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On the Inequality $w(AB) \leq c\|A\|w(B)$ where A is a Positive Operator

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Abstract. Abu-Omar and Kittaneh [Numerical radius inequalities for products of Hilbert space operators, J. Operator Theory **72**(2) (2014), 521–527], wonder what is the smallest constant c such that $w(AB) \leq c\|A\|w(B)$ for all bounded linear operators A, B on a complex Hilbert space with A is positive. Here, $w(\cdot)$ stands for the numerical radius. In this paper, we prove that $c = \dfrac{3\sqrt{3}}{4}$.

1. Introduction

Let $\mathcal{H}$ denote a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denotes the induced norm. Let $\mathcal{B}(\mathcal{H})$ denote the collection of all bounded linear operators acting on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, the *numerical range* of T is given by

$$W(T) = \Big\{\langle Tx, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\Big\}.$$

It is known that $W(T)$ is a nonempty bounded convex subset (not necessarily closed) of the complex plane. To measure the location and relative size of $W(T)$, one frequently used quantity; *numerical radius* of T. It is denoted and given by

$$w(T) = \sup\Big\{|\lambda| : \lambda \in W(T)\Big\}.$$

It is well-known that

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\| \tag{1}$$

for all $T \in \mathcal{B}(\mathcal{H})$, that is $w(\cdot)$ defines an equivalent norm to $\|\cdot\|$ on $\mathcal{B}(\mathcal{H})$. Also, it is a basic fact that the norm $w(\cdot)$ is self-adjoint $\big($i.e., $w(T^*) = w(T)$ for all $T \in \mathcal{B}(\mathcal{H})$ where T^* is the adjoint of $T\big)$. For more material about the numerical radius and other information on the basic theory of numerical range, we refer the reader to [3].

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The problem of the numerical radius of a product of operators consists in finding the best constant c, which satisfies the following inequality

$$w(AB) \leq c\|A\|w(B), \tag{2}$$

where $A, B \in \mathcal{B}(\mathcal{H})$ satisfy some given conditions. It follows readily from the inequalities (1) that if $A, B \in \mathcal{B}(\mathcal{H})$, then

$$w(AB) \leq 2\|A\|w(B). \tag{3}$$

The constant 2 in the inequality (3) is the best possible. Indeed, the sharpness of the inequality (3) is evident by taking $A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The question of whether, when A and B commute,

$$w(AB) \leq \|A\|w(B), \tag{4}$$

was open for about twenty years. In [4], Müller proved by a counterexample that the inequality (4) fails to be true. The related question of the best constants for the inequality (2) for commuting A and B has also been considered (see [5]), the best known result is that $1 < c \leq \frac{1}{2}\sqrt{2 + 2\sqrt{3}}$. In [1], Abu-Omar and Kittaneh wonder what is the smallest constant c such that the inequality

$$w(AB) \leq c\|A\|w(B)$$

holds for all $A, B \in \mathcal{B}(\mathcal{H})$ with A is positive $\left(\text{i.e., } \langle Ax, x\rangle \geq 0 \text{ for all } x \in \mathcal{H}\right)$. They proved that $\sqrt{5} - 1 \leq c \leq 3/2$.

In this paper, we prove that for any $A, B \in \mathcal{B}(\mathcal{H})$ with A is positive, we have

$$w(AB) \leq \frac{3\sqrt{3}}{4}\|A\|w(B).$$

Moreover, we show by giving an example, that the constant $\frac{3\sqrt{3}}{4}$ is the smallest possible.

2. Main result

In order to prove our result, we need the following lemma.

Lemma 2.1. *Let A, B be two 2×2 matrices with A is positive non-invertible. Then*

$$w(AB) \leq \frac{3\sqrt{3}}{4}\|A\|w(B).$$

Proof. Without loss of generality we may assume that $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $b = 0$, then $AB = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $w(AB) = |a| = \left\langle B\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\rangle \leq w(B)$ and we are done.

Therefore, suppose that $b \neq 0$. We may assume that $|b| = 1$ and $a \geq 0$. So, $AB = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, and then $w(AB) = \dfrac{a + \sqrt{a^2 + 1}}{2}$ (see, [2]). If $1 \leq a$, we have

$$w(AB) \leq \frac{1 + \sqrt{2}}{2}a \leq \frac{1 + \sqrt{2}}{2}w(B).$$

Let $0 \leq a < 1$. According to [6],

$$\begin{aligned} w(B) &= \sup_{\theta\in\mathbb{R}} \left\| \mathrm{Re}\left(e^{i\theta}B\right)\right\| \\ &= \sup_{\theta\in\mathbb{R}} \frac{|\,\mathrm{Re}\left(e^{i\theta}a\right) + \mathrm{Re}\left(e^{i\theta}d\right)| + \sqrt{(\mathrm{Re}\,(e^{i\theta}a) - \mathrm{Re}\,(e^{i\theta}d))^2 + \left|e^{i\theta}b + e^{-i\theta}\bar{c}\right|^2}}{2} \\ &\geq \sup_{\theta\in\mathbb{R}} \sqrt{a^2\cos^2\theta + \frac{1}{4}\left|e^{i\theta}b + e^{-i\theta}\bar{c}\right|^2} \\ &= w\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\right). \end{aligned}$$

