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Bornological Spaces in the Context of Fuzzy Soft Sets

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Abstract. The aim of this study is to present the concept of an (L, M)-fuzzy (E, K)-soft bornology as a parameterized extension of the LM-valued bornology. By this way, we describe the notions of boundedness and the parameterized degree of boundedness for L-fuzzy soft sets. We examine several fundamental properties of the proposed structures. In addition, we induce a $(2, M)$-fuzzy (E, K)-soft bornology in a given $(2, M)$-fuzzy (E, K)-soft topological space with the help of the measures of compactness of a soft set.

1. Introduction

Since Molodtsov [18] introduced the soft set theory to overcome some of the difficulties involving the parametrization process in handling uncertainties, many researchers have applied soft set theory in different directions [3, 7, 24, 25]. In 2001, Maji et al.[17] proposed the fuzzy soft set theory which is the combination of fuzzy set and soft set theories. Later, many researchers focused on the theory of fuzzy soft sets and they applied this theory to their own branches such as algebra, topology, decision making and so on [4, 5, 8, 9, 21].

General topology, with its emphasis on neighborhoods, entourages, and proximity, primarily deals with local phenomena. Through the years, there have been attempts to build frameworks to discuss macroscopic phenomena and their interplay with topology. Bounded sets described in metric spaces play an important role in some applications, but in general topological spaces, the notion of a "bounded set" makes no sense by the absence of the distance function. Hence in order to identify bounded sets independently from the distance function, a structure named bornology (or so called abstract boundedness), has been constructed by Hu [15]. And hence, Hu's work opened a new perspective to discuss macroscopic phenomena in general topological spaces. According to this definition a bornology is a collection of sets which satisfies some certain conditions: closed for finite unions, closed hereditary and contains all singletons. The sets which belong to a bornology are called as the bounded sets of the space. The families $CL(X)$, $\mathbb{F}(X)$ and $\mathbb{K}(X)$ of all nonempty closed, all nonempty finite and all nonempty compact subsets of a Hausdorff topological space X, the family of all (totally) bounded subsets of a metric or uniform space are examples of boundedness. At present the theory of bornological spaces is developed in various directions. Most of the research involving bornologies is done in the context of topological linear spaces [14] and in topological algebras, that is in case when the underlying set, in addition to topology, is endowed with a certain algebraic structure. Maio and Kočinac [12] studied the notion of boundedness in a topological space and demonstrated the importance

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of this notion in selection principles theory. Caserta et al. [6] investigated some properties of the function spaces endowed with bornologies in the view of selection principles. Abel and Šostak [1] proposed the notion of an L-bornology which is a family of L-fuzzy sets satisfies some certain conditions. Hence they described the concept of classical boundedness for L-fuzzy sets. Later, Šostak and Uljane [22, 23] studied the fuzzy-crisp and fuzzy-fuzzy approaches of the bornologies named as L-valued bornology and LM-valued bornology, respectively. Paseka et al. [19] concerned and studied some categorical properties of the Abel and Šostak's fuzzy bornology.

The main intention of the present paper is to handle the soft interpretation of bornological spaces and also shed light to the way of describing boundedness in parameterized spaces. This paper is arranged in the following manner. In section 2, we recall some basic notions and notations for fuzzy soft sets. In Section 3, we describe the notions of (E, K)-soft L-bornology and (L, M)-fuzzy (E, K)-soft bornology. We investigate the relations between these notions and we also study fundamental features of the (L, M)-fuzzy (E, K)-soft bornological spaces. We introduce bounded fuzzy soft mappings and construct a category of such spaces. In Section 4, we induce a $(2, M)$-fuzzy (E, K)-soft bornology in a $(2, M)$-fuzzy (E, K)-soft topological space by using the measures of parameterized compactness of a soft set in the corresponding space.

2. Preliminaries

In our work two lattices L and M, will play the fundamental role. The first one is a complete DeMorgan algebra $L = (L, \leq, \wedge, \vee, ')$, satisfying the infinite distributivity law

$$\alpha \wedge (\bigvee_{i\in I} \beta_i) = \bigvee_{i\in I} (\alpha \wedge \beta_i), \forall \alpha \in L, \{\beta_i\}_{i\in I} \subset L.$$

The top and the bottom elements of L are denoted by 1_L and 0_L, respectively. By M we denote the complete completely distributive lattice $M = (M, \leq, \wedge, \vee)$ whose the bottom and the top elements are denoted by 0_M and 1_M, respectively. For a complete lattice M and $\alpha, \beta \in M$, the wedge-below relation $\lhd$ is defined on M as follows: $\beta \lhd \alpha \Leftrightarrow$ if $K \subseteq M$ and $\alpha \leq \bigvee K$ then $\exists \gamma \in K, \beta \leq \gamma$.

As shown in [20] a lattice M is completely distributive if and only if the wedge-below relation has the following property, $\alpha = \bigvee\{\beta \in M \mid \beta \lhd \alpha\}$, for each $\alpha \in M$.

An element α in M is said to be coprime if $\alpha \leq \beta \vee \gamma$ implies that $\alpha \leq \beta$ or $\alpha \leq \gamma$. The set of all nonzero coprime elements of M is denoted by $c(M)$. We also denote $M^o = \{\alpha \in M \mid \alpha \lhd 1_M\}$. For more details about the lattices, we refer [13, 20].

Throughout this work, X refers to a nonempty initial universe and E denotes an arbitrary nonempty set viewed on the sets of parameters.

The parameterized extension of an L-fuzzy set is called an L-fuzzy soft set and it is defined as follows.

Definition 2.1. ([21]) An L-fuzzy soft set (f, E) over the universe X with the set of parameters E is defined by the set of ordered pairs

$$(f, E) = \{(e, f_e) : e \in E, f_e := f(e) \in L^X\},$$

where $f : E \to L^X$, is a mapping. Hence, for an L-fuzzy soft set (f, E) it is clear that $f \in (L^X)^E$.

The notation $FS(X, E)$ denotes the family of all L-fuzzy soft sets on X with the set of parameters E.

Definition 2.2. ([21]) Let (f, E) and (g, E) be two L-fuzzy soft sets on X, then

(1) we say that (f, E) is an L-fuzzy soft subset of (g, E) and write $(f, E) \sqsubseteq (g, E)$ if $f_e \leq g_e$, for each $e \in E$. (f, E) and $(, E)g$ are called equal if $(f, E) \sqsubseteq (g, E)$ and $(g, E) \sqsubseteq (f, E)$.

