we provided some properties of them and discussed the relationship of them in L-convex spaces and induced L-convex spaces. All the concepts and the relevant relationships in the framework of L-convex spaces are shown to be proper generalizations of those in the classical case. Following separation axioms in this paper, we will consider S_4 separation axiom in L-convex spaces and induced L-convex spaces in the future.

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