Filomat 36:4 (2022), 1269–1293
https://doi.org/10.2298/FIL2204269D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Matrix Pencils Completions Under Double Rank Restrictions

Marija Dodig[a,b]

[a]*CEAFEL, Departamento de Matématica, Universidade de Lisboa, Edificio C6, Campo Grande, 1749-016 Lisbon, Portugal*
[b]*Mathematical Institute SANU, Knez Mihajlova 36, 11000 Beograd, Serbia.*

Abstract. In this paper we study the General Matrix Pencil Completion Problem under double rank restrictions. Considered rank restrictions are not structural, hence the obtained results deal with full sets of Kronecker invariants of the involved matrix pencils, and they generalize many of the existing results in the literature, for example [1, 5, 7, 9, 11, 12, 21, 24, 45, 46]. Main methods consist of combining the celebrated Sá-Thompson's result [1, 39, 44] with novel results on rank restrictions in completions of matrix pencils. All of the obtained results are explicit and constructive.

1. Introduction

Let $\mathbb{F}$ be an algebraically closed field. Let $C(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, and $D(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$ be matrix pencils, i.e. matrix polynomials of degree 1. We say that matrix pencils $C(\lambda)$ and $D(\lambda)$ are *strictly equivalent* if there exist invertible matrices $P \in \mathbb{F}^{n\times n}$ and $Q \in \mathbb{F}^{m\times m}$, such that

$$PC(\lambda)Q = D(\lambda).$$

One of the fundamental open topics in Linear Algebra is the General Matrix Pencil Completion Problem. Its roots date back from 1970's [35], and it has been posed as a *Challenge* of Linear Algebra in [30].

Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ and $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+y)\times(n+m+x)}$ be two matrix pencils. Then the General Matrix Pencil Completion Problem is the following:

Problem 1.1. *Find necessary and sufficient conditions for the existence of pencils $X(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times x}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{y\times(n+m)}$ and $Z(\lambda) \in \mathbb{F}[\lambda]^{y\times x}$ such that the pencil*

$$\begin{bmatrix} A(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix},$$

is strictly equivalent to $M(\lambda)$.

2020 *Mathematics Subject Classification*. Primary 15A83; Secondary 05A17
Keywords. Completion of matrix pencils, rank restrictions
Received: 20 April 2021; Accepted: 19 July 2021
Communicated by Dragan S. Djordjević
This work was done within the activities of CEAFEL and was partially supported by FCT project UIDB/04721/2020, and by the Ministry of Education, Science and Technological Development of the Republic of Serbia through the activities of the Mathematical Institute of the Serbian Academy of Sciences and Arts.
Email address: msdodig@fc.ul.pt (Marija Dodig)

Apart from purely theoretical importance, Problem 1.1 has strong connections and applications in Control Theory, Graph Theory, Perturbation Theory and Representation Theory of Quivers, see e.g. [3, 4, 8, 20, 22, 25, 29, 31–34, 41–43, 46]. Although many authors through decades have studied Problem 1.1 and its particular cases, it still remains open and presents a big challenge in Matrix Theory. For the most important contributions towards solving the General Matrix Pencil Completion Problem see e.g. [2, 5, 7, 9, 14, 16, 21, 24, 37, 39, 44–47].

Strict equivalence invariants of a matrix pencil are usually called *Kronecker invariants* and they consist of invariant factors, infinite elementary divisors, and column and row minimal indices. For all details see [23].

Until recently, the main method to attack and resolve Problem 1.1 was by considering structural restrictions, i.e. by limiting the types of Kronecker invariants of involved pencils. For example, the most celebrated classical results on this topic deal with pencils $A(\lambda)$ and $M(\lambda)$ that are both regular [1, 39, 44], or such that one of them is regular [7, 12, 21, 24], or such that one, or both, of them is quasi-regular [5, 10, 19], etc.

In [9] for the first time in the literature, Problem 1.1 has been considered and solved without any structural restrictions, i.e. the obtained necessary and sufficient conditions involve all the possible eight types of Kronecker invariants (four of them coming from the matrix pencil $A(\lambda)$, and four of them coming from the matrix pencil $M(\lambda)$). Hence, the solution to Problem 1.1 in [9] is different and more general concerning the types of Kronecker invariants involved, than all the existing ones. However, the solution in [9] isn't the final one, since we consider restrictions on the dimensions of the involved matrix pencils. In fact, the solution from [9] corresponds to the minimal possible values of x and y in order the prescribed set of Kronecker invariants could be reached. For all details see [9, 11].

In this paper we improve this novel approach from [9], by stretching the existing methods to the limits, combining the results from [9] with the classical celebrated Sá-Thompson's result [1, 39, 44]. In addition, we use technical results on rank restrictions in completions of matrix pencils (Lemmas 3.1-3.9), as well as combinatorial results on generalized majorization (Lemma 2.3).

Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$, $\operatorname{rank} A(\lambda) = n$, and $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+y)\times(n+m+x)}$, $\operatorname{rank} M(\lambda) = n+s$, be two matrix pencils. The main result of the paper is a solution to General Matrix Pencils Completion Problem, Problem 1.1, under any of the following double rank constraints:

$$\operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \end{bmatrix} = n+s-y, \quad \text{and} \quad \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} = n+s-x. \tag{1}$$

$$\operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \end{bmatrix} = n, \quad \text{and} \quad \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} = n. \tag{2}$$

$$\operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \end{bmatrix} = n, \quad \text{and} \quad \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} = n+s-x. \tag{3}$$

$$\operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \end{bmatrix} = n+s-y, \quad \text{and} \quad \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} = n. \tag{4}$$

We note that restrictions (1) imply

$$\max(x,y) \le s \le x+y,$$

restrictions (2) imply

$$0 \le s \le \min(x,y),$$

restrictions (3) imply

$$x \le s \le y,$$

and restrictions (4) imply

$$y \le s \le x.$$

By considering and solving Problem 1.1 under these restrictions, we come one step closer to its general solution. The obtained results directly generalize the main result in [9], as well as the main results in [5, 7, 11, 12, 21, 24, 46]. Moreover, novel restrictions that we are considering in this paper (1)-(4), are given in the form of *double rank restrictions*. This is important due to various reasons. First, these restrictions are not structural and they allow considering all eight types of Kronecker invariants. Also, these constraints directly connect completion problems with problems of rank restrictions, and thus open the door to possible applications of Problem 1.1 into Perturbation Theory and Control Theory. Next, the rank restrictions are easily comprehensible and of more interest to wider audience, comparing to structural restrictions which involve Kronecker invariants. Also, these kind of restrictions can be checked more easily. Finally, they are more general than majority of the restrictions considered to Problem 1.1 in the past, both structural and minimal.

In particular, the main results in [9] are just special cases of Theorems 1–3. More precisely, [9, Theorem 4.1] is a special case of Theorem 1 when $s = x + y$; [9, Theorem 4.2] is a special case of Theorem 2 when $s = 0$; [9, Theorem 4.3] is a special case of Theorem 3 when $x = 0$; and [9, Theorem 4.4] is a special case of the transposed version of Theorem 3 when $y = 0$. Even more, the main result in [7] is a special case of Theorem 1 when $M(\lambda)$ is regular. The main result in [12] is a special case of Theorem 2 when $A(\lambda)$ is regular, while the main result in [5] is a special case of Theorem 3 if both $A(\lambda)^T$ and $M(\lambda)$ are quasi-regular pencils. Analogously, one can easily see that [1, 11, 21, 24, 45, 46] are also direct corollaries of the main results in this paper.

The paper is organised in six sections. In Section 2 we give notation that is used throughout the paper, as well as the most important previous results on minimal case completions, and basic concepts on Matrix Pencils and General Majorizations. Section 3 consists of auxiliary lemmas that are essential in the proofs of the main results. Lemma 3.8 is particularly useful and challenging, and it has strong impact in solving Problem 1.1. More applications of this particular result are expected.

We solve Problem 1.1 under restriction (1) in Theorem 4.3 in Section 4, and we solve Problem 1.1 under restriction (2) in Theorem 5.2 in Section 5. Finally, it is straightforward to see that restrictions (3) and (4) are transposed versions of each other, so it is enough to solve Problem 1.1 under one of them. A solution to Problem 1.1 under restriction (3) is given in Theorem 6.1 in Section 6.

All of the obtained results are explicit and constructive, and are valid over algebraically closed fields.

2. Notation and previous results

2.1. Matrix pencils

Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be a matrix pencil. In this paper we shall consider invariant factors and infinite elementary divisors unified as *homogeneous invariant factors*, for all details see e.g. [7]. Hence, the Kronecker invariants of $A(\lambda)$ consist of homogeneous invariant factors and column and row minimal indices. Also, their number can be expressed in terms of the size and rank of a matrix pencil as follows. Let $\operatorname{rank} A(\lambda) = n$, then $A(\lambda)$ has n homogeneous invariant factors, p row minimal indices, and m column minimal indices. Also, if we denote the sum of the column minimal indices of $A(\lambda)$ by c, the sum of the row minimal indices of $A(\lambda)$ by r, and the sum of the degrees of the homogeneous invariant factors of $A(\lambda)$ by h, then

$$c + r + h = n.$$

For more details on Kronecker invariants see chapter XII of [23].

The canonical form for strict equivalence is called Kronecker canonical form. We shall consider the Kronecker canonical form of $A(\lambda)$ as given in [18, 23]. Moreover, since in this paper we unify the invariant factors and infinite elementary divisors as homogeneous invariant factors, we have that the Kronecker canonical form of $A(\lambda)$ has the following shape:

$$\left[\begin{array}{c|c|c} N(\lambda) & 0 & 0 \\ \hline 0 & C(\lambda) & 0 \\ \hline 0 & 0 & R(\lambda) \end{array}\right] \tag{5}$$

where $N(\lambda) \in \mathbb{F}[\lambda]^{h\times h}$ is the matrix pencil block that corresponds to homogeneous invariant factors (made out of two blocks, one corresponding to companion matrices of the invariant factors, and the other one corresponding to infinite elementary divisors), $C(\lambda) \in \mathbb{F}[\lambda]^{c\times(c+m)}$ corresponds to the column minimal indices and $R(\lambda) \in \mathbb{F}[\lambda]^{(r+p)\times r}$ corresponds to the row minimal indices of $A(\lambda)$. For the purpose of this paper we do not need the specific forms of $N(\lambda)$, $C(\lambda)$ and $R(\lambda)$, however they are well known, and can be found e.g. in [18, 23].

By I_k we denote the identity matrix of size k. All polynomials throughout the paper are assumed to be homogeneous polynomials from $\mathbb{F}[\lambda,\mu]$, and monic with respect to λ. Let $\delta_1|\cdots|\delta_n$ be homogeneous invariant factors of a matrix pencil $D(\lambda)$ (and so $\operatorname{rank} D(\lambda) = n$). We assume $\delta_i = 1$, for all $i < 1$ and $\delta_i = 0$, for all $i > n$. By $d(\delta_i)$ we denote the degree of a polynomial δ_i.

2.2. *Generalized majorizations*

By a partition we mean a non-increasing sequence of integers. For any sequence of non-increasing integers $a_1 \geq \cdots \geq a_s$, we define the corresponding partition by $\mathbf{a} = (a_1,\ldots,a_s)$. Moreover for any such sequence, we assume $a_i = +\infty$, for $i \leq 0$, and $a_i = -\infty$, for $i > s$. Also, we put $\sum_{i=a}^{b} a_i = 0$ whenever $a > b$.

We recall the definition of the classical majorization:

Definition 2.1. *[26] Let $\mathbf{a} = (a_1,\ldots,a_s)$ and $\mathbf{g} = (g_1,\ldots,g_s)$ be two partitions. If $\sum_{i=1}^{s} g_i = \sum_{i=1}^{s} a_i$ and*

$$\sum_{i=1}^{j} g_i \leq \sum_{i=1}^{j} a_i, \quad j = 1,\ldots,s-1,$$

then we say that $\mathbf{g}$ is majorized by $\mathbf{a}$ and write $\mathbf{g} \prec \mathbf{a}$.