(2) the union of (f, E) and (g, E) is an L-fuzzy soft set $(h, E) = (f, E) \sqcup (g, E)$, where $h_e = f_e \vee g_e$, for each $e \in E$.

(3) the intersection of (f, E) and (g, E) is an L-fuzzy soft set $(h, E) = (f, E) \sqcap (g, E)$, where $h_e = f_e \wedge g_e$, for each $e \in E$.

(4) the complement of an L-fuzzy soft set (f, E) is denoted by $(f, E)' = (f', E)$, where $f' : E \to L^X$ is a mapping given by $f'_e = (f_e)'$, for each $e \in E$. Clearly $(f', E)' = (f, E)$.

Definition 2.3. ([21])

(1) (Null L-fuzzy soft set) An L-fuzzy soft set (f, E) on X is called a null L-fuzzy soft set and denoted by $\widetilde{0}$, if $f_e(x) = 0$, for each $e \in E, x \in X$.

(2) (Absolute L-fuzzy soft set) An L-fuzzy soft set (f, E) on X is called an absolute L-fuzzy soft set and denoted by $\widetilde{1}$, if $f_e(x) = 1$, for each $e \in E, x \in X$. Clearly $(\widetilde{1})' = \widetilde{0}$ and $\widetilde{0}' = \widetilde{1}$.

Proposition 2.4. ([2]) *Let Δ be an index set and $(f, E), (f_i, E), (g_i, E) \in FS(X, E)$, for all $i \in \Delta$. Then the following properties are satisfied.*

(1) $(f, E) \sqcap \left(\bigsqcup_{i\in\Delta} (g_i, E) \right) = \bigsqcup_{i\in\Delta} ((f, E) \sqcap (g_i, E))$ *and* $(f, E) \sqcup \left(\bigsqcap_{i\in\Delta} (g_i, E) \right) = \bigsqcap_{i\in\Delta} ((f, E) \sqcup (g_i, E))$.

(2) $\left(\bigsqcap_{i\in\Delta} (f_i, E) \right)' = \bigsqcup_{i\in\Delta} (f'_i, E)$ *and* $\left(\bigsqcup_{i\in\Delta} (f_i, E) \right)' = \bigsqcap_{i\in\Delta} (f'_i, E)$.

Definition 2.5. ([4, 16]) Let (f, E_1) and (g, E_2) be two L-fuzzy soft sets over X_1 and X_2, respectively. A fuzzy soft mapping between $FS(X_1, E_1)$ and $FS(X_2, E_2)$ is a pair (φ, ψ), denoted also by simply φ_ψ, of crisp mappings $\varphi : X_1 \to X_2$ and $\psi : E_1 \to E_2$ such that:

(1) The image of (f, E_1) under φ_ψ is an L-fuzzy soft set over X_2, defined by

$$\varphi_\psi((f, E_1))_k(y) = \bigvee_{\varphi(x)=y} \bigvee_{\psi(a)=k} f_a(x), \qquad \text{for all } k \in E_2,\, y \in X_2.$$

(2) The pre-image of (g, E_2) under φ_ψ is an L-fuzzy soft set over X_1, defined by

$$\varphi_\psi^{-1}((g, E_2))_e(x) = g_{\psi(e)}(\varphi(x)), \qquad \text{for all } e \in E_1,\, x \in X_1.$$

If φ and ψ are both injective (or surjective), then φ_ψ is said to be injective (or surjective).

Proposition 2.6. ([16]) *Let $(f_i, E_1) \in FS(X_1, E_1)$ and $(g_i, E_2) \in FS(X_2, E_2)$ for all $i \in \Gamma$, where Γ is an index set. Then the following properties are satisfied.*

(1) $\varphi_\psi \left(\bigsqcup_{i\in\Gamma} (f_i, E_1) \right) = \bigsqcup_{i\in\Gamma} \varphi_\psi((f_i, E_1))$.

(2) $\varphi_\psi \left(\bigsqcap_{i\in\Gamma} (f_i, E_1) \right) \sqsubseteq \bigsqcap_{i\in\Gamma} \varphi_\psi((f_i, E_1))$, *the equality holds if φ_ψ is injective.*

(3) $\varphi_\psi^{-1} \left(\bigsqcup_{i\in\Gamma} (g_i, E_2) \right) = \bigsqcup_{i\in\Gamma} \varphi_\psi^{-1}((g_i, E_2))$ *and* $\varphi_\psi^{-1} \left(\bigsqcap_{i\in\Gamma} (g_i, E_2) \right) = \bigsqcap_{i\in\Gamma} \varphi_\psi^{-1}((g_i, E_2))$.

3. Boundedness in the fuzzy soft universe

In this section, we describe the notions of "boundedness" and "the parameterized degree of boundedness" for L-fuzzy soft sets. In order to achieve this goal, we define the parameterized extensions of the L-bornology and LM-valued bornology in the framework of mathematics of fuzzy sets. Besides, we observe some elementary features of the (L, M)-fuzzy (E, K)-soft bornological spaces.

Definition 3.1. A parameterized family $B = \{B_k\}_{k\in K}$ of mappings $B_k : FS(X, E) \to 2$ is called an (E, K)-soft L-bornology if it satisfies the following conditions.

(B1) $\bigsqcup\{(f, E) \in FS(X, E) \mid (f, E) \in B_k\} = \widetilde{1}$.

(B2) If $(f, E) \in B_k$ and $(g, E) \sqsubseteq (f, E)$, then $(g, E) \in B_k$.

(B3) If $(f_1, E), (f_2, E) \in B_k$, then $(f_1, E) \sqcup (f_2, E) \in B_k$.

Then the pair (X, B) is said to be an (E, K)-soft L-bornological space. If $(f, E) \in B_k$, then the L-fuzzy soft set (f, E) is called a bounded L-fuzzy soft set with respect to the parameter $k \in K$.

Definition 3.2. A mapping $\mathcal{B} : K \to M^{FS(X,E)}$ is called an (L, M)-fuzzy (E, K)-soft bornology on X if it satisfies the following conditions.

(SB1) $\forall \alpha \in M^o, \forall k \in K, \exists \mathcal{U} \subseteq FS(X, E)$ s.t. $\bigsqcup \mathcal{U} = \widetilde{1}$ and $\mathcal{B}_k((f, E)) \geq \alpha, \forall (f, E) \in \mathcal{U}$.