We claim that for any scalar c there is $\theta \in \mathbb{R}$ such that

$$\left(1 + a^2\right)^2 \leq 4a^2\cos^2\theta + \left|1 + e^{-2i\theta}\bar{c}\right|^2.$$

If $a^2 \leq |c|$, the result follows immediately. Now let $|c| < a^2$, then $\left|1 + e^{-2i\theta}\bar{c}\right| \geq 1 - a^2$, hence by taking $\theta = 0$ we have $\left(1 + a^2\right)^2 = 4a^2 + (1 - a^2)^2 \leq 4a^2 + |1 + \bar{c}|^2$. Our claim is then proved. It follows that $\frac{a^2+1}{2} \leq w\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\right) \leq w(B)$ and since $\frac{a + \sqrt{a^2+1}}{a^2+1} \leq \frac{3\sqrt{3}}{4}$ for all $0 \leq a < 1$, we derive that

$$w(AB) = \frac{a + \sqrt{a^2+1}}{2} = \frac{a + \sqrt{a^2+1}}{a^2+1}\frac{a^2+1}{2} \leq \frac{3\sqrt{3}}{4}w(B)$$

as desired. □

Now, we are ready to state and prove our main result.

Theorem 2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ with A is positive. Then*

$$w(AB) \leq \frac{3\sqrt{3}}{4}\|A\|w(B). \tag{5}$$

Moreover, the constant $\frac{3\sqrt{3}}{4}$ is the smallest possible.

Proof. We prove that for all unit vector $x \in \mathcal{H}$, we have

$$\left|\langle ABx, x\rangle\right| \leq \frac{3\sqrt{3}}{4}\|A\|w(B).$$

Let $x \in \mathcal{H}$ be a unit vector. We may assume that x and Bx are linearly independent. Otherwise, $|\langle ABx, x\rangle| \leq w(A)w(B) = \|A\|w(B)$. Therefore, let $\mathcal{Y}$ be the subspace spanned by x and Bx, and let P be the orthogonal projection of $\mathcal{H}$ on $\mathcal{Y}$. Put $\lambda := \langle Bx, x\rangle$, $\beta := \|Bx - \langle Bx, x\rangle x\|$ and $y := \frac{1}{\beta}(Bx - \lambda x)$. Then $\{y, x\}$ is an orthonormal basis of $\mathcal{Y}$. We identify the operators PAP and PBP with their restrictions to $\mathcal{Y}$. With respect to the basis $\{y, x\}$, PAP and PBP may be represented by the matrices $\begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}$ and $\begin{bmatrix} u & \beta \\ v & \lambda \end{bmatrix}$, respectively, where u, v, a, b and c are scalars. Since PAP is positive, the scalars a and c are non-negative. Furthermore, we may assume that $c \neq 0$, otherwise $b = 0$ (reason: $ac \geq |b|^2$), $Ax = 0$ and $\langle ABx, x\rangle = 0$. Therefore, as $Px = x$ and $PBx = Bx$, we have

$$\begin{aligned}
\left|\langle ABx, x\rangle\right| &= \left|\langle ABPx, Px\rangle\right| \\
&= \left|\langle PAPPBPx, x\rangle\right| \\
&= \left|\left\langle \begin{bmatrix} |b|^2/c & b \\ \bar{b} & c \end{bmatrix} \begin{bmatrix} u & \beta \\ v & \lambda \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle\right| \\
&\leq w\left(\begin{bmatrix} |b|^2/c & b \\ \bar{b} & c \end{bmatrix} \begin{bmatrix} u & \beta \\ v & \lambda \end{bmatrix} \right) \\
&\leq \frac{3\sqrt{3}}{4} \left\| \begin{bmatrix} |b|^2/c & b \\ \bar{b} & c \end{bmatrix} \right\| w\left(\begin{bmatrix} u & \beta \\ v & \lambda \end{bmatrix} \right) \quad \text{(by Lemma 2.1).}
\end{aligned}$$

Since $ac \geq |b|^2$, it is easy to verify that

$$\left\| \begin{bmatrix} |b|^2/c & b \\ \bar{b} & c \end{bmatrix} \right\| = \frac{|b|^2}{c} + c \leq \left\| \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \right\|.$$

It follows that

$$\begin{aligned}
\left|\langle ABx, x\rangle\right| &\leq \frac{3\sqrt{3}}{4} \|PAP\| w(PBP) \\
&\leq \frac{3\sqrt{3}}{4} \|A\| w(B).
\end{aligned}$$

Consequently, for any unit vector $x \in \mathcal{H}$,

$$\left|\langle ABx, x\rangle\right| \leq \frac{3\sqrt{3}}{4} \|A\| w(B),$$

and the inequality (5) is obtained by taking the supremum over all unit vectors $x \in \mathcal{H}$.

The sharpness of the inequality (5) is evident by taking $A = \begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Indeed, A is positive, $\|A\| = 4$, $w(B) = 1/2$, and $w(AB) = 3\sqrt{3}/2$, that is, $w(AB) = \frac{3\sqrt{3}}{4}\|A\| w(B)$. This completes the proof. □

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