In [16] we have introduced the concept of generalized majorization. We recall it here since it will be used in Section 6:

Definition 2.2. *Let $d_1 \geq \cdots \geq d_{m+k-s}$, $g_1 \geq \cdots \geq g_{m+k}$, $a_1 \geq \cdots \geq a_s$ be integers. Consider partitions $\mathbf{d} = (d_1,\ldots,d_{m+k-s})$, $\mathbf{g} = (g_1,\ldots,g_{m+k})$ and $\mathbf{a} = (a_1,\ldots,a_s)$. If*

$$d_i \geq g_{i+s}, \qquad i = 1,\ldots,m+k-s, \tag{6}$$

$$\sum_{i=1}^{h_j} g_i - \sum_{i=1}^{h_j-j} d_i \leq \sum_{i=1}^{j} a_i, \qquad j = 1,\ldots,s \tag{7}$$

$$\sum_{i=1}^{m+k} g_i = \sum_{i=1}^{m+k-s} d_i + \sum_{i=1}^{s} a_i, \tag{8}$$

where

$$h_j := \min\{i | d_{i-j+1} < g_i\}, \quad j = 1,\ldots,s,$$

then we say that $\mathbf{g}$ is majorized by $\mathbf{d}$ and $\mathbf{a}$. This type of majorization we call the generalized majorization, and we write

$$\mathbf{g} \prec' (\mathbf{d},\mathbf{a}).$$

Notice that, if (8) is satisfied, then (7) is equivalent to the following:

$$\sum_{i=h_j+1}^{m+k} g_i \geq \sum_{i=h_j-j+1}^{m+k-s} d_i + \sum_{i=j+1}^{s} a_i, \quad j = 1,\ldots,s. \tag{9}$$

The following result follows directly from Definitions 2.1 and 2.2:

Lemma 2.3. *Let* $\mathbf{a}, \mathbf{c}$ *and* $\mathbf{d}$ *be partitions such that*

$$\mathbf{c} \prec' (\mathbf{d}, \mathbf{a}).$$

Let $\bar{\mathbf{a}}$ *be a partition of the same length as* $\mathbf{a}$, *such that* $\mathbf{a} \prec \bar{\mathbf{a}}$, *then*

$$\mathbf{c} \prec' (\mathbf{d}, \bar{\mathbf{a}}).$$

2.3. Notation

The following notation will be used throughout the paper:
Let n, m, p, x, y and s be nonnegative integers.
Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be a matrix pencil, such that $\operatorname{rank} A(\lambda) = n$.
We denote the Kronecker invariants of the pencil $A(\lambda)$ by:

$$\begin{array}{lcl} \alpha_1 | \cdots | \alpha_n & - & \text{homogeneous invariant factors} \\ c_1 \geq \cdots \geq c_\rho > c_{\rho+1} = \cdots = c_m = 0 & - & \text{column minimal indices} \\ r_1 \geq \cdots \geq r_\theta > r_{\theta+1} = \cdots = r_p = 0 & - & \text{row minimal indices} \end{array}$$

Then

$$\sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{m} c_i + \sum_{i=1}^{p} r_i = n. \tag{10}$$

Let $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+y)\times(n+m+x)}$ be a matrix pencil, such that $\operatorname{rank} M(\lambda) = n + s$.
We denote the Kronecker invariants of the pencil $M(\lambda)$ by:

$$\begin{array}{lcl} \gamma_1 | \cdots | \gamma_{n+s} & - & \text{homogeneous invariant factors} \\ d_1 \geq \cdots \geq d_{\bar{\rho}} > d_{\bar{\rho}} = \cdots = d_{m+x-s} = 0 & - & \text{column minimal indices} \\ \bar{r}_1 \geq \cdots \geq \bar{r}_{\bar{\theta}} > \bar{r}_{\bar{\theta}+1} = \cdots = \bar{r}_{p+y-s} = 0 & - & \text{row minimal indices} \end{array}$$

Then

$$\sum_{i=1}^{n+s} d(\gamma_i) + \sum_{i=1}^{m+x-s} d_i + \sum_{i=1}^{p+y-s} \bar{r}_i = n + s. \tag{11}$$

The following theorem will be useful in the proofs of the main results:

Theorem 2.4. *[9, Theorem 4.3][15, Theorem 2] Let* $A(\lambda)$ *and* $M(\lambda)$ *be pencils as given in Section 2.3, with* $x = 0$. *There exist a pencil* $Y(\lambda)$ *such that the pencil*

$$\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} \tag{12}$$

is strictly equivalent to $M(\lambda)$ *if and only if*

$$\bar{\theta} \geq \theta, \tag{13}$$

$$\mathbf{c} \prec' (\mathbf{d}, \mathbf{a}), \tag{14}$$

$$\bar{\mathbf{r}} \prec' (\mathbf{r}, \mathbf{b}), \tag{15}$$

$$\gamma_i \mid \alpha_i \mid \gamma_{i+y}, \quad i = 1, \ldots, n, \tag{16}$$

$$\sum_{i=1}^{n+s} d(\operatorname{lcm}(\alpha_{i-s}, \gamma_i)) \leq \sum_{i=1}^{n+s} d(\gamma_i) - \sum_{i=1}^{p} r_i + \sum_{i=1}^{p+y-s} \bar{r}_i. \tag{17}$$

Here the partitons $\mathbf{a} = (a_1, \ldots, a_s)$ *and* $\mathbf{b} = (b_1, \ldots, b_{y-s})$ *are given by*

$$\begin{aligned}\sum_{i=1}^{j} a_i &= \sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{m}(c_i+1) - \sum_{i=1}^{m-s}(d_i+1) - \sum_{i=1}^{n+s-j} d(\operatorname{lcm}(\alpha_{i-s+j}, \gamma_i)) - j, \quad j = 1, \ldots, s,\\ \sum_{i=1}^{k} b_i &= \sum_{i=1}^{n+s} d(\gamma_i) - \sum_{i=1}^{p} r_i + \sum_{i=1}^{p+y-s} \bar{r}_i - \sum_{i=1}^{n+s} d(\operatorname{lcm}(\alpha_{i-k-s}, \gamma_i)), \quad k = 1, \ldots, y-s.\end{aligned}$$

3. Auxiliary results on ranks and completions of matrix pencils

In this section we give eight lemmas, all of them dealing with ranks of matrix pencils and completions. Some of them (Lemmas 3.2, 3.3 and 3.4) are known, but we cite them in stronger form than comparing to the paper they have appeared in, [18]. In fact, by following their original proofs in [18], one can conclude much stronger result than stated in those lemmas originally. We stress this out here, and we cite those lemmas in these more powerful forms. The results in Lemmas 3.1, 3.6 and 3.7 are straightforward. Finally, we give some involved technical lemmas based on the results of Lemmas 3.3 and 3.4. Novel powerful technical Lemmas 3.8 and 3.9 will be essential in proving the main results of the paper.

Lemma 3.1. *Let* $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ *and* $X(\lambda) \in \mathbb{F}[\lambda]^{(x+s)\times(n+m)}$ *be matrix pencils such that* $\operatorname{rank} A(\lambda) = n$ *and*

$$\operatorname{rank}\begin{bmatrix} A(\lambda) \\ X(\lambda) \end{bmatrix} = n + s.$$

Then there exists an invertible matrix $P \in \mathbb{F}^{(x+s)\times(x+s)}$ *such that*

$$\begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix}\begin{bmatrix} A(\lambda) \\ X(\lambda) \end{bmatrix} = \begin{bmatrix} A(\lambda) \\ X_1(\lambda) \\ X_2(\lambda) \end{bmatrix},$$

where $X_1(\lambda) \in \mathbb{F}[\lambda]^{s\times(n+m)}$ *and*

$$\operatorname{rank}\begin{bmatrix} A(\lambda) \\ X_1(\lambda) \end{bmatrix} = n + s.$$

As announced, we cite a stronger version of [18, Lemmas 9 and 10] comparing to [18]. In fact, the proofs of [18, Lemmas 9 and 10] prove these stronger versions, and we write them below in Lemmas 3.2, 3.3 and 3.4 explicitly:

Lemma 3.2. *[18, Lemma 9] Let*

$$C_c(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & \lambda & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{c\times(c+1)},$$

and

$$R_r(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots \\ 0 & \lambda & 1 & \ddots \\ & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \lambda \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(r+1)\times r},$$

be matrix pencils. Let $t(\lambda) \in \mathbb{F}[\lambda]^{1\times(c+1)}$ and $s(\lambda) \in \mathbb{F}[\lambda]^{1\times r}$ be matrix pencils such that

$$\operatorname{rank}\left[\begin{array}{cc} C_c(\lambda) & 0 \\ 0 & R_r(\lambda) \\ \hline t(\lambda) & s(\lambda) \end{array}\right] = c + r + 1.$$

Then there exist matrices $P_1 \in \mathbb{F}^{c\times(r+1)}$, $P_2 \in \mathbb{F}^{1\times(r+1)}$, and $P_3 \in \mathbb{F}^{(c+1)\times r}$, such that

$$\begin{bmatrix} I_c & P_1 & 0 \\ 0 & I_{r+1} & 0 \\ 0 & P_2 & 1 \end{bmatrix}\left[\begin{array}{cc} C_c(\lambda) & 0 \\ 0 & R_r(\lambda) \\ \hline t(\lambda) & s(\lambda) \end{array}\right]\begin{bmatrix} I_{c+1} & P_3 \\ 0 & I_r \end{bmatrix} = \left[\begin{array}{cc} C_c(\lambda) & 0 \\ 0 & R_r(\lambda) \\ \hline t(\lambda) & 0 \end{array}\right].$$

Lemma 3.3. *[18, Lemma 10] Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ and $B(\lambda) \in \mathbb{F}[\lambda]^{y\times(n+m)}$ be matrix pencils, $n = \operatorname{rank} A(\lambda)$. Let $s = \operatorname{rank}\begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix} - \operatorname{rank} A(\lambda)$.*

Then there exist invertible matrices $P \in \mathbb{F}^{(n+p)\times(n+p)}$, $Q \in \mathbb{F}^{y\times y}$ and $S \in \mathbb{F}^{(n+m)\times(n+m)}$, and a matrix $R \in \mathbb{F}^{y\times(n+p)}$ such that

$$\begin{bmatrix} P & 0 \\ R & Q \end{bmatrix}\begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix} S = \begin{bmatrix} A(\lambda) \\ X(\lambda) \\ Y(\lambda) \end{bmatrix}$$

where $X(\lambda) \in \mathbb{F}[\lambda]^{s\times(n+m)}$, and $Y(\lambda) \in \mathbb{F}[\lambda]^{(y-s)\times(n+m)}$, such that the pencil $\begin{bmatrix} A(\lambda) \\ X(\lambda) \end{bmatrix}$ has rank equal to $n + s$, has the same row minimal indices as $A(\lambda)$, and the same column minimal indices as $\begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix}$.

In particular, this implies that $\begin{bmatrix} A(\lambda) \\ X(\lambda) \\ Y(\lambda) \end{bmatrix}$ is strictly equivalent to $\begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix}$.

Having in mind the stronger version of [18, Lemma 9] given as Lemma 3.2, in [18] we have also proved the following result:

Lemma 3.4. *[18, Lemma 10] Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ be a matrix pencil, $n = \operatorname{rank} A(\lambda)$. Let*

$$\begin{bmatrix} N(\lambda) & 0 & 0 \\ 0 & C(\lambda) & 0 \\ 0 & 0 & R(\lambda) \end{bmatrix}$$

be the Kronecker canonical form of the matrix pencil $A(\lambda)$. Let us denote by h the sum of degrees of its homogeneous invariant factors, by c the sum of its column minimal indices, and by r the sum of its row minimal indices. Let $x(\lambda) \in \mathbb{F}[\lambda]^{s\times h}$, $t(\lambda) \in \mathbb{F}[\lambda]^{s\times(c+m)}$ and $s(\lambda) \in \mathbb{F}[\lambda]^{s\times r}$ be matrix pencils, such that

$$\operatorname{rank}\left[\begin{array}{ccc} N(\lambda) & 0 & 0 \\ 0 & C(\lambda) & 0 \\ 0 & 0 & R(\lambda) \\ \hline x(\lambda) & t(\lambda) & s(\lambda) \end{array}\right] = n + s.$$

Then there exist matrices $P_1 \in \mathbb{F}^{c\times(r+p)}$, $P_2 \in \mathbb{F}^{s\times(r+p)}$, and $P_3 \in \mathbb{F}^{(c+m)\times r}$, such that

$$\begin{bmatrix} I_h & 0 & 0 & 0 \\ 0 & I_c & P_1 & 0 \\ 0 & 0 & I_{r+p} & 0 \\ 0 & 0 & P_2 & I_s \end{bmatrix}\left[\begin{array}{ccc} N(\lambda) & 0 & 0 \\ 0 & C(\lambda) & 0 \\ 0 & 0 & R(\lambda) \\ \hline x(\lambda) & t(\lambda) & s(\lambda) \end{array}\right]\begin{bmatrix} I_h & 0 & 0 \\ 0 & I_{c+m} & P_3 \\ 0 & 0 & I_r \end{bmatrix} = \left[\begin{array}{ccc} N(\lambda) & 0 & 0 \\ 0 & C(\lambda) & 0 \\ 0 & 0 & R(\lambda) \\ \hline x(\lambda) & t(\lambda) & 0 \end{array}\right].$$