(SB2) If $(f, E) \sqsubseteq (g, E)$, then $\mathcal{B}_k((f, E)) \geq \mathcal{B}_k((g, E))$, for all $k \in K$.

(SB3) $\mathcal{B}_k((f, E) \sqcup (g, E)) \geq \mathcal{B}_k((f, E)) \wedge \mathcal{B}_k((g, E))$, for all $(f, E), (g, E) \in FS(X, E), k \in K$.

Then the pair $(X, \mathcal{B})$ is called an (L, M)-fuzzy (E, K)-soft bornological space and the value $\mathcal{B}_k((f, E))$ is interpreted as the parameterized degree of boundedness of an L-fuzzy soft set (f, E) in this space.

One can prefer to consider the following stronger version of the first axiom:

(SB1*) $\bigsqcup \{(f, E) \in FS(X, E) \mid \mathcal{B}_k((f, E)) = 1_M\} = \widetilde{1}, \ \forall k \in K.$

The mapping $\mathcal{B} : K \to M^{FS(X,E)}$ which satisfies the axioms ($SB1^*$), (SB2) and (SB3) is said to be a strong (L, M)-fuzzy (E, K)-soft bornology on X.

Remark 3.3. (1) In case when both parameter sets are one-point, then we come to the definition of an LM-valued bornology [23].

(2) If the parameter sets are both singletons and if besides $M = 2$, then we return to the definition of an L-bornology [1].

(3) If the parameter sets are both singletons and if besides $L = 2$, then we return to the definition of an M-valued bornology [22].

(4) If the parameter sets are both singletons and if besides $L = M = 2$, then we return to the original definition of a bornology [15].

(5) If the parameter set K is singleton, then we get the crisp bornologies for the soft and the fuzzy soft sets.

(6) If the parameter set E is singleton, then we get the soft bornologies for the crisp and L-fuzzy sets [11].

In the light of the above discussion, one may conclude that (L, M)-fuzzy (E, K)-soft bornology definition is the general case of all proposed boundedness types given not only for the crisp but also for the fuzzy universes.

Remark 3.4. It is noted that if $B = \{B_k\}_{k \in K}$ is an (E, K)-soft L-bornology on X, then the mapping $\mathcal{B} : K \to 2^{FS(X,E)}$ defined by $\mathcal{B}(k) := \mathcal{B}_k = \chi_{B_k}$ is an $(L, 2)$-fuzzy (E, K)-soft bornology on X, where

$$\chi_{B_k}((h, E)) = \begin{cases} 1, & \text{if } (h, E) \in B_k, \\ 0, & \text{if } (h, E) \notin B_k. \end{cases}$$

Example 3.5. Let $K = \{k_1, k_2, k_3\}$, E be a non-empty set and define a family of mappings $B = \{B_k\}_{k \in K}$ as follows: $\mathcal{B}_{k_1}((f, E)) = 1, \forall (f, E) \in SS(X, E)$, $\mathcal{B}_{k_2} = \{(f, E) \in SS(X, E) \mid |(f, E)| = \sup_{e \in E} |f(e)| < n, \text{ for some } n \in \mathbb{N}\}$ and $\mathcal{B}_{k_3} = \{(f, E) \in SS(X, E) \mid |(F, E)| < \aleph_0\}$, where $SS(X, E) = \{(f, E) \mid f : E \to 2^X\}$ denotes the set of all soft sets over X with the set of parameters E. Since the parameterized family $B = \{B_k\}_{k \in K}$ of mappings $B_k : SS(X, E) \to 2$ is an (E, K)-soft 2-bornology on X, then $\mathcal{B}_k = \chi_{B_k}$ is an $(2, 2)$-fuzzy (E, K)-soft bornology on X.

Example 3.6. Let $L = \{(0,0),(1,1)\} \cup \{(a,0),(0,b),(a,a) \mid a,b \in (0,1)\}$ and let the relation $'' \leq ''$ on the set L be defined by $(m,b) \leq (n,d)$ if and only if $m \leq n$ and $b \leq d$.
Define an order reversing involution $' : L \to L$ as follows:

For each $x,y \in (0,1)$, $(x,0)' = (1-x,0),(0,y)' = (0,1-y),(x,x)' = (1-x,1-x)$ and $(1,1)' = (0,0)$. Then $(L,\leq,')$ is a complete DeMorgan algebra. Let $X = \{x,y\}, E = (0,0.5], L = M$ and $f_e(x) = f_e(y) = (e,0)$, $g_e(x) = g_e(y) = (0,e)$ for each $e \in E$.
Define a mapping $\mathcal{B} : E \to M^{FS(X,E)}$ as follows:

$$\mathcal{B}_e((h,E)) = \begin{cases} (e,0), & \text{if } (h,E) = (f,E) \\ (0,e), & \text{if } (h,E) = (g,E) \\ (1,1), & \text{otherwise} \end{cases}.$$

Then the mapping $\mathcal{B}$ is an (L,M)-fuzzy (E,E)-soft bornology on X.

Example 3.7. Let $L = \{0,a,b,1\}$ be a diamond-type lattice with the order reversing involution $' : L \to L$ defined by $0' = 1, 1' = 0, a' = a$ and $b' = b$. Then $(L,\leq,')$ is a completely distributive DeMorgan algebra. Let $K = \{k_1,k_2\}$, $X = \{x,y\}$, $E = \{1,2\}$ and $(f,E),(g,E)$ be two L-fuzzy soft sets defined by follows: $f_1(x) = f_1(y) = a, f_2(x) = f_2(y) = b$ and $g_1(x) = g_1(y) = b, g_2(x) = g_2(y) = a$. Define a mapping $\mathcal{B} : \{k_1,k_2\} \to L^{FS(X,E)}$ as follows:

$$\mathcal{B}_{k_1}((h,E)) = \begin{cases} a, & \text{if } (h,E) = (f,E) \text{ or } (h,E) \sqsubseteq (f,E) \\ b, & \text{if } (h,E) = (g,E) \text{ or } (h,E) \sqsubseteq (g,E) \\ 0, & \text{if } (h,E) \sqsupseteq (f,E) \text{ or } (h,E) \sqsupseteq (g,E) \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad \mathcal{B}_{k_2}((h,E)) = 1, \text{ for all } (h,E) \in FS(X,E).$$

Then the mapping $\mathcal{B}$ is an (L,L)-fuzzy (E,K)-soft bornology on X.

Definition 3.8. Let $\mathcal{L}$ be a subset of $FS(X,E)$ that is closed under finite unions, then the mapping $\mathcal{D} : K \to M^{\mathcal{L}}$ is said to be an (L,M)-fuzzy (E,K)-soft bornology base if the followings are satisfied.

(1) $\forall \alpha \in M^o, \forall k \in K, \exists \mathcal{U} \subseteq \mathcal{L}$ s.t. $\bigsqcup \mathcal{U} = \widetilde{1}$ and $\mathcal{D}_k((f,E)) \geq \alpha, \forall (f,E) \in \mathcal{U}$.

(2) $\mathcal{D}_k((f,E) \sqcup (g,E)) \geq \mathcal{D}_k((f,E)) \wedge \mathcal{D}_k((g,E))$, for each $(f,E),(g,E) \in \mathcal{L}$ and for each $k \in K$.

Proposition 3.9. *Let $\mathcal{D} : K \to M^{\mathcal{L}}$ be an (L,M)-fuzzy (E,K)-soft bornology base on X. Then the mapping $\langle \mathcal{D} \rangle : K \to M^{FS(X,E)}$ defined by $\langle \mathcal{D} \rangle_k((f,E)) = \bigvee\{\mathcal{D}_k((g,E)) \mid (g,E) \in \mathcal{L}, (f,E) \sqsubseteq (g,E)\}$ is an (L,M)-fuzzy (E,K)-soft bornology on X.*

Proof. (SB1) Ir is evident.
(SB2) Let $(f,E),(g,E) \in FS(X,E)$ and $k \in K$ be chosen. Then it is evident that

$$\langle \mathcal{D} \rangle_k((f,E)) = \bigvee\{\mathcal{D}_k((h,E)) \mid (f,E) \sqsubseteq (h,E)\} \geq \bigvee\{\mathcal{D}_k((h,E)) \mid (g,E) \sqsubseteq (h,E)\} = \langle \mathcal{D} \rangle_k((g,E)).$$

(SB3)

$$\begin{aligned} \langle \mathcal{D} \rangle_k((f_1,E) \sqcup (f_2,E)) &= \bigvee\{\mathcal{D}_k((g,E)) \mid ((f_1,E) \sqcup (f_2,E)) \sqsubseteq (g,E)\} \\ &= \bigvee\{\mathcal{D}_k((g_1,E) \sqcup (g_2,E) \mid (f_i,E) \sqsubseteq (g_i,E), i = 1,2)\} \\ &\geq \bigvee\{\mathcal{D}_k((g_1,E)) \wedge \mathcal{D}_k((g_2,E)) \mid (f_i,E) \sqsubseteq (g_i,E), i = 1,2\} \\ &\geq \bigvee\{\mathcal{D}_k((g_1,E)) \mid (f_1,E) \sqsubseteq (g_1,E)\} \wedge \bigvee\{\mathcal{D}_k((g_2,E)) \mid (f_2,E) \sqsubseteq (g_2,E)\} \\ &= \langle \mathcal{D} \rangle_k((f_1,E)) \wedge \langle \mathcal{D} \rangle_k((f_2,E)), \text{ for each } k \in K. \quad \square \end{aligned}$$

Definition 3.10. Let $(X_1,\mathcal{B}^1)$ and $(X_2,\mathcal{B}^2)$ be an (L,M)-fuzzy (E_1,K_1)-soft and an (L,M)-fuzzy (E_2,K_2)-soft bornological spaces, respectively. Then the fuzzy soft mapping $\varphi_{\psi,\eta} : (X_1,\mathcal{B}^1) \to (X_2,\mathcal{B}^2)$ is said to be bounded if $\mathcal{B}^1_k((f,E_1)) \leq \mathcal{B}^2_{\eta(k)}(\varphi_\psi((f,E_1)))$ for all $(f,E_1) \in FS(X_1,E_1)$ and for all $k \in K_1$.
Here $\varphi : X_1 \to X_2, \psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ are crisp functions.

Proposition 3.11. *Composition of two bounded fuzzy soft mappings is bounded, too.*

Proof. The proof is evident. $\square$

Example 3.12. Let $\mathcal{B}^1$ be the discrete (L,M)-fuzzy (E,K)-soft bornology on X, that is $\mathcal{B}^1_k((f,E)) = 1_M$, for all $(f,E) \in FS(X,E)$ and for all $k \in K$. Let us consider the identity fuzzy soft mapping $(id_X)_{(id_E,id_K)} : (X,\mathcal{B}^1) \to (X,\mathcal{B}^2)$, where $\mathcal{B}^2$ is the (L,M)-fuzzy (E,K)-soft bornology given in Example 3.6. Since $\mathcal{B}^1_e((f,E)) = (1,1) \not\leq (e,0) = \mathcal{B}^2_e(\varphi_\psi((f,E)))$, for some $e \in E$, the mapping $(id_X)_{(id_E,id_K)} : (X,\mathcal{B}^1) \to (X,\mathcal{B}^2)$ is not bounded as different from $(id_X)_{(id_E,id_K)} : (X,\mathcal{B}^2) \to (X,\mathcal{B}^1)$ that is a bounded fuzzy soft mapping.

It is also easily seen that the identity fuzzy soft mapping $(id_X)_{(id_E,id_K)} : (X,\mathcal{B}) \to (X,\mathcal{B})$ is bounded, where $(X,\mathcal{B})$ is any (L,M)-fuzzy (E,K)-soft bornology. Hence (L,M)-fuzzy $(E,K)-$soft bornological spaces and bounded fuzzy soft mappings between them form a category which is denoted by **SBOR**(L,M,E,K).