In addition, if $X(\lambda) \in \mathbb{F}[\lambda]^{(s+x)\times h}$, $T(\lambda) \in \mathbb{F}[\lambda]^{(s+x)\times(c+m)}$ *and* $S(\lambda) \in \mathbb{F}[\lambda]^{(s+x)\times r}$ *are matrix pencils, such that*

$$\operatorname{rank}\left[\begin{array}{ccc} N(\lambda) & 0 & 0 \\ 0 & C(\lambda) & 0 \\ 0 & 0 & R(\lambda) \\ \hline X(\lambda) & T(\lambda) & S(\lambda) \end{array}\right] = n+s,$$

then there exist matrices $P_1 \in \mathbb{F}^{c\times(r+p)}$, $P_2 \in \mathbb{F}^{(s+x)\times(r+p)}$, $P_3 \in \mathbb{F}^{(c+m)\times r}$, *and an invertible matrix* $P \in \mathbb{F}^{(s+x)\times(s+x)}$ *such that*

$$\begin{bmatrix} I_h & 0 & 0 & 0 \\ 0 & I_c & P_1 & 0 \\ 0 & 0 & I_{r+p} & 0 \\ 0 & 0 & P_2 & P \end{bmatrix}\left[\begin{array}{ccc} N(\lambda) & 0 & 0 \\ 0 & C(\lambda) & 0 \\ 0 & 0 & R(\lambda) \\ \hline X(\lambda) & T(\lambda) & S(\lambda) \end{array}\right]\begin{bmatrix} I_h & 0 & 0 \\ 0 & I_{c+m} & P_3 \\ 0 & 0 & I_r \end{bmatrix} = \left[\begin{array}{ccc} N(\lambda) & 0 & 0 \\ 0 & C(\lambda) & 0 \\ 0 & 0 & R(\lambda) \\ \hline PX(\lambda) & PT(\lambda) & \begin{array}{c} 0 \\ \hline \bar{S}(\lambda) \end{array} \end{array}\right].$$

Here $0 \in \mathbb{F}^{s\times p}$ *is a zero matrix with s rows. Also, matrix P is the one obtained from Lemma 3.1.*

Remark 3.5. *We also note that if* $X(\lambda) \in \mathbb{F}[\lambda]^{s\times h}$, $T(\lambda) \in \mathbb{F}[\lambda]^{s\times(c+m)}$ *and* $S(\lambda) \in \mathbb{F}[\lambda]^{s\times r}$ *are matrix pencils, such that*

$$\operatorname{rank}\left[\begin{array}{ccc} N(\lambda) & 0 & 0 \\ 0 & C(\lambda) & 0 \\ 0 & 0 & R(\lambda) \\ \hline X(\lambda) & T(\lambda) & S(\lambda) \end{array}\right] = \operatorname{rank}\begin{bmatrix} N(\lambda) & 0 & 0 \\ 0 & C(\lambda) & 0 \\ 0 & 0 & R(\lambda) \end{bmatrix} = n,$$

then there exists a matrix $P \in \mathbb{F}^{s\times c}$, *such that*

$$\begin{bmatrix} I_h & 0 & 0 & 0 \\ 0 & I_c & 0 & 0 \\ 0 & 0 & I_{r+p} & 0 \\ 0 & P & 0 & I_s \end{bmatrix}\left[\begin{array}{ccc} N(\lambda) & 0 & 0 \\ 0 & C(\lambda) & 0 \\ 0 & 0 & R(\lambda) \\ \hline X(\lambda) & T(\lambda) & S(\lambda) \end{array}\right] = \left[\begin{array}{ccc} N(\lambda) & 0 & 0 \\ 0 & C(\lambda) & 0 \\ 0 & 0 & R(\lambda) \\ \hline X(\lambda) & 0 & S(\lambda) \end{array}\right].$$

The following two lemmas are straightforward:

Lemma 3.6. *Let* $A(\lambda) \in \mathbb{F}^{(n+p)\times(n+m)}$, $X(\lambda) \in \mathbb{F}^{(n+p)\times x}$, $Y(\lambda) \in \mathbb{F}^{y\times(n+m)}$ *and* $Z(\lambda) \in \mathbb{F}^{y\times x}$ *be matrix pencils such that* $\operatorname{rank} A(\lambda) = n$,

$$\operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \end{bmatrix} = n+x, \text{ and } \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} = n+y,$$

then

$$\operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix} = n+x+y.$$

Lemma 3.7. *Let* $A(\lambda) \in \mathbb{F}^{(n+p)\times(n+m)}$, $X(\lambda) \in \mathbb{F}^{(n+p)\times x}$, $Y(\lambda) \in \mathbb{F}^{y\times(n+m)}$ *and* $Z(\lambda) \in \mathbb{F}^{y\times x}$ *be matrix pencils. Then*

$$x \geq \operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix} - \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} \geq \operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \end{bmatrix} - \operatorname{rank} A(\lambda).$$

By using Lemmas 3.3 and 3.4, we can prove the following result:

Lemma 3.8. *Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$, $X(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times x}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{y\times(n+m)}$ and $Z(\lambda) \in \mathbb{F}[\lambda]^{y\times x}$ be matrix pencils such that*

$$\operatorname{rank} A(\lambda) = \operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} = n, \tag{18}$$

and such that

$$\operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix} = n + s. \tag{19}$$

Then

$$\begin{bmatrix} A(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix} \tag{20}$$

is strictly equivalent to

$$\left[\begin{array}{cc|c} A(\lambda) & X_1(\lambda) & X_2(\lambda) \\ Y_1(\lambda) & Z_1(\lambda) & Z_2(\lambda) \\ \hline Y_2(\lambda) & Z_3(\lambda) & Z_4(\lambda) \end{array}\right]$$

where

$$\begin{bmatrix} A(\lambda) & X_1(\lambda) \\ Y_1(\lambda) & Z_1(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p+s)\times(n+m+s)} \tag{21}$$

has rank equal to $n + s$, and has the same column and row minimal indices as $A(\lambda)$, respectively.

In particular, if $s = x = y$, then column and row minimal indices of $A(\lambda)$ and (20) coincide, respectively.

Proof. Let us denote by h the sum of degrees of homogeneous invariant factors of $A(\lambda)$, by c the sum of its column minimal indices, and by r the sum of its row minimal indices.

We start by putting $A(\lambda)$ in its Kronecker canonical form. Hence there exist invertible matrices $\bar{P} \in \mathbb{F}^{(n+p)\times(n+p)}$ and $\bar{Q} \in \mathbb{F}^{(n+m)\times(n+m)}$ such that

$$\begin{bmatrix} \bar{P} & 0 \\ 0 & I_y \end{bmatrix}\begin{bmatrix} A(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix}\begin{bmatrix} \bar{Q} & 0 \\ 0 & I_x \end{bmatrix} = \left[\begin{array}{ccc|c} N(\lambda) & 0 & 0 & \hat{X}_1(\lambda) \\ 0 & C(\lambda) & 0 & \hat{X}_2(\lambda) \\ 0 & 0 & R(\lambda) & \hat{X}_3(\lambda) \\ \hline \hat{Y}_1(\lambda) & \hat{Y}_2(\lambda) & \hat{Y}_3(\lambda) & Z(\lambda) \end{array}\right]. \tag{22}$$

Since (18) holds, by Remark 3.5 there exist matrices $P_1 \in \mathbb{F}^{y\times c}$ and $Q_1 \in \mathbb{F}^{r\times x}$ such that

$$\begin{bmatrix} I_h & 0 & 0 & 0 \\ 0 & I_c & 0 & 0 \\ 0 & 0 & I_{r+p} & 0 \\ 0 & P_1 & 0 & I_y \end{bmatrix}\left[\begin{array}{ccc|c} N(\lambda) & 0 & 0 & \hat{X}_1(\lambda) \\ 0 & C(\lambda) & 0 & \hat{X}_2(\lambda) \\ 0 & 0 & R(\lambda) & \hat{X}_3(\lambda) \\ \hline \hat{Y}_1(\lambda) & \hat{Y}_2(\lambda) & \hat{Y}_3(\lambda) & Z(\lambda) \end{array}\right]\begin{bmatrix} I_h & 0 & 0 & 0 \\ 0 & I_{c+m} & 0 & 0 \\ 0 & 0 & I_r & Q_1 \\ 0 & 0 & 0 & I_x \end{bmatrix} =$$

$$= \left[\begin{array}{ccc|c} N(\lambda) & 0 & 0 & \hat{X}_1(\lambda) \\ 0 & C(\lambda) & 0 & \hat{X}_2(\lambda) \\ 0 & 0 & R(\lambda) & 0 \\ \hline \hat{Y}_1(\lambda) & 0 & \hat{Y}_3(\lambda) & \hat{Z}(\lambda) \end{array}\right]. \tag{23}$$

We note that the subpencil

$$\left[\begin{array}{ccc|c} N(\lambda) & 0 & 0 & \hat{X}_1(\lambda) \\ 0 & C(\lambda) & 0 & \hat{X}_2(\lambda) \end{array}\right]$$

is quasi-regular, i.e. it has only homogeneous invariant factors and column minimal indices as the Kronecker invariants.

Thus, by applying Lemma 3.4 on the pencil (23) we have that there exist matrices $P_1 \in \mathbb{F}^{h\times(r+p)}$, $P_2 \in \mathbb{F}^{c\times(r+p)}$, $P_3 \in \mathbb{F}^{y\times(r+p)}$, $P_4 \in \mathbb{F}^{h\times r}$, $P_5 \in \mathbb{F}^{(c+m)\times r}$ and $P_6 \in \mathbb{F}^{x\times r}$, and an invertible matrix $P \in \mathbb{F}^{y\times y}$, such that

$$\begin{bmatrix} I_h & 0 & P_1 & 0 \\ 0 & I_c & P_2 & 0 \\ 0 & 0 & I_{r+p} & 0 \\ 0 & 0 & P_3 & P \end{bmatrix}\left[\begin{array}{ccc|c} N(\lambda) & 0 & 0 & \hat{X}_1(\lambda) \\ 0 & C(\lambda) & 0 & \hat{X}_2(\lambda) \\ 0 & 0 & R(\lambda) & 0 \\ \hline \hat{Y}_1(\lambda) & 0 & \hat{Y}_3(\lambda) & \hat{Z}(\lambda) \end{array}\right]\begin{bmatrix} I_h & 0 & P_4 & 0 \\ 0 & I_{c+m} & P_5 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & P_6 & I_x \end{bmatrix} =$$

$$\left[\begin{array}{ccc|c} N(\lambda) & 0 & 0 & \hat{X}_1(\lambda) \\ 0 & C(\lambda) & 0 & \hat{X}_2(\lambda) \\ 0 & 0 & R(\lambda) & 0 \\ \hline P\hat{Y}_1(\lambda) & 0 & \begin{array}{c} 0 \\ \hline \bar{Y}_3(\lambda) \end{array} & P\hat{Z}(\lambda) \end{array}\right], \tag{24}$$

and such that the subpencil of (24) made of its first $n + p + s$ rows has rank equal to $n + s$. Here the number of rows in the zero matrix in the subpencil $\left[\begin{array}{c} 0 \\ \hline \bar{Y}_3(\lambda) \end{array}\right]$ is exactly s.