Proposition 3.13. *Let $\mathcal{B} : K \to M^{FS(X,E)}$ (where, $\mathcal{B}_k ::= \mathcal{B}(k) : FS(X,E) \to M$ are mappings for all $k \in K$) be an (L,M)-fuzzy (E,K)-soft bornology on X and $\alpha \in M$, then the family $\mathcal{B}^\alpha = \{\mathcal{B}^\alpha_k\}_{k\in K}$ of mappings $\mathcal{B}^\alpha_k : FS(X,E) \to 2$ which are defined by, $\mathcal{B}^\alpha_k = \{(f,E) \in FS(X,E) \mid \mathcal{B}_k((f,E)) \geq \alpha\}$ is an (E,K)-soft L-bornology on X, for each $k \in K$.*

Proof. The proof can be easily verified by the construction of the level bornologies and by the axioms of Definition 3.2. In addition the collection of α-levels $\{\mathcal{B}^\alpha \mid \alpha \in c(M)\}$ of an (L,M)-fuzzy (E,K)-soft bornology is lower semi-continuous in the following sense:
$\mathcal{B}^\alpha_k = \bigcap\{\mathcal{B}^\beta_k \mid \beta \lhd \alpha,\ \beta \in M\}$, for each $\alpha \in c(M)$ and for all $k \in K$,
where $\mathcal{B}^{0_M}_k = FS(X,E)$ as the intersection of the empty-set. Hence each (L,M)-fuzzy (E,K)-soft bornology $\mathcal{B}$ can be characterized by the lower semi-continuous decomposition into the level soft bornologies by for each $k \in K$; $\{\mathcal{B}^\alpha_k = \bigvee_{\beta\lhd\alpha} \mathcal{B}^\beta_k \mid \alpha \in c(M)\}$. □

Now, let us consider the converse as follows.

Proposition 3.14. *Let $\{\mathcal{D}^\alpha \mid \alpha \in c(M)\}$ be an indexed family of (E,K)-soft L-bornologies on X, such that $\alpha \leq \beta$ implies $\mathcal{D}^\beta_k \subseteq \mathcal{D}^\alpha_k$, for each $k \in K$. Then the mapping $\mathcal{B} : K \to M^{FS(X,E)}$ defined by*

*$\mathcal{B}_k((f,E)) = \bigvee\{\alpha \in c(M) \mid (f,E) \in \mathcal{D}^\alpha_k\}$, for all $f \in FS(X,E)$ and $k \in K$,
is an (L,M)-fuzzy (E,K)-soft bornology on X.*

Proof. Since each $\mathcal{D}^\alpha = \{\mathcal{D}^\alpha_k\}_{k\in K}$ satisfies the axioms of Definition 3.1, then $\bigsqcup \mathcal{D}^\alpha_k = \widetilde{1}$ for any $k \in K$. Hence the axiom (SB1) is ensured by the construction.
(SB2) Let $(f,E),(g,E) \in FS(X,E)$ be given such that $(f,E) \sqsubseteq (g,E)$, and $k \in K$ be fixed. Then we have $\mathcal{B}_k((f,E)) = \bigvee\{\alpha \in c(M) \mid (f,E) \in \mathcal{D}^\alpha_k\} \geq \bigvee\{\alpha \in c(M) \mid (g,E) \in \mathcal{D}^\alpha_k\} = \mathcal{B}_k((g,E))$.
(SB3) Let $(f,E),(g,E) \in FS(X,E)$ and $k \in K$ be given. Let $\alpha := \mathcal{B}_k((f,E)) \wedge \mathcal{B}_k((g,E))$. We need to show that $\mathcal{B}_k((f,E)\sqcup(g,E)) \geq \beta$ for all $\beta\lhd\alpha$. Since $\alpha \leq \mathcal{B}_k((f,E))$ and $\alpha \leq \mathcal{B}_k((g,E))$, for each $\beta\lhd\alpha$, there exists $\gamma_1,\gamma_2 \in c(M)$ such that $\beta \leq \gamma_1 \wedge \gamma_2$ and $(f,E) \in \mathcal{D}^{\gamma_1}_k$, $(g,E) \in \mathcal{D}^{\gamma_2}_k$. By the hypothesis, we get $(f,E),(g,E) \in \mathcal{D}^\beta_k$, and hence $(f,E) \sqcup (g,E) \in \mathcal{D}^\beta_k$. This witnesses the following inequality $\mathcal{B}_k((f,E) \sqcup (g,E)) \geq \mathcal{B}_k((f,E)) \wedge \mathcal{B}_k((g,E))$. □

Proposition 3.15. $\mathcal{B}^\alpha_k = \bigcap\{\mathcal{D}^\beta_k \mid \beta \in c(M), \beta \lhd \alpha\}$*, for each $\alpha \in M$ and $k \in K$.*

Proof. Let $k \in K$ be a fixed parameter and $(f,E) \in FS(X,E)$ be given such that $(f,E) \notin \mathcal{B}^\alpha_k$. Then there exists $\beta \in c(M)$ with $\beta \lhd \alpha$ such that $(f,E) \in \mathcal{D}_k(\beta)$, and hence $(f,E) \notin \bigcap\{\mathcal{D}^\beta_k \mid \beta \in c(M), \beta \lhd \alpha\}$. This proves that $\mathcal{B}^\alpha_k \subseteq \bigcap\{\mathcal{D}^\beta_k \mid \beta \in c(M), \beta \lhd \alpha\}$ and hence also the equality $\mathcal{B}^\alpha_k = \bigcap\{\mathcal{D}^\beta_k \mid \beta \in c(M), \beta \lhd \alpha\}$ since the converse inequality is clear from the construction of the (L,M)-fuzzy (E,K)-soft bornology $\mathcal{B}$.

a given indexed collection of (E,K)-soft L-bornologies $\{\mathcal{D}^\alpha \mid \alpha \in c(M)\}$ (where $\mathcal{D}^\alpha = \{D^\alpha_k : FS(X,E) \to 2\}_{k\in K}$) on X, let the mapping $\mathcal{B} : K \to M^{FS(X,E)}$ be the (L,M)-fuzzy (E,K)-soft bornology described as above. Let us define a new indexed collection of (E,K)-soft L-bornologies $\{\overline{\mathcal{D}}^\alpha \mid \alpha \in M\}$ by describing $\overline{\mathcal{D}}^\alpha_k := \bigcap\{\mathcal{D}^\beta_k \mid \beta \lhd \alpha, \beta \in c(M)\}$, for any $k \in K$. Let the mapping $\overline{\mathcal{B}} : K \to M^{FS(X,E)}$ be the (L,M)-fuzzy (E,K)-soft bornology which is defined by follows:

$\overline{\mathcal{B}}_k((f,E)) = \bigvee\{\alpha \in M \mid (f,E) \in \overline{D}_k^{\alpha}\}$, for all $(f,E) \in FS(X,E)$ and $k \in K$.
Then for each $\alpha \in M$ and $k \in K$, we have

$$\overline{\mathcal{B}}_k^{\alpha} = \bigcap_{\beta \lhd \alpha, \beta \in M} \overline{D}_k^{\beta} = \bigcap_{\gamma \lhd \beta, \gamma \in c(M)} \left(\bigcap_{\beta \lhd \alpha} \mathcal{D}_k^{\beta} \right) = \bigcap_{\gamma \lhd \alpha, \gamma \in c(M)} \mathcal{D}_k^{\gamma} = \mathcal{B}_k^{\alpha}.$$

As a result, we have that $\overline{\mathcal{B}} = \mathcal{B}$. □

Definition 3.16. Let $\mathfrak{B}(L,M,E,K,X)$ be the family of all (L,M)-fuzzy (E,K)-soft bornologies on X. Define a partial order $'' \leq ''$ by setting for $\mathcal{B}^1, \mathcal{B}^2 \in \mathfrak{B}(L,M,E,K,X)$:

$\mathcal{B}^1 \leq \mathcal{B}^2 :\Leftrightarrow \mathcal{B}_k^1((f,E)) \geq \mathcal{B}_k^2((f,E))$ for all $k \in K$ and $(f,E) \in FS(X,E)$.

In this case, we say that $\mathcal{B}^1$ is coarser (or stronger) than $\mathcal{B}^2$, or $\mathcal{B}^2$ is said to be finer than $\mathcal{B}^1$.

Proposition 3.17. *The partially ordered set* $(\mathfrak{B}(L,M,E,K,X), \leq)$ *is a complete lattice.*

Proof. The mapping $\mathcal{B}^{\perp} : K \to M^{FS(X,E)}$ which is described as $\mathcal{B}_k^{\perp}((f,E)) = 1_M$ for each $(f,E) \in FS(X,E)$ and $k \in K$, is an (L,M)-fuzzy (E,K)-soft bornology on X. Besides it is obvious that $\mathcal{B}^{\perp}$ is the coarsest element of $\mathfrak{B}(L,M,E,K,X)$. Now let us identify the finest element of the $\mathfrak{B}(L,M,E,K,X)$. Let $S \subseteq X$ and $\alpha : E \to c(L)$ be given, where $c(L)$ denotes the set of all coprimes of L. Define $Pt(S,\beta) = \{\bigvee_{x \in S} x^{\beta} \mid x \in S\}$, for all $\beta \in c(L)$ and define a fuzzy soft set $P : E \to FS(X,E); P(e) \in Pt(S, \alpha(e))$. Here x^{β} denotes a fuzzy point which is defined by

$$x^{\beta}(y) = \begin{cases} \beta, & \text{if } y = x \\ 0_L, & \text{if } y \neq x \end{cases}, \text{ for some } \beta \in c(L).$$

Then the mapping $\mathcal{B}^{\top} : K \to M^{FS(X,E)}$ defined by

$$\mathcal{B}_k^{\top}((f,E)) = \begin{cases} 1_M, & \text{if } \exists S \subseteq X, |S| < \aleph_0, \exists P : E \to Pt(S,\beta) \text{ for some } \beta \in c(L) \text{ such that } (f,E) \sqsubseteq (P,E), \\ 0_M, & \text{otherwise} \end{cases}$$

is the finest (L,M)-fuzzy (E,K)-soft bornology in $\mathfrak{B}(L,M,E,K,X)$. Further for a given family of $\{\mathcal{B}^i : K \to M^{FS(X,E)} \mid i \in \Gamma\}$ of (L,M)-fuzzy (E,K)-soft bornologies on X, define a mapping $\mathcal{B}^* : K \to M^{FS(X,E)}$ by setting $\mathcal{B}_k^*((f,E)) = \bigwedge_{i \in \Gamma} \mathcal{B}_k^i((f,E))$ for all $(f,E) \in FS(X,E)$ and $k \in K$. Then the mapping $\mathcal{B}^*$ is an (L,M)-fuzzy (E,K)-soft bornology on X. Since the first two axioms of Definition 3.2 are easy to verify, we only check the third axiom:

(SB3) Let $(f,E),(g,E) \in FS(X,E)$ and $k \in K$ be given. Then we have

$$\mathcal{B}_k^*((f,E) \sqcup (g,E)) = \bigwedge_{i \in I} \mathcal{B}_k^i((f,E) \sqcup (g,E)) \geq \bigwedge_{i \in I} (\mathcal{B}_k^i((f,E)) \wedge \mathcal{B}_k^i((g,E))) = (\bigwedge_{i \in I} \mathcal{B}_k^i((f,E))) \wedge (\bigwedge_{i \in I} \mathcal{B}_k^i((g,E))) = \mathcal{B}_k^*((f,E)) \wedge \mathcal{B}_k^*((g,E)).$$

By the construction we see that the mapping $\mathcal{B}^* = \curlyvee_{i \in I} \mathcal{B}^i$ is the least upper bound of the family $\{\mathcal{B}^i : K \to M^{FS(X,E)} \mid i \in \Gamma\}$ in $\mathfrak{B}(L,M,E,K,X)$. Hence it is a complete join semi-lattice. Now let us build the greatest lower bound of the family $\{\mathcal{B}^i : K \to M^{FS(X,E)} \mid i \in \Gamma\}$ in $\mathfrak{B}(L,M,E,K,X)$ by follows:

$$\curlywedge_{i \in I} \mathcal{B}^i = \curlyvee_{i \in I} \{\mathcal{B} \in \mathfrak{B}(L,M,E,K,X) \mid \mathcal{B}_k \leq \bigwedge_{i \in I} \mathcal{B}_k^i, \forall k \in K\}. \quad \square$$

Theorem 3.18. *Let* $\varphi_{\psi,\eta} : (X_1, \mathcal{B}^1) \to (X_2, \mathcal{B}^2)$ *be a fuzzy soft mapping, where the* (L,M)*-fuzzy* (E_1,K_1)*-soft bornology* $\mathcal{B}^1$ *is an induced fuzzy soft bornology from an* (L,M)*-fuzzy* (E_1,K_1)*-soft bornology base* $\mathcal{D}$ *on* X_1*. Then the mapping* $\varphi_{\psi,\eta}$ *is bounded if and only if* $\mathcal{D}_k((f,E_1)) \leq \mathcal{B}_{\eta(k)}^2(\varphi_{\psi}((f,E_1)))$ *for all* $k \in K_1$ *and* $(f,E_1) \in FS(X_1,E_1)$.