Hence, if we denote

$$P\hat{Y}_1(\lambda) = \left[\begin{array}{c} \bar{Y}_1(\lambda) \\ \hline \bar{Y}_2(\lambda) \end{array}\right], \quad \bar{Y}_1(\lambda) \in \mathbb{F}[\lambda]^{s\times h},$$

and

$$P\hat{Z}(\lambda) = \left[\begin{array}{c} \hat{Z}_1(\lambda) \\ \hline \hat{Z}_2(\lambda) \end{array}\right], \quad \hat{Z}_1(\lambda) \in \mathbb{F}[\lambda]^{s\times x},$$

(24) becomes

$$\left[\begin{array}{ccc|c} N(\lambda) & 0 & 0 & \hat{X}_1(\lambda) \\ 0 & C(\lambda) & 0 & \hat{X}_2(\lambda) \\ 0 & 0 & R(\lambda) & 0 \\ \hline \bar{Y}_1(\lambda) & 0 & 0 & \hat{Z}_1(\lambda) \\ \hline \bar{Y}_2(\lambda) & 0 & \bar{Y}_3(\lambda) & \hat{Z}_2(\lambda) \end{array}\right], \tag{25}$$

where

$$\operatorname{rank}\left[\begin{array}{ccc|c} N(\lambda) & 0 & 0 & \hat{X}_1(\lambda) \\ 0 & C(\lambda) & 0 & \hat{X}_2(\lambda) \\ 0 & 0 & R(\lambda) & 0 \\ \hline \bar{Y}_1(\lambda) & 0 & 0 & \hat{Z}_1(\lambda) \end{array}\right] = n + s. \tag{26}$$

Now, we shall apply the transposed version of Lemma 3.4 on the pencil in (26). So, there exist matrices $Q_1 \in \mathbb{F}^{c\times h}$, $Q_2 \in \mathbb{F}^{c\times(r+p)}$, $Q_3 \in \mathbb{F}^{c\times s}$, $Q_4 \in \mathbb{F}^{(c+m)\times h}$, $Q_5 \in \mathbb{F}^{(c+m)\times r}$ and $Q_6 \in \mathbb{F}^{(c+m)\times x}$, and an invertible matrix $Q \in \mathbb{F}^{x\times x}$, such that

$$\begin{bmatrix} I_h & 0 & 0 & 0 \\ Q_1 & I_c & Q_2 & Q_3 \\ 0 & 0 & I_{r+p} & 0 \\ 0 & 0 & 0 & I_s \end{bmatrix}\left[\begin{array}{ccc|c} N(\lambda) & 0 & 0 & \hat{X}_1(\lambda) \\ 0 & C(\lambda) & 0 & \hat{X}_2(\lambda) \\ 0 & 0 & R(\lambda) & 0 \\ \hline \bar{Y}_1(\lambda) & 0 & 0 & \hat{Z}_1(\lambda) \end{array}\right]\begin{bmatrix} I_h & 0 & 0 & 0 \\ Q_4 & I_{c+m} & Q_5 & Q_6 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & Q \end{bmatrix} = \tag{27}$$

$$\left[\begin{array}{ccc|c} N(\lambda) & 0 & 0 & \hat{X}_1(\lambda)Q \\ 0 & C(\lambda) & 0 & 0 \mid \bar{X}_3(\lambda) \\ 0 & 0 & R(\lambda) & 0 \\ \hline \bar{Y}_1(\lambda) & 0 & 0 & \hat{Z}_1(\lambda)Q \end{array}\right]. \tag{28}$$

Here the number of columns of the zero submatrix in the pencil $\left[\begin{array}{c|c} 0 & \bar{X}_3(\lambda) \end{array}\right]$ is exactly s, and the rank of the subpencil of (28) made of its first $n+m+s$ columns is $n+s$.

Hence, if we denote

$$\hat{X}_1(\lambda)Q = \left[\begin{array}{c|c} \bar{X}_1(\lambda) & \bar{X}_2(\lambda) \end{array}\right], \quad \bar{X}_1(\lambda) \in \mathbb{F}[\lambda]^{h\times s},$$

and

$$\hat{Z}_1(\lambda)Q = \left[\begin{array}{c|c} Z_1(\lambda) & Z_2(\lambda) \end{array}\right], \quad Z_1(\lambda) \in \mathbb{F}[\lambda]^{s\times s},$$

the pencil (28) becomes

$$\left[\begin{array}{ccc|c|c} N(\lambda) & 0 & 0 & \bar{X}_1(\lambda) & \bar{X}_2(\lambda) \\ 0 & C(\lambda) & 0 & 0 & \bar{X}_3(\lambda) \\ 0 & 0 & R(\lambda) & 0 & 0 \\ \hline \bar{Y}_1(\lambda) & 0 & 0 & Z_1(\lambda) & Z_2(\lambda) \end{array}\right],$$

and

$$\operatorname{rank}\left[\begin{array}{ccc|c} N(\lambda) & 0 & 0 & \bar{X}_1(\lambda) \\ 0 & C(\lambda) & 0 & 0 \\ 0 & 0 & R(\lambda) & 0 \\ \hline \bar{Y}_1(\lambda) & 0 & 0 & Z_1(\lambda) \end{array}\right] = n+s. \tag{29}$$

Moreover, the pencil in (29) has the same column and row minimal indices as $A(\lambda)$.

Operations (27) transform the pencil (25) into its strictly equivalent form

$$\left[\begin{array}{ccc|c|c} N(\lambda) & 0 & 0 & \bar{X}_1(\lambda) & \bar{X}_2(\lambda) \\ 0 & C(\lambda) & 0 & 0 & \bar{X}_3(\lambda) \\ 0 & 0 & R(\lambda) & 0 & 0 \\ \hline \bar{Y}_1(\lambda) & 0 & 0 & Z_1(\lambda) & Z_2(\lambda) \\ \hline \bar{Y}_2(\lambda) & 0 & \bar{Y}_3(\lambda) & Z_3(\lambda) & Z_4(\lambda) \end{array}\right],$$

where

$$\hat{Z}_2(\lambda)Q = \left[\begin{array}{c|c} Z_3(\lambda) & Z_4(\lambda) \end{array}\right], \quad Z_3(\lambda) \in \mathbb{F}[\lambda]^{(y-s)\times s}.$$

Finally,

$$\begin{bmatrix} \bar{P}^{-1} & 0 \\ 0 & I_y \end{bmatrix} \left[\begin{array}{ccc|c|c} N(\lambda) & 0 & 0 & \bar{X}_1(\lambda) & \bar{X}_2(\lambda) \\ 0 & C(\lambda) & 0 & 0 & \bar{X}_3(\lambda) \\ 0 & 0 & R(\lambda) & 0 & 0 \\ \hline \bar{Y}_1(\lambda) & 0 & 0 & Z_1(\lambda) & Z_2(\lambda) \\ \hline \bar{Y}_2(\lambda) & 0 & \bar{Y}_3(\lambda) & Z_3(\lambda) & Z_4(\lambda) \end{array}\right] \begin{bmatrix} \bar{Q}^{-1} & 0 \\ 0 & I_x \end{bmatrix} = \left[\begin{array}{cc|c} A(\lambda) & X_1(\lambda) & X_2(\lambda) \\ Y_1(\lambda) & Z_1(\lambda) & Z_2(\lambda) \\ \hline Y_2(\lambda) & Z_3(\lambda) & Z_4(\lambda) \end{array}\right],$$

where $X_1(\lambda) = \bar{P}^{-1}\begin{bmatrix} \bar{X}_1(\lambda) \\ 0 \\ 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times s}$, $X_2(\lambda) = \bar{P}^{-1}\begin{bmatrix} \bar{X}_2(\lambda) \\ \bar{X}_3(\lambda) \\ 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(x-s)}$, $Y_1(\lambda) = \begin{bmatrix} \bar{Y}_1(\lambda) & 0 & 0 \end{bmatrix}\bar{Q}^{-1} \in \mathbb{F}[\lambda]^{s\times(n+m)}$, and $Y_2(\lambda) = \begin{bmatrix} \bar{Y}_2(\lambda) & 0 & \bar{Y}_3(\lambda) \end{bmatrix}\bar{Q}^{-1} \in \mathbb{F}[\lambda]^{(y-s)\times(n+m)}$. Here the subpencil

$$\begin{bmatrix} A(\lambda) & X_1(\lambda) \\ Y_1(\lambda) & Z_1(\lambda) \end{bmatrix},$$

is strictly equivalent to (29), hence has the rank equal to $n+s$, and its column and row minimal indices coincide with the column and row minimal indices of $A(\lambda)$, respectively, as wanted. □

In the following lemma we give a generalization to Sá-Thompson's theorem [1, 39, 44]:

Lemma 3.9. *Let $A(\lambda) \in \mathbb{F}^{(n+p)\times(n+m)}$ and $M(\lambda) \in \mathbb{F}^{(n+p+x)\times(n+m+x)}$ be matrix pencils, with $n = \operatorname{rank} A(\lambda)$ and $n+x = \operatorname{rank} M(\lambda)$, such that their column and row minimal indices coincide. Let $\alpha_1|\cdots|\alpha_n$ and $\gamma_1|\cdots|\gamma_{n+x}$ be homogeneous invariant factors of $A(\lambda)$ and $M(\lambda)$, respectively. If*

$$\gamma_i|\alpha_i|\gamma_{i+2x}, \quad i = 1,\dots,n,$$

then there exist pencils $X(\lambda) \in \mathbb{F}^{(n+p)\times x}$, $Y(\lambda) \in \mathbb{F}^{x\times(n+m)}$ and $Z(\lambda) \in \mathbb{F}^{x\times x}$ such that the pencil

$$\begin{bmatrix} A(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix} \tag{30}$$

is strictly equivalent to $M(\lambda)$, and such that

$$\operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \end{bmatrix} = n \quad \text{and} \quad \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} = n.$$

Proof. Let $P \in \mathbb{F}^{(n+p)\times(n+p)}$ and $Q \in \mathbb{F}^{(n+m)\times(n+m)}$ be invertible matrices that put $A(\lambda)$ in its Kronecker canonical form:

$$PA(\lambda)Q = \left[\begin{array}{c|c|c} C(\lambda) & 0 & 0 \\ \hline 0 & R(\lambda) & 0 \\ \hline 0 & 0 & N(\lambda) \end{array}\right].$$

Let $h := \sum_{i=1}^{n} d(\alpha_i)$. Now since

$$\gamma_i|\alpha_i|\gamma_{i+2x}, \quad i = 1,\dots,n$$

by Sá-Thompson's theorem [1, 39, 44] there exist pencils $\bar{X}(\lambda) \in \mathbb{F}[\lambda]^{h\times x}$, $\bar{Y}(\lambda) \in \mathbb{F}[\lambda]^{x\times h}$, and $\bar{Z}(\lambda) \in \mathbb{F}[\lambda]^{x\times x}$ such that the pencil

$$\left[\begin{array}{cc|cc} C(\lambda) & 0 & 0 & 0 \\ 0 & R(\lambda) & 0 & 0 \\ \hline 0 & 0 & N(\lambda) & \bar{X}(\lambda) \\ 0 & 0 & \bar{Y}(\lambda) & \bar{Z}(\lambda) \end{array}\right]$$

is strictly equivalent to $M(\lambda)$. Thus

$$\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{ccc|c} C(\lambda) & 0 & 0 & 0 \\ 0 & R(\lambda) & 0 & 0 \\ 0 & 0 & N(\lambda) & \bar{X}(\lambda) \\ \hline 0 & 0 & \bar{Y}(\lambda) & \bar{Z}(\lambda) \end{array}\right] \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix}$$

has the form (30), with

$$X(\lambda) = P^{-1}\begin{bmatrix} 0 \\ \bar{X}(\lambda) \end{bmatrix}, \quad Y(\lambda) = \begin{bmatrix} 0 & \bar{Y}(\lambda) \end{bmatrix} Q^{-1}, \quad Z(\lambda) = \bar{Z}(\lambda).$$

In addition,

$$\operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \end{bmatrix} = n, \quad \text{and} \quad \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} = n,$$

as wanted. □

4. Problem 1.1 under restrictions (1)

Now we can pass to solving Problem 1.1. We start by resolving it under restriction (1). To that aim we shall use some additional notation and previous results given in the following subsection:

4.1. Additional notation

Let us consider matrix pencils $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ and $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+y)\times(n+m+x)}$ as given in Section 2.3. Let

$$\max\{x, y\} \leq s \leq x + y.$$

Let

$$h_j = \min\{i | d_{i-j+1} < c_i\}, \quad j = 1, \dots, s - x, \quad h_0 = 0,$$

and

$$v_k = \min\{i | \bar{r}_{i-k+1} < r_i\}, \quad k = 1, \dots, s - y, \quad v_0 = 0.$$

Then we define

$$\hat{x}_j := \sum_{i=h_j+1}^{m} (c_i + 1) - \sum_{i=h_j-j+1}^{m+x-s} (d_i + 1), \quad j = 0, \dots, s - x,$$

and

$$\hat{y}_k := \sum_{i=v_k+1}^{p} (r_i + 1) - \sum_{i=v_k-k+1}^{p+y-s} (\bar{r}_i + 1), \quad k = 0, \dots, s - y.$$

By using this notation we also cite [9, Theorem 4.1], since it will be used further on in this section:

Theorem 4.1. *[9, Theorem 4.1] Let $A(\lambda)$ and $M(\lambda)$ be matrix pencils given in Section 2.3, with*

$$s = x + y.$$

There exist pencils $X(\lambda)$, $Y(\lambda)$, and $Z(\lambda)$, such that the pencil

$$\begin{bmatrix} A(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix}$$

is strictly equivalent to $M(\lambda)$ if and only if

$$\bar{r}_i \geq r_{i+x}, \quad i = 1, \dots, p - y, \quad \text{and} \quad d_i \geq c_{i+y}, \quad i = 1, \dots, m - y, \tag{31}$$

$$\gamma_i | \alpha_i | \gamma_{i+x+y}, \quad i = 1, \dots, n, \tag{32}$$

$$\sum_{i=1}^{n+x+y-k-j} d(\operatorname{lcm}(\alpha_{i-x-y+k+j}, \gamma_i)) - \sum_{i=1}^{n} d(\alpha_i) \leq \hat{x}_j + \hat{y}_k, \quad j = 0, \dots, y, \quad \text{and} \quad k = 0, \dots, x. \tag{33}$$

Remark 4.2. *We note that the restriction $s = x + y$ in Theorem 4.1 is equivalent to the following double rank restrictions:*

$$\operatorname{rank} \begin{bmatrix} A(\lambda) & X(\lambda) \end{bmatrix} = n + x, \text{ and } \operatorname{rank} \begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} = n + y.$$

4.2. A solution to Problem 1.1 under restriction (1)

In the proof of Problem 1.1 under restriction (1) we shall use the results from Theorem 4.1, the classical Sá-Thompson's result on interlacing inequalities [1, 39, 44], Lemmas 3.3, 3.7 and 3.9, and the notation from Sections 2.3 and 4.1. The solution is given in the following theorem:

Theorem 4.3. *Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ and $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+y)\times(n+m+x)}$ be matrix pencils as given in Section 2.3. There exist pencils $X(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times x}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{y\times(n+m)}$ and $Z(\lambda) \in \mathbb{F}[\lambda]^{y\times x}$ such that*

$$\operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \end{bmatrix} = n+s-y, \text{ and } \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} = n+s-x, \tag{34}$$

and such that the pencil

$$\begin{bmatrix} A(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix} \tag{35}$$

is strictly equivalent to $M(\lambda)$, if and only if

(*o*) $\max\{x, y\} \leq s \leq x + y$

(*i*) $\bar{r}_i \geq r_{i+s-y}, \quad i = 1, \dots, p+y-s,$

(*ii*) $d_i \geq c_{i+s-x}, \quad i = 1, \dots, m+x-s,$

(*iii*) $\gamma_i | \alpha_i | \gamma_{i+x+y}, \quad i = 1, \dots, n,$

(*iv*) $\sum_{i=1}^{n+2s-x-y-k-j} d(\operatorname{lcm}(\alpha_{i-2s+x+y+k+j}, \gamma_i)) - \sum_{i=1}^{n} d(\alpha_i) \leq \hat{x}_j + \hat{y}_k, \quad$ *for all $j = 0, \dots, s-x, \quad k = 0, \dots, s-y$.*

Proof. Necessity: Let us suppose that there exist pencils $X(\lambda)$, $Y(\lambda)$, and $Z(\lambda)$, such that (34) is satisfied, and such that (35) is strictly equivalent to $M(\lambda)$.

Conditions (34) imply that $s \geq \max\{x, y\}$. Also, from the size and ranks of the pencils $A(\lambda)$ and $M(\lambda)$, we have that $s \leq x + y$. Altogether we have (*o*).

By (34) and by applying Lemma 3.3 first on the pencil $\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix}$, and then the transposed version of Lemma 3.3 on $\left[\begin{array}{c|c} A(\lambda) & X(\lambda) \end{array}\right]$, we get that (35) is strictly equivalent to

$$\left[\begin{array}{c|c|c} A(\lambda) & X_1(\lambda) & X_2(\lambda) \\ \hline Y_1(\lambda) & Z_1(\lambda) & Z_2(\lambda) \\ \hline Y_2(\lambda) & Z_3(\lambda) & Z_4(\lambda) \end{array}\right], \tag{36}$$

where $X_1(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(s-y)}$ and $Y_1(\lambda) \in \mathbb{F}[\lambda]^{(s-x)\times(n+m)}$,

$$\operatorname{rank}\left[\begin{array}{c|c} A(\lambda) & X_1(\lambda) \end{array}\right] = \left[\begin{array}{c|c|c} A(\lambda) & X_1(\lambda) & X_2(\lambda) \end{array}\right] = n+s-y, \tag{37}$$

and

$$\operatorname{rank}\left[\begin{array}{c} A(\lambda) \\ \hline Y_1(\lambda) \end{array}\right] = \left[\begin{array}{c} A(\lambda) \\ \hline Y_1(\lambda) \\ \hline Y_2(\lambda) \end{array}\right] = n+s-x. \tag{38}$$

Next, let us consider the subpencil

$$\left[\begin{array}{c|c} A(\lambda) & X_1(\lambda) \\ \hline Y_1(\lambda) & Z_1(\lambda) \end{array}\right] \in \mathbb{F}[\lambda]^{(n+p+s-x)\times(n+m+s-y)}. \tag{39}$$

By Lemma 3.6 the rank of (39) is equal to $n+2s-x-y$. Since the rank of (36) is $n+s$, and (37) and (38) hold, by Lemma 3.7 we get

$$\operatorname{rank}\left[\begin{array}{c|c|c} A(\lambda) & X_1(\lambda) & X_2(\lambda) \\ \hline Y_1(\lambda) & Z_1(\lambda) & Z_2(\lambda) \end{array}\right] = n+2s-x-y, \tag{40}$$

and

$$\operatorname{rank}\left[\begin{array}{c|c} A(\lambda) & X_1(\lambda) \\ \hline Y_1(\lambda) & Z_1(\lambda) \\ \hline Y_2(\lambda) & Z_3(\lambda) \end{array}\right] = n+2s-x-y. \tag{41}$$

By (40), (41), and the facts that the rank of (39) is $n+2s-x-y$, and the rank of (36) is $n+s$, by the second part of Lemma 3.8, we have that both the column and row minimal indices of (39) and (36) coincide, respectively.

Let us denote the homogeneous invariant factors of (39) by

$$\beta_1 | \cdots | \beta_{n+2s-x-y}.$$

Then by the ranks of the pencils $A(\lambda)$ and (39) we have

$$\sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{m}(c_i+1) + \sum_{i=1}^{p}(r_i+1) = \sum_{i=1}^{n+2s-x-y} d(\beta_i) + \sum_{i=1}^{m+x-s}(d_i+1) + \sum_{i=1}^{p+y-s}(\bar{r}_i+1).$$

By applying Sá-Thompson's result [1, 39, 44] on the completion of (39) up to (36), we obtain

$$\gamma_i \mid \beta_i \mid \gamma_{i+2(x+y-s)}, \quad i=1,\ldots,n+2s-x-y. \tag{42}$$

The completion of $A(\lambda)$ up to (39) is a minimal completion Case I, since it is a completion by exactly $s-x$ rows and $s-y$ columns (for all details see [9]). Hence, by applying Theorem 4.1, i.e. [9, Theorem 4.1], to this completion we get that the following holds:

$$\bar{r}_i \geq r_{i+s-y}, \quad i=1,\ldots,p+y-s, \tag{43}$$

$$d_i \geq c_{i+s-x}, \quad i=1,\ldots,m+x-s, \tag{44}$$

$$\beta_i | \alpha_i | \beta_{i+2s-x-y}, \quad i=1,\ldots,n, \tag{45}$$

$$\sum_{i=1}^{n+2s-x-y-k-j} d(\operatorname{lcm}(\alpha_{i-2s+x+y+k+j},\beta_i)) - \sum_{i=1}^{n} d(\alpha_i) \leq \hat{x}_j + \hat{y}_k, \tag{46}$$

for all $j=0,\ldots,s-x$, and $k=0,\ldots,s-y$.

Conditions (43) and (44) are (*i*) and (*ii*), respectively. Also, (42) and (45) together give (*iii*). We are left with proving condition (*iv*). Since

$$\operatorname{lcm}(\alpha_{i-2s+x+y},\gamma_i) \mid \beta_i, \quad i=1,\ldots,n+2s-x-y,$$

we have

$$\sum_{i=1}^{n+2s-x-y-k-j} d(\operatorname{lcm}(\alpha_{i-2s+x+y+k+j},\gamma_i)) \leq \sum_{i=1}^{n+2s-x-y-k-j} d(\operatorname{lcm}(\alpha_{i-2s+x+y+k+j},\beta_i)), \tag{47}$$

for all $j=0,\ldots,s-x$, $k=0,\ldots,s-y$. Hence (46) implies (*iv*).

Sufficiency:

Let us assume that conditions $(o)-(iv)$ are valid. Let us denote by

$$\beta_i := \mathrm{lcm}(\alpha_{i-2s+x+y}, \gamma_i), \quad i = 1, \dots, n+2s-x-y-1,$$

and let

$$\beta_{n+2s-x-y} := \psi \, \mathrm{lcm}(\alpha_n, \gamma_{n+2s-x-y}),$$

where ψ is a monic polynomial with the degree, $d(\psi)$, equal to

$$\sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{m}(c_i+1) + \sum_{i=1}^{p}(r_i+1) - \sum_{i=1}^{m+x-s}(d_i+1) - \sum_{i=1}^{p+y-s}(\bar{r}_i+1) - \sum_{i=1}^{n+2s-x-y} d(\mathrm{lcm}(\alpha_{i-2s+x+y}, \gamma_i)). \tag{48}$$

By (iv) for $j=0$ and $k=0$, we get $d(\psi) \geq 0$, i.e. β_i's are well defined. With such defined β_i's, condition (iv) gives (46).

Also, by $(o)-(iv)$ and from the definition of β_i, we have that (42) and (45) are satisfied. Indeed, from the definition of β_i we have,

$$\alpha_i \mid \beta_{i+2s-x-y}, \quad i = 1, \dots, n, \quad \text{and} \quad \gamma_i \mid \beta_i, \quad i = 1, \dots, n+2s-x-y,$$

and together with (iii):

$$\beta_i \mid \alpha_i, \quad i = 1, \dots, n+2s-x-y-1, \quad \text{and} \quad \beta_i \mid \gamma_{i+2(x+y-s)}, \quad i = 1, \dots, n+2s-x-y-1.$$

If $n+2s-x-y > n$ (i.e. by (o), if $s > x$ or $s > y$), we automatically have $\beta_i | \alpha_i$, for $i = 1, \dots, n$. If $n+2s-x-y = n$, then $s = x = y$, and $d(\psi) = 0$. Indeed, conditions (i), (ii), and condition (iv) for $j = 0$ and $k = 0$, give $r_i = \bar{r}_i$, $i = 1, \dots, p$, and $c_i = d_i$, $i = 1, \dots, m$. So condition (iii) implies (45).

Similarly, if $s < x+y$, we have that by convention $\gamma_{n+x+y} = 0$, and so $\beta_i \mid \gamma_{i+2(x+y-s)}$, $\quad i = 1, \dots, n+2s-x-y$ is automatically satisfied. If $s = x+y$, condition (iii) gives $\mathrm{lcm}(\alpha_{i-2s+x+y}, \gamma_i) = \mathrm{lcm}(\alpha_{i-x-y}, \gamma_i) = \gamma_i$. The last equality, together with (10) and (11), gives that $d(\psi) = 0$, i.e. in this case we also have $\psi = 1$. And so, again by (iii) we have that (42) is satisfied.

Hence, by Theorem 4.1 conditions (i), (ii), (45) and (46), together with (48) give the existence of pencils $X_1(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(s-y)}$, $Y_1(\lambda) \in \mathbb{F}[\lambda]^{(s-x)\times(n+m)}$, and $Z_1(\lambda) \in \mathbb{F}[\lambda]^{(s-x)\times(s-y)}$, such that

$$\left[\begin{array}{c|c} A(\lambda) & X_1(\lambda) \\ \hline Y_1(\lambda) & Z_1(\lambda) \end{array}\right] \tag{49}$$

has $\beta_1 | \cdots | \beta_{n+2s-x-y}$ as homogeneous invariant factors, $d_1 \geq \cdots \geq d_{m+x-s}$ as column minimal indices, and $\bar{r}_1 \geq \cdots \geq \bar{r}_{p+y-s}$ as row minimal indices, and such that

$$\mathrm{rank}\left[\begin{array}{c|c} A(\lambda) & X_1(\lambda) \end{array}\right] = n+s-y, \quad \text{and} \quad \mathrm{rank}\left[\begin{array}{c} A(\lambda) \\ \hline Y_1(\lambda) \end{array}\right] = n+s-x. \tag{50}$$

Also, condition (42) by Lemma 3.9 implies the existence of matrix pencils $X_2(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(x+y-s)}$, $Y_2(\lambda) \in \mathbb{F}[\lambda]^{(x+y-s)\times(n+m)}$, and $Z_2(\lambda) \in \mathbb{F}[\lambda]^{(s-y)\times(x+y-s)}$, $Z_3(\lambda) \in \mathbb{F}[\lambda]^{(x+y-s)\times(s-y)}$, and $Z_4(\lambda) \in \mathbb{F}[\lambda]^{(x+y-s)\times(x+y-s)}$, such that

$$\left[\begin{array}{c|c|c} A(\lambda) & X_1(\lambda) & X_2(\lambda) \\ \hline Y_1(\lambda) & Z_1(\lambda) & Z_2(\lambda) \\ \hline Y_2(\lambda) & Z_3(\lambda) & Z_4(\lambda) \end{array}\right], \tag{51}$$

is strictly equivalent to $M(\lambda)$, and such that

$$\mathrm{rank}\left[\begin{array}{cc|c} A(\lambda) & X_1(\lambda) & X_2(\lambda) \\ Y_1(\lambda) & Z_1(\lambda) & Z_2(\lambda) \end{array}\right] = n+2s-x-y, \tag{52}$$

and

$$\operatorname{rank}\left[\begin{array}{cc} A(\lambda) & X_1(\lambda) \\ Y_1(\lambda) & Z_1(\lambda) \\ \hline Y_2(\lambda) & Z_3(\lambda) \end{array}\right] = n + 2s - x - y. \tag{53}$$

Altogether, since the pencil (49) has rank equal to $n + 2s - x - y$, by Lemma 3.7 we conclude that

$$\operatorname{rank}\left[\begin{array}{c|cc} A(\lambda) & X_1(\lambda) & X_2(\lambda) \end{array}\right] = \operatorname{rank}\left[\begin{array}{c|c} A(\lambda) & X_1(\lambda) \end{array}\right] = n + s - y,$$

and

$$\operatorname{rank}\left[\begin{array}{c} A(\lambda) \\ \hline Y_1(\lambda) \\ Y_2(\lambda) \end{array}\right] = \operatorname{rank}\left[\begin{array}{c} A(\lambda) \\ Y_1(\lambda) \end{array}\right] = n + s - x,$$

as wanted. This finishes the proof. □

5. Problem 1.1 under restrictions (2)

5.1. Additional notation

Let us consider matrix pencils $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ and $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+y)\times(n+m+x)}$ as given in Section 2.3. Let

$$0 \le s \le \min\{x, y\}.$$

Let

$$\bar{h}_j = \min\{i | c_{i-j+1} < d_i\}, \quad j = 1, \ldots, x - s, \quad \bar{h}_0 = 0,$$

and

$$\bar{v}_k = \min\{i | r_{i-k+1} < \bar{r}_i\}, \quad k = 1, \ldots, y - s, \quad \bar{v}_0 = 0.$$

Then we define

$$\bar{x}_j := \sum_{i=\bar{h}_j+1}^{m+x-s} d_i - \sum_{i=\bar{h}_j-j+1}^{m} c_i, \quad j = 0, \ldots, x - s,$$

and

$$\bar{y}_k := \sum_{i=\bar{v}_k+1}^{p+y-s} \bar{r}_i - \sum_{i=\bar{v}_k-k+1}^{p} r_i, \quad k = 0, \ldots, y - s.$$

We also cite [9, Theorem 4.2] in this notation, since it will be used further on in this section:

Theorem 5.1. *[9, Theorem 4.2] Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ and $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+y)\times(n+m+x)}$ be matrix pencils as given in Section 2.3, with*

$$s = 0.$$

There exist pencils $X(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times x}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{y\times(n+m)}$, and $Z(\lambda) \in \mathbb{F}[\lambda]^{y\times x}$, such that the pencil

$$\left[\begin{array}{c|c} A(\lambda) & X(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array}\right]$$

is strictly equivalent to $M(\lambda)$ if and only if

$$\bar{\rho} \ge \rho, \quad \text{and} \quad \bar{\theta} \ge \theta, \tag{54}$$

$$r_i \ge \bar{r}_{i+y}, \quad i = 1, \ldots, p, \quad \text{and} \quad c_i \ge d_{i+x}, \quad i = 1, \ldots, m, \tag{55}$$

$$\gamma_i | \alpha_i | \gamma_{i+x+y}, \quad i = 1, \ldots, n, \tag{56}$$

$$\sum_{i=1}^{n} d(\operatorname{lcm}(\alpha_{i-j-k}, \gamma_i)) - \sum_{i=1}^{n} d(\gamma_i) \le \bar{x}_j + \bar{y}_k, \quad \text{for all} \quad j = 0, \ldots, x, \quad k = 0, \ldots, y. \tag{57}$$

5.2. A solution to Problem 1.1 under restriction (2)

In the following theorem we solve Problem 1.1 under restriction (2). In the solution we use the notation from Sections 2.3 and 5.1, and the results of Theorem 5.1, the classical Sá-Thompson's theorem [1, 39, 44], and Lemmas 3.7, 3.8 and 3.9.

Theorem 5.2. *Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ and $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+y)\times(n+m+x)}$ be matrix pencils as given in Section 2.3. There exist pencils $X(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times x}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{y\times(n+m)}$ and $Z(\lambda) \in \mathbb{F}[\lambda]^{y\times x}$ such that*

$$\operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \end{bmatrix} = n, \quad \textit{and} \quad \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} = n, \tag{58}$$

and such that the pencil

$$\begin{bmatrix} A(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix} \tag{59}$$

is strictly equivalent to $M(\lambda)$, if and only if

$$\begin{array}{ll} (o) & 0 \le s \le \min\{x, y\} \\ (i) & \bar{\rho} \ge \rho, \quad \textit{and} \quad \bar{\theta} \ge \theta, \\ (ii) & r_i \ge \bar{r}_{i+y-s}, \quad i = 1, \dots, p, \\ (iii) & c_i \ge d_{i+x-s}, \quad i = 1, \dots, m, \\ (iv) & \gamma_i | \alpha_i | \gamma_{i+x+y}, \quad i = 1, \dots, n, \\ (v) & \sum_{i=1}^{n+s} d(\operatorname{lcm}(\alpha_{i-2s-j-k}, \gamma_i)) - \sum_{i=1}^{n+s} d(\gamma_i) \le \bar{x}_j + \bar{y}_k, \quad \textit{for all} \quad j = 0, \dots, x-s, \quad k = 0, \dots, y-s. \end{array}$$

Proof. Necessity: Let us suppose that there exist pencils $X(\lambda)$, $Y(\lambda)$, and $Z(\lambda)$, such that (58) is satisfied, and such that (59) is strictly equivalent to $M(\lambda)$.

By condition (58), we directly obtain (o).

By Lemma 3.8, (59) is strictly equivalent to the pencil

$$\left[\begin{array}{c|c|c} A(\lambda) & X_1(\lambda) & X_2(\lambda) \\ \hline Y_1(\lambda) & Z_1(\lambda) & Z_2(\lambda) \\ \hline Y_2(\lambda) & Z_3(\lambda) & Z_4(\lambda) \end{array}\right] \tag{60}$$

whose subpencil

$$\left[\begin{array}{c|c} A(\lambda) & X_1(\lambda) \\ \hline Y_1(\lambda) & Z_1(\lambda) \end{array}\right] \in \mathbb{F}[\lambda]^{(n+p+s)\times(n+m+s)} \tag{61}$$

has the rank equal to $n + s$, and has the same column and row minimal indices as $A(\lambda)$, respectively. Let us denote the homogeneous invariant factors of (61) by

$$\beta_1 | \cdots | \beta_{n+s}.$$

By applying Sá-Thompson's result [1, 39, 44] on the completion from $A(\lambda)$ up to (61) we obtain

$$\beta_i \mid \alpha_i \mid \beta_{i+2s}, \quad i = 1, \dots, n. \tag{62}$$

Also, from the ranks of the pencils (60) and (61), we get

$$\sum_{i=1}^{n+s} d(\beta_i) + \sum_{i=1}^{m} c_i + \sum_{i=1}^{p} r_i = \sum_{i=1}^{n+s} d(\gamma_i) + \sum_{i=1}^{m+x-s} d_i + \sum_{i=1}^{p+y-s} \bar{r}_i.$$

The completion of (61) up to (60) is a minimal completion Case II, since it is a completion by exactly $y-s$ rows and $x-s$ columns (for all details see [9]). Hence, by applying Theorem 5.1, i.e. [9, Theorem 4.2], to this completion we get that the following conditions hold:

$$\bar{\rho} \geq \rho, \quad \text{and} \quad \bar{\theta} \geq \theta, \tag{63}$$

$$r_i \geq \bar{r}_{i+y-s}, \quad i = 1, \ldots, p, \tag{64}$$

$$c_i \geq d_{i+x-s}, \quad i = 1, \ldots, m, \tag{65}$$

$$\gamma_i | \beta_i | \gamma_{i+x+y-2s}, \quad i = 1, \ldots, n+s, \tag{66}$$

$$\sum_{i=1}^{n+s} d(\operatorname{lcm}(\beta_{i-j-k}, \gamma_i)) - \sum_{i=1}^{n+s} d(\gamma_i) \leq \bar{x}_j + \bar{y}_k, \quad \text{for all} \quad j = 0, \ldots, x-s, \quad k = 0, \ldots, y-s. \tag{67}$$

Conditions (63), (64) and (65) are (*i*), (*ii*), and (*iii*) respectively. Also, (66) and (62) together give (*iv*), and we are left with proving condition (*v*). Since

$$\operatorname{lcm}(\alpha_{i-2s}, \gamma_i) \mid \beta_i, \quad i = 1, \ldots, n+s,$$

we have

$$\sum_{i=1}^{n+s} d(\operatorname{lcm}(\alpha_{i-2s-j-k}, \gamma_i)) \leq \sum_{i=1}^{n+s} d(\operatorname{lcm}(\beta_{i-j-k}, \gamma_i)), \quad \text{for all} \quad j = 0, \ldots, x-s, \quad k = 0, \ldots, y-s. \tag{68}$$

Hence (67) implies (*v*).

Sufficiency:
Let us assume that conditions (*o*) – (*v*) are valid. Let us denote by

$$\beta_i := \operatorname{lcm}(\alpha_{i-2s}, \gamma_i), \quad i = 1, \ldots, n+s-1,$$

and let

$$\beta_{n+s} := \bar{\psi} \operatorname{lcm}(\alpha_{n-s}, \gamma_{n+s}),$$

where $\bar{\psi}$ is a monic polynomial such that its degree $d(\bar{\psi})$ is equal to

$$\sum_{i=1}^{n+s} d(\gamma_i) + \sum_{i=1}^{m+x-s} d_i + \sum_{i=1}^{p+y-s} \bar{r}_i - \sum_{i=1}^{m} c_i - \sum_{i=1}^{p} r_i - \sum_{i=1}^{n+s} \operatorname{lcm}(\alpha_{i-2s}, \gamma_i). \tag{69}$$

By (*v*) for $j = 0$ and $k = 0$, we get $d(\bar{\psi}) \geq 0$, and so the β_i's are well defined.
For such defined β_i's, condition (*v*) gives (67).

Furthermore, condition (*iv*) for such defined $\beta_1 | \cdots | \beta_{n+s}$ implies (62) and (66). Indeed, by the definition of β_i we have

$$\gamma_i | \beta_i, \quad i = 1, \ldots, n+s, \quad \text{and} \quad \alpha_i \mid \beta_{i+2s}, \quad i = 1, \ldots, n,$$

and together with (*iv*)

$$\beta_i | \gamma_{i+x+y-2s}, \quad i = 1, \ldots, n+s-1, \quad \text{and} \quad \beta_i \mid \alpha_i, \quad i = 1, \ldots, n+s-1.$$

If $s > 0$, we automatically have $\beta_i | \alpha_i$, for $i = 1, \ldots, n$. If $s = 0$, (*iv*) implies $\operatorname{lcm}(\alpha_{i-2s}, \gamma_i) = \alpha_i$, $i = 1, \ldots, n$. Hence by (10) and (11) we get that $d(\bar{\psi}) = 0$, i.e. in this case $\bar{\psi} := 1$, and so (62) follows from (*iv*).