Proof. Let $\varphi_{\psi,\eta} : (X_1, \mathcal{B}^1) \to (X_2, \mathcal{B}^2)$ be a bounded fuzzy soft mapping. Then $\mathcal{B}_k^1((f,E_1)) \leq \mathcal{B}_{\eta(k)}^2(\varphi_{\psi}((f,E_1)))$, for all $k \in K_1$ and $(f,E_1) \in FS(X_1,E_1)$. Then from Proposition 3.9, it is easy to see that $\mathcal{D}_k((g,E_1)) \leq \mathcal{B}_{\eta(k)}^2(\varphi_{\psi}((g,E_1)))$ for all $k \in K_1$ and $(g,E_1) \in \mathcal{L} \subseteq FS(X_1,E_1)$. For the converse implication, take $k \in K_1$ and $(f,E_1) \in FS(X_1,E_1)$, then we have

$$\begin{aligned} \mathcal{B}_k^1((f,E_1)) &= \bigvee\{\mathcal{D}_k((g,E_1)) \mid (g,E_1) \in \mathcal{L}, (f,E_1) \sqsubseteq (g,E_1)\} \\ &\leq \bigvee\{\mathcal{B}_{\eta(k)}^2(\varphi_{\psi}((g,E_1))) \mid (g,E_1) \in \mathcal{L}, \varphi_{\psi}((f,E_1)) \sqsubseteq \varphi_{\psi}((g,E_1))\} \\ &\leq \bigvee\{\mathcal{B}_{\eta(k)}^2((h,E_2)) \mid (h,E_2) \in \mathcal{L}^* \subseteq FS(X_2,E_2), \varphi_{\psi}(f,E_1) \sqsubseteq (h,E_2)\} = \mathcal{B}_{\eta(k)}^2(\varphi_{\psi}((f,E_1))). \quad \square \end{aligned}$$

Theorem 3.19. *Let $\varphi_{\psi,\eta} : (X_1, E_1, K_1) \to (X_2, E_2, K_2, \mathcal{B}^2)$ be a fuzzy soft mapping, where $(X_2, \mathcal{B}^2)$ is an (L, M)-fuzzy (E_2, K_2)-soft bornology and $\eta : K_1 \to K_2$ is a surjective crisp function. Let $\mathcal{L} := \{(f, E_1) = \varphi_\psi^{-1}((g, E_2)) \mid (g, E_2) \in FS(X_2, E_2)\}$. Then the mapping $\mathcal{D} : K_1 \to M^{\mathcal{L}}$ which is defined by $\mathcal{D}_k((f, E_1)) = \mathcal{B}^2_{\eta(k)}((g, E_2))$, is an (L, M)-fuzzy (E_1, K_1)-soft bornology base on X_1. In addition, the induced (L, M)-fuzzy (E_1, K_1)-soft bornology $\mathcal{B}^1 = \langle \mathcal{D} \rangle$ is the coarsest (L, M)-fuzzy (E_1, K_1)-soft bornology on X_1 for which $\varphi_{\psi,\eta} : (X_1, \mathcal{B}^1) \to (X_2, \mathcal{B}^2)$ is bounded.*

Proof. (1) Let $\alpha \in M^o$ and $k \in K_1$ be given. Then since $\mathcal{B}^2$ is a fuzzy soft bornology on X_2, then there exists a family $\mathcal{V} = \{(g_\lambda, E_2) \mid \lambda \in \Lambda\} \subseteq (L^{X_2})^{E_2}$ such that $\bigsqcup_{\lambda \in \Lambda}(g_\lambda, E_2) = \widetilde{1}$ and $\mathcal{B}^2_{\eta(k)}((g_\lambda, E_2)) \geq \alpha$, for each $\lambda \in \Lambda$. Let $\mathcal{U} = \{(f_\lambda, E_1) \mid (f_\lambda, E_1) = \varphi_\psi^{-1}((g_\lambda, E_2)),\ (g_\lambda, E_2) \in \mathcal{V}\}$. Then $\mathcal{D}_k((f_\lambda, E_1)) \geq \alpha$ for each $\lambda \in \Lambda$ and also it is easily seen that $\bigsqcup_{\lambda \in \Lambda}(f_\lambda, E_1) = \bigsqcup_{\lambda \in \Lambda} \varphi_\psi^{-1}((g_\lambda, E_2)) = \varphi_\psi^{-1}(\bigsqcup_{\lambda \in \Lambda}(g_\lambda, E_2)) = \varphi_\psi^{-1}(\widetilde{1}) = \widetilde{1}$.

(2) Let $(f_1, E_1) = \varphi_\psi^{-1}((g_1, E_2)), (f_2, E_1) = \varphi_\psi^{-1}((g_2, E_2))$ and $k \in K_1$ be chosen. Then we have

$\mathcal{D}_k((f_1, E_1)) \wedge \mathcal{D}_k((f_2, E_1)) = \mathcal{B}^2_{\eta(k)}((g_1, E_2)) \wedge \mathcal{B}^2_{\eta(k)}((g_2, E_2)) \leq \mathcal{B}^2_{\eta(k)}((g_1, E_2) \sqcup (g_2, E_2)) = \mathcal{D}_k(\varphi_\psi^{-1}((g_1, E_2) \sqcup (g_2, E_2))) = \mathcal{D}_k((f_1, E_1) \sqcup (f_2, E_1))$ is satisfied by the equality of $\varphi_\psi^{-1}((g_1, E_2) \sqcup (g_2, E_2)) = \varphi_\psi^{-1}((g_1, E_2)) \sqcup \varphi_\psi^{-1}((g_2, E_2))$.