Analogously, if $x + y - 2s > 0$, we have by convention that $\gamma_{n+x+y-s} = 0$, and so $\beta_{n+s} | \gamma_{n+x+y-s}$. If $x + y - 2s \leq 0$, by condition (*o*), we have $s = x = y$. Then by (*iv*) we have $\operatorname{lcm}(\alpha_{i-2s}, \gamma_i) = \gamma_i$, $i = 1, \ldots, n+s$. Also, in this case conditions (*ii*), (*iii*), and (*v*) for $j = 0$ and $k = 0$, together give $c_i = d_i$, $i = 1, \ldots, m$, and

$r_i = \bar{r}_i$, $i = 1, \dots, p$. Altogether we conclude that $d(\bar{\psi}) = 0$, i.e. in this case $\bar{\psi} := 1$, and so (67) follows from (*iv*).

So, by Lemma 3.9 and condition (62), there exist pencils $X_1(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times s}$, $Y_1(\lambda) \in \mathbb{F}[\lambda]^{s\times(n+m)}$, and $Z_1(\lambda) \in \mathbb{F}[\lambda]^{s\times s}$, such that

$$\left[\begin{array}{c|c} A(\lambda) & X_1(\lambda) \\ \hline Y_1(\lambda) & Z_1(\lambda) \end{array}\right] \in \mathbb{F}[\lambda]^{(n+p+s)\times(n+m+s)} \tag{70}$$

has $\beta_1 | \cdots | \beta_{n+s}$ as homogeneous invariant factors, $c_1 \geq \cdots \geq c_m$ as column minimal indices and $r_1 \geq \cdots \geq r_p$ as row minimal indices, and such that

$$\operatorname{rank}\begin{bmatrix} A(\lambda) & X_1(\lambda) \end{bmatrix} = n \quad \text{and} \quad \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y_1(\lambda) \end{bmatrix} = n.$$

Hence, by Theorem 5.1, conditions (*i*), (*ii*), (*iii*), (66) and (67), together with (69), give the existence of pencils $X_2(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(x-s)}$, $Y_2(\lambda) \in \mathbb{F}[\lambda]^{(y-s)\times(n+m)}$, $Z_2(\lambda) \in \mathbb{F}[\lambda]^{s\times(x-s)}$, $Z_3(\lambda) \in \mathbb{F}[\lambda]^{(y-s)\times s}$, and $Z_4(\lambda) \in \mathbb{F}[\lambda]^{(y-s)\times(x-s)}$, such that the matrix pencil

$$\left[\begin{array}{c|c|c} A(\lambda) & X_1(\lambda) & X_2(\lambda) \\ \hline Y_1(\lambda) & Z_1(\lambda) & Z_2(\lambda) \\ \hline Y_2(\lambda) & Z_3(\lambda) & Z_4(\lambda) \end{array}\right] \tag{71}$$

has $\gamma_1 | \cdots | \gamma_{n+s}$ as homogeneous invariant factors, $d_1 \geq \cdots \geq d_{m+x-s}$ as column minimal indices and $\bar{r}_1 \geq \cdots \geq \bar{r}_{p+y-s}$ as row minimal indices. Since the rank of (70) and the rank of (71) are both equal to $n+s$, we also have

$$\operatorname{rank}\left[\begin{array}{cc|c} A(\lambda) & X_1(\lambda) & X_2(\lambda) \\ Y_1(\lambda) & Z_1(\lambda) & Z_2(\lambda) \end{array}\right] = n+s, \quad \text{and} \quad \operatorname{rank}\left[\begin{array}{cc} A(\lambda) & X_1(\lambda) \\ Y_1(\lambda) & Z_1(\lambda) \\ \hline Y_2(\lambda) & Z_3(\lambda) \end{array}\right] = n+s.$$

Finally, by Lemma 3.7 we conclude

$$\operatorname{rank}\left[\begin{array}{cc|c} A(\lambda) & X_1(\lambda) & X_2(\lambda) \end{array}\right] = \operatorname{rank}\begin{bmatrix} A(\lambda) & X_1(\lambda) \end{bmatrix} = n,$$

and

$$\operatorname{rank}\left[\begin{array}{c} A(\lambda) \\ Y_1(\lambda) \\ \hline Y_2(\lambda) \end{array}\right] = \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y_1(\lambda) \end{bmatrix} = n,$$

as wanted. This finishes the proof. □

6. Problem 1.1 under restrictions (3) and (4)

6.1. *A solution to Problem 1.1 under restriction (3)*

By using the notation from Section 2.3, we give a solution to Problem 1.1 under restriction (3) in the following theorem:

Theorem 6.1. *Let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ and $M(\lambda) \in \mathbb{F}[\lambda]^{(n+p+y)\times(n+m+x)}$ be matrix pencils as given in Section 2.3. There exist pencils $X(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times x}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{y\times(n+m)}$ and $Z(\lambda) \in \mathbb{F}[\lambda]^{y\times x}$ such that*

$$\operatorname{rank}\begin{bmatrix} A(\lambda) & X(\lambda) \end{bmatrix} = n, \quad \textit{and} \quad \operatorname{rank}\begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} = n+s-x, \tag{72}$$

and such that the pencil

$$\begin{bmatrix} A(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix} \tag{73}$$

is strictly equivalent to $M(\lambda)$*, if and only if*

$$\begin{array}{ll} (o) & x \leq s \leq y, \\ (i) & \bar{\theta} \geq \theta, \\ (ii) & \gamma_i | \alpha_i | \gamma_{i+x+y}, \quad i = 1, \ldots, n, \\ (iii) & \mathbf{c} \prec' (\mathbf{d}, \bar{\mathbf{a}}), \\ (iv) & \bar{\mathbf{r}} \prec' (\mathbf{r}, \bar{\mathbf{b}}), \\ (v) & \sum_{i=1}^{n+s-x} d(\mathrm{lcm}(\alpha_{i-s+x}, \gamma_i)) \leq \sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{m}(c_i+1) - \sum_{i=1}^{m+x-s}(d_i+1). \\ (vi) & \sum_{i=1}^{n+s} d(\mathrm{lcm}(\alpha_{i-s-x}, \gamma_i)) \leq \sum_{i=1}^{n+s} d(\gamma_i) + \sum_{i=1}^{p+y-s} \bar{r}_i - \sum_{i=1}^{p} r_i. \end{array}$$

Here partitions $\bar{\mathbf{a}} = (\bar{a}_1, \ldots, \bar{a}_{s-x})$ *and* $\bar{\mathbf{b}} = (\bar{b}_1, \ldots, \bar{b}_{y-s})$ *are given by*

$$\sum_{i=1}^{j} \bar{a}_i = \sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{m}(c_i+1) - \sum_{i=1}^{m+x-s}(d_i+1) - \sum_{i=1}^{n+s-x-j} d(\mathrm{lcm}(\alpha_{i-s+x+j}, \gamma_i)) - j, \quad j = 1, \ldots, s-x,$$

$$\sum_{i=1}^{k} \bar{b}_i = \sum_{i=1}^{n+s} d(\gamma_i) + \sum_{i=1}^{p+y-s} \bar{r}_i - \sum_{i=1}^{p} r_i - \sum_{i=1}^{n+s} d(\mathrm{lcm}(\alpha_{i-s-x-k}, \gamma_i)), \quad k = 1, \ldots, y-s.$$

Remark 6.2. *We note that by [19, Lemma 4.1] (see also [17, Lemma 2]) together with conditions* (v) *and* (vi)*, the partitions* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *are well defined, i.e.* $\bar{a}_1 \geq \cdots \geq \bar{a}_{s-x}$ *and* $\bar{b}_1 \geq \cdots \geq \bar{b}_{y-s}$*.*

Proof. Necessity: Let us suppose that there exist pencils $X(\lambda)$, $Y(\lambda)$, and $Z(\lambda)$, such that (72) is satisfied, and such that (73) is strictly equivalent to $M(\lambda)$.

By the ranks and sizes of the involved pencils, we obtain (o).

By Lemma 3.3, there exist invertible matrices $P \in \mathbb{F}^{(n+p)\times(n+p)}$, $Q \in \mathbb{F}^{y\times y}$ and $S \in \mathbb{F}^{(n+m)\times(n+m)}$, and a matrix $R \in \mathbb{F}^{y\times(n+p)}$ such that

$$\begin{bmatrix} P & 0 \\ R & Q \end{bmatrix} \begin{bmatrix} A(\lambda) \\ Y(\lambda) \end{bmatrix} S = \begin{bmatrix} A(\lambda) \\ Y_1(\lambda) \\ Y_2(\lambda) \end{bmatrix},$$

where $Y_1(\lambda) \in \mathbb{F}[\lambda]^{(s-x)\times(n+m)}$ and such that

$$\mathrm{rank} \begin{bmatrix} A(\lambda) \\ Y_1(\lambda) \end{bmatrix} = n + s - x.$$

Thus,

$$\begin{bmatrix} P & 0 \\ R & Q \end{bmatrix} \begin{bmatrix} A(\lambda) & X(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} = \left[\begin{array}{c|c} A(\lambda) & \bar{X}(\lambda) \\ Y_1(\lambda) & \bar{Z}_1(\lambda) \\ \hline Y_2(\lambda) & \bar{Z}_2(\lambda) \end{array}\right]. \tag{74}$$

Moreover by the dimensions of the involved pencils (see also Lemma 3.7), we conclude that

$$\mathrm{rank} \begin{bmatrix} A(\lambda) & \bar{X}(\lambda) \\ Y_1(\lambda) & \bar{Z}_1(\lambda) \end{bmatrix} = n + s - x.$$

Moreover, by Lemma 3.3, the row minimal indices of $A(\lambda)$ and $\begin{bmatrix} A(\lambda) \\ Y_1(\lambda) \end{bmatrix}$ coincide ($r_1 \geq \cdots \geq r_p$). Also, by the same lemma, the column minimal indices of $\begin{bmatrix} A(\lambda) \\ Y_1(\lambda) \end{bmatrix}$ coincide with the column minimal indices of

$\begin{bmatrix} A(\lambda) \\ Y_1(\lambda) \\ Y_2(\lambda) \end{bmatrix}$. Furthermore, by the transposed version of Lemma 3.3, the column minimal indices of $\begin{bmatrix} A(\lambda) \\ Y_1(\lambda) \\ Y_2(\lambda) \end{bmatrix}$ and (74) coincide. Hence the column minimal indices of

$$\begin{bmatrix} A(\lambda) \\ Y_1(\lambda) \end{bmatrix} \tag{75}$$

are $d_1 \geq \cdots \geq d_{m+x-s}$.

Let us denote the homogeneous invariant factors of (75) by

$$\beta_1 | \cdots | \beta_{n+s-x}.$$

Now, we can apply Theorem 5.2 for a completion of (75) up to (74), and thus obtain that the following hold:

$$0 \leq x \leq \min\{x + y - s, x\}, \tag{76}$$
$$\bar{\theta} \geq \theta, \tag{77}$$
$$r_i \geq \bar{r}_{i+y-s}, \quad i = 1, \ldots, p, \tag{78}$$
$$\gamma_i | \beta_i | \gamma_{i+2x+y-s}, \quad i = 1, \ldots, n+s-x, \tag{79}$$
$$\sum_{i=1}^{n+s} d(\operatorname{lcm}(\beta_{i-2x-k}, \gamma_i)) - \sum_{i=1}^{n+s} d(\gamma_i) \leq \bar{y}_k, \quad k = 0, \ldots, y-s. \tag{80}$$

From the ranks of the pencils $A(\lambda)$ and (75) we have

$$\sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{m} (c_i + 1) = \sum_{i=1}^{n+s-x} d(\beta_i) + \sum_{i=1}^{m+x-s} (d_i + 1). \tag{81}$$

The row completion of $A(\lambda)$ up to (75), by Theorem 2.4, i.e. [15, Theorem 2], implies:

$$\beta_i | \alpha_i | \beta_{i+s-x}, \quad i = 1, \ldots, n, \tag{82}$$
$$\mathbf{c} \prec' (\mathbf{d}, \mathbf{a}), \tag{83}$$

where $\mathbf{a} = (a_1, \ldots, a_{s-x})$ is given by

$$\sum_{i=1}^{j} a_i = \sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{m} (c_i + 1) - \sum_{i=1}^{m+x-s} (d_i + 1) - \sum_{i=1}^{n+s-x-j} d(\operatorname{lcm}(\alpha_{i-s+x+j}, \beta_i)) - j, \quad \text{for all} \quad j = 1, \ldots, s-x.$$

Condition (77) is (i), while conditions (82) and (79) together give (ii). Since

$$\operatorname{lcm}(\gamma_i, \alpha_{i-s+x}) \mid \beta_i, \quad i = 1, \ldots, n+s-x, \tag{84}$$

we have that $\mathbf{a} \prec \bar{\mathbf{a}}$. Hence by Lemma 2.3 we have that (83) implies (iii). Also, (81) and (84) together give (v). Finally, (vi) follows from (80) for $k = 0$ and (84).