From the above theorem and by the construction of $\mathcal{B}^1$, the fuzzy soft mapping $\varphi_{\psi,\eta} : (X_1, \mathcal{B}^1 = \langle \mathcal{D} \rangle) \to (X_2, \mathcal{B}^2)$ is bounded. □

Theorem 3.20. *Let $\varphi_{\psi,\eta} : (X_1, E_1, K_1, \mathcal{B}^1) \to (X_2, E_2, K_2)$ be a surjective fuzzy soft mapping, where $(X_1, \mathcal{B}^1)$ is an (L, M)-fuzzy (E_1, K_1)-soft bornological space. Then the mapping $\mathcal{B} : K_2 \to M^{FS(X_2, E_2)}$ which is defined by $\mathcal{B}_{\eta(k)}((g, E_2)) = \mathcal{B}^1_k(\varphi_\psi^{-1}((g, E_2)))$ is an (L, M)-fuzzy (E_2, K_2)-soft bornology on X_2. Besides $\mathcal{B}$ is the finest (L,M)-fuzzy (E_2, K_2)-soft bornology on X_2 for which $\varphi_{\psi,\eta}$ is bounded.*

Proof. (SB1) Let $\alpha \in M^o$ and $k^* \in K_2$ be given. Since η is surjective there exists $k \in K_1$ such that $k^* = \eta(k)$. Since $\mathcal{B}^1$ is a fuzzy soft bornology on X_1, then there exists a family $\mathcal{U} \subseteq FS(X_1, E_1)$ such that $\mathcal{B}^1_k((f, E_1)) \geq \alpha$ for every $(f, E_1) \in \mathcal{U}$ and $\bigsqcup_{(f,E_1) \in \mathcal{U}}(f, E_1) = \widetilde{1}$. For each $(f, E_1) \in \mathcal{U}$, let $(g_f, E_2) = \varphi_\psi((f, E_1))$ and let $\mathcal{V} = \{(g_f, E_2) \mid (f, E_1) \in \mathcal{U}\}$. Then

$\bigsqcup\{(g_f, E_2) \mid (f, E_1) \in \mathcal{U}\} = \bigsqcup\{\varphi_\psi((f, E_1)) \mid (f, E_1) \in \mathcal{U}\} = \varphi_\psi(\bigsqcup\{(f, E_1) \mid (f, E_1) \in \mathcal{U}\}) = \varphi_\psi(\widetilde{1}) = \widetilde{1}$. So, by the definition of $\mathcal{B}$, for each $(g_f, E_2) \in \mathcal{V}$, it is provided that $\mathcal{B}_{\eta(k)}((g_f, E_2)) = \mathcal{B}^1_k(\varphi_\psi^{-1}((g, E_1))) = \mathcal{B}^1_k(\varphi_\psi^{-1}(\varphi_\psi((f, E_1)))) = \mathcal{B}^1_k((f, E_1)) \geq \alpha$.

(SB2) It is easy to verify by the definition of the mapping $\mathcal{B}$.

(SB3) Let $(g_1, E_2), (g_2, E_2) \in (L^{X_2})^{E_2}$ and $k^* \in K_2$ such that $k^* = \eta(k)$, for some $k \in K_1$. Then we have

$\mathcal{B}_{\eta(k)}((g_1, E_2) \sqcup (g_2, E_2)) = \mathcal{B}^1_k(\varphi_\psi^{-1}(((g_1, E_2) \sqcup (g_2, E_2)))) = \mathcal{B}^1_k(\varphi_\psi^{-1}((g_1, E_2)) \sqcup \varphi_\psi^{-1}((g_2, E_2))) \geq \mathcal{B}^1_k(\varphi_\psi^{-1}((g_1, E_2))) \wedge \mathcal{B}^1_k(\varphi_\psi^{-1}((g_2, E_2))) = \mathcal{B}_{\eta(k)}((g_1, E_2)) \wedge \mathcal{B}_{\eta(k)}((g_2, E_2))$.

From the construction of the mapping $\mathcal{B}$, it is easy to verify that $\mathcal{B}$ is the finest (L, M)-fuzzy (E_2, K_2)-soft bornology on X_2 for which the fuzzy soft mapping $\varphi_{\psi,\eta} : (X_1, \mathcal{B}^1) \to (X_2, \mathcal{B})$ is bounded. □

4. Relations between parameterized degree of compactness and boundedness

In this section, we build a $(2, M)$-fuzzy (E, K)-soft bornology in a given $(2, M)$-fuzzy (E, K)-soft topological space by using the concept of the measures of compactness.

Definition 4.1. ([10]) Let $\tau : K \to M^{FS(X,E)}$ be a map and $(g, E) \in FS(X, E)$. Define such a map $com_\tau : K \to M^{FS(X,E)}$ as follows.

$$com_\tau(k, (g, E)) = \bigwedge_{\mathcal{U} \subseteq FS(X,E)} [\tau_k(\mathcal{U}) \leq [[(g, E) \widetilde{\sqsubseteq} \bigvee \mathcal{U}] \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [(g, E) \widetilde{\sqsubseteq} \bigvee \mathcal{V}]]]$$

If (X, τ) is an (L, M)-fuzzy (E, K)-soft topological space, then the value $com_\tau(k, (g, E))$ is called the compactness degree of (g, E) with respect to the parameter k. So (g, E) is said to be compact L-fuzzy soft set with respect to k if $com_\tau(k,(g,E)) = 1_M$. In this manner, the compactness degree of (g,E) in the whole space (X,τ) is computed by the value $com_\tau((g,E)) = \bigwedge_{k\in K} com_\tau(k,(g,E))$. So the L-fuzzy soft set (g,E) is said to be compact in the fuzzy soft space (X,τ) if $com_\tau((g,E)) = 1_M$.

Here, the inclusion $[\,\widetilde{\sqsubseteq}\,] : FS(X,E) \times FS(X,E) \to L$ is described by $[(f,E)\widetilde{\sqsubseteq}(g,E)] = \bigwedge_{x\in X} \bigwedge_{e\in E} (f_e'(x) \vee g_e(x))$

Theorem 4.2. ([10]) *Let (X,τ) be an (L,M)-fuzzy (E,K)-soft topological space and $(g,E),(h,E) \in FS(X,E)$. Then the following inequality is satisfied for each $k \in K$,*

$$com_\tau(k,(g,E) \sqcup (h,E)) \geq com_\tau(k,(g,E)) \wedge com_\tau(k,(h,E)).$$

Theorem 4.3. ([10]) *Let $\varphi_{\psi,\eta} : (X_1,\tau^1) \to (X_2,\tau^2)$ be a continuous fuzzy soft mapping between (L,M)-fuzzy (E_1,K_1)-soft and (L,M)-fuzzy (E_2,K_2)-soft topological spaces. Then for each $k \in K_1$ and $(g,E_1) \in FS(X_1,E_1)$, we have $com_{\tau^1}(k,(g,E_1)) \leq com_{\tau^2}(\eta(k),\varphi_\psi((g,E_1)))$.*

Theorem 4.4. *Let (X,τ) be a $(2,M)$-fuzzy (E,K)-soft topological space. Then the mapping $\mathcal{B}^\tau : K \to M^{SS(X,E)}$ which is defined by