Also, by the definition of the generalized majorization conditions (78), (79), and (80) give

$$\bar{\mathbf{r}} \prec' (\mathbf{r}, \mathbf{b}), \tag{85}$$

where $\mathbf{b} = (b_1, \ldots, b_{y-s})$ is given by

$$\sum_{i=1}^{k} b_i = \sum_{i=1}^{n+s} d(\gamma_i) + \sum_{i=1}^{p+y-s} \bar{r}_i - \sum_{i=1}^{p} r_i - \sum_{i=1}^{n+s} d(\operatorname{lcm}(\beta_{i-2x-k}, \gamma_i)), \quad k = 1, \ldots, y-s.$$

Note that, as in Remark 6.2, by [19, Lemma 4.1] (see also [17, Lemma 2]) and (80) for $k = 0$, $\mathbf{b}$ is a partition.

By (84) we obtain that $\mathbf{b} \prec \bar{\mathbf{b}}$. Hence by Lemma 2.3 we have that (85) implies (*iv*).

Sufficiency:

Let us assume that conditions (*o*) – (*vi*) are valid. Let us denote by

$$\beta_i := \operatorname{lcm}(\alpha_{i-s+x}, \gamma_i), \quad i = 1, \ldots, n+s-x-1,$$

and let

$$\beta_{n+s-x} := \bar{\psi} \operatorname{lcm}(\alpha_n, \gamma_{n+s-x}),$$

where $\bar{\psi}$ is a monic polynomial such that its degree $d(\bar{\psi})$ is equal to

$$\sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{m} (c_i + 1) - \sum_{i=1}^{m+x-s} (d_i + 1) - \sum_{i=1}^{n+s-x} \operatorname{lcm}(\alpha_{i-s+x}, \gamma_i). \tag{86}$$

By (*v*) we have that $d(\bar{\psi}) \geq 0$, i.e. β_i's are well defined.

Also with such defined β_i's, we have $\mathbf{a} = \bar{\mathbf{a}}$ as well as $\mathbf{b} = \bar{\mathbf{b}}$.

Now, condition (*iii*) coincide with (83), while (*iv*) coincide with (85). Also (*ii*) implies that conditions (82) and (79) are satisfied. Indeed, from the definition of β_i's together with conditions (*o*) and (*ii*) we have

$$\alpha_i | \beta_{i+s-x}, \quad i = 1, \ldots, n, \tag{87}$$

$$\gamma_i | \beta_i, \quad i = 1, \ldots, n+s-x, \tag{88}$$

$$\beta_i | \alpha_i, \quad i = 1, \ldots, n+s-x-1, \tag{89}$$

$$\beta_i | \gamma_{i+2x+y-s}, \quad i = 1, \ldots, n+s-x-1. \tag{90}$$

If $s > x$, from (89) we get $\beta_i | \alpha_i$, $i = 1, \ldots, n$.
If $s = x$, then conditions (*ii*), (*iii*) and (*v*) give that $d(\bar{\psi}) = 0$, i.e. in this case $\bar{\psi} := 1$, and so $\beta_i | \alpha_i$, $i = 1, \ldots, n$, and consequently (82) holds.

If $s < x + y$ then we also have $\beta_i | \gamma_{i+2x+y-s}$, $i = 1, \ldots, n+s-x$.
If $s = x + y$, then by (*o*) we have $x = 0$ and $s = y$, and then conditions (*ii*)–(*v*) give that $d(\bar{\psi}) = 0$, i.e. in this case $\bar{\psi} := 1$, and so $\beta_i | \gamma_{i+2x+y-s}$, $i = 1, \ldots, n+s-x$, and consequently (79) holds.

Now, by Theorem 2.4, conditions (82) and (83) give the existence of a pencil $Y_1(\lambda) \in \mathbb{F}[\lambda]^{(s-x)\times(n+m)}$ such that

$$\begin{bmatrix} A(\lambda) \\ Y_1(\lambda) \end{bmatrix} \tag{91}$$

has $\beta_1 | \cdots | \beta_{n+s-x}$ as homogeneous invariant factors, $d_1 \geq \cdots \geq d_{m+x-s}$ as column minimal indices and $r_1 \geq \cdots \geq r_p$ as row minimal indices.

Next, by Theorem 5.2, by conditions (*o*), (*i*), (*iv*), (*vi*) and (79) there exist matrix pencils $X(\lambda)$, $Y(\lambda)$, and $Z(\lambda)$, such that the pencil

$$\left[\begin{array}{c|c} \begin{matrix} A(\lambda) \\ Y_1(\lambda) \end{matrix} & X(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array}\right] \tag{92}$$

is strictly equivalent to $M(\lambda)$.
This finishes the proof. $\square$

6.2. A Solution to Problem 1.1 under restrictions (4)

It is straightforward to see that restrictions 3 and 4 are transposed one to another. So, the transposed version of Theorem 6.1 solves Problem 1.1 under restriction (4).

References

[1] I. Baragaña, Interlacing inequalities for regular pencils, Linear Algebra Appl. 121 (1989) 521-535.
[2] I. Baragaña, I. Zaballa, Column completion of a pair of matrices, Linear Multilinear Algebra, 27 (1990), pp. 243-273.
[3] G. Basile and G. Marro, Controlled and Conditional Invariants in Linear Systems Theory. Prentice-Hall, Englewood Cliffs, NJ, 1992.
[4] L. Batzke, Generic rank-one perturbations of structured regular matrix pencils, Linear Algebra Appl., 458 (2014), pp. 638-670.
[5] I. Cabral, F. C. Silva, Similarity invariants of completions of submatrices, J. Linear Algebra Appl., 169 (1992) 151-161.
[6] D. Carlson, Inequalities relating the degrees of elementary divisors within a matrix, Simon Stevin 44 (1970) 3-10.
[7] M. Dodig, M. Stošić, Similarity class of a matrix with prescribed submatrix, Linear and Multilinear Algebra, 57 (2009) 217-245.
[8] M. Dodig, Descriptor systems under feedback and output injection, Operator Theory, Operator Algebras, and Matrix Theory, 2018, 141-166.
[9] M. Dodig, M. Stošić, The general matrix pencil completion problem - a minimal case, SIAM J. Matrix Anal. & Appl. Vol. 40, No. 1, (2019) 347-369.
[10] M. Dodig, Completion of quasi-regular matrix pencils, Linear Algebra Appl. 501 (2016) 198-241.
[11] M. Dodig, Minimal completion problem for quasi-regular matrix pencils, Linear Algebra Appl. 525 (2017) 84-104.
[12] M. Dodig, Matrix pencils completion problems II, Linear Algebra Appl. 429 (2008) 633-648.
[13] M.Dodig, M.Stošić, Rank one perturbations of matrix pencils, SIAM J. Matrix Anal. & Appl. Vol. 41, No. 4, (2020) 1889-1911.
[14] M. Dodig, Explicit solution of the row completion problem for matrix pencils, Linear Algebra Appl. 432 (2010) 1299-1309.
[15] M. Dodig, Completion up to a matrix pencil with column minimal indices as the only nontrivial Kronecker invariants, Linear Algebra Appl. 438 (2013) 3155-3173.
[16] M. Dodig, M. Stošić, Combinatorics of column minimal indices and matrix pencil completion problems, SIAM J. Matrix Anal. & Appl. 31, (2010) 2318-2346.
[17] M. Dodig, M. Stošić, On convexity of polynomial paths and generalized majorizations, The Electronic Journal of Combinatorics, Vol. 17, No.1, R61 (2010).
[18] M. Dodig, Matrix pencils completion problems, Linear Algebra Appl. 428 (2008), no. 1, 259-304.
[19] M. Dodig, Complete characterisation of Kronecker invariants of a patrix pencil with a prescribed quasi-regular subpencil, Electronic Journal of Linear Algebra 36, (2020) 430-445.
[20] J.C. Doyle, Guaranteed margins for LQG regulators. IEEE Trans. on AC 23 (1978), 756-757.
[21] S. Furtado, F. C. Silva, Embedding a regular subpencil into a general linear pencil, Linear Algebra Appl. 295 (1999) 61-72.
[22] W. Fulton, Eigenvalues, invariant factors, highest weights, and Schubert calculus, Bull. Amer. Math. Soc. 37 (3) (2000) 209-249.
[23] F. R. Gantmacher, The Theory of Matrices, vol.2, Chelsea Publishing Comp., New York (1960).
[24] I. Gohberg, M. A. Kaashoek, F. van Schagen, Eigenvalues of completions of submatrices, Linear and Multilinear Algebra, 25 (1989) 55-70.
[25] Y. Han, Subrepresentations of Kronecker representations, Linear Algebra Appl., 402 (2005), pp. 150-164.
[26] G. Hardy, J. E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, 1991.
[27] M. Heymann, Controllability indices and feedback simulation, SIAM J. Control Optim., 14 (1976), pp. 769-789.
[28] V. Kučera, Assigning the invariant factors by feedback, Kybernetika, 17 (1981), pp. 118-127.
[29] V. Kučera, P. Zagalak, Fundamental theorem of state feedback for singular systems, Automatica 24 (1988) 653-658.
[30] J. Loiseau, S. Mondié, I. Zaballa, P. Zagalak, Assigning the Kronecker invariants of a matrix pencil by row or column completion, Linear Algebra and Appl. 278 (1998) 327-336.
[31] C. Mehl, V. Mehrmann, A. C. M. Ran, and L. Rodman, Eigenvalue perturbation theory of classes of structured matrices under generic structured rank one perturbations, Linear Algebra Appl., 435 (2011), pp. 687-716.
[32] C. Mehl, V. Mehrmann, A. C. M. Ran, and L. Rodman, Jordan forms of real and complex matrices under rank one perturbations, Oper. Matrices, 7 (2013), pp. 381-398.
[33] S. Mondié, P. Zagalak, V. Kučera, State feedback in linear control theory, Linear Algebra and its Appl. 317 (2000) 177-192.
[34] J. Moro and F. Dopico, Low rank perturbation of Jordan structure, SIAM Journal on Matrix Analysis and Applications 25, 2 (2003), 495-506.
[35] G. de Oliveira, Matrices with prescribed characteristic polynomial and a prescribed submatrix III, Monatsh. Math. 75 (1971), 441-446.
[36] G. de Oliveira, Matrices with prescribed entries and eigenvalues II, SIAM J. Appl. Math. 24 (1983), 414-417.
[37] A. Roca. I. Zaballa, Invariant factor assignment under state feedback and output injection, Linear Algebra Appl. 332-334 (2001) 401-436.
[38] H. H. Rosenbrock, State-Space and Multivariable Theory, Thomas Nelson and Sons, London, 1970.
[39] E. M. Sá, Imbedding conditions for λ -matrices, Linear Algebra Appl. 24 (1979) 33-50.
[40] E. M. Sá, Imersão de Matrizes e Entrelaçamento de Factores Invariantes, Ph.D. Thesis, Univ. Coimbra (1979).
[41] C. Szanto, Submodules of Kronecker modules via extension monoid products, J. Pure Appl. Algebra, 222 (2018), pp. 3360-3378.
[42] S. V. Savchenko, Typical changes in spectral properties under perturbations by a rank-one operator, Mathematical Notes 74, 4 (2003), 557-568.

[43] F. De Terán, F. Dopico and J. Moro, Low rank perturbation of Weierstrass structure, SIAM Journal on Matrix Analysis and Applications 30, 2 (2008), 538-547.
[44] R. C. Thompson, Interlacing inequalities for invariant factors, Linear Algebra Appl. 24 (1979) 1-31.
[45] H. K. Wimmer, Existenzsatze in Theorie der Matrizen und Lineare Kontrolltheorie, Monatsh. Math., 78 (1974), pp. 256-263.
[46] I. Zaballa, Matrices with prescribed rows and invariant factors, Linear Algebra Appl. 87 (1987) 113-146.
[47] I. Zaballa, Interlacing inequalities and control theory, Linear Algebra Appl. 101 (1988), 9-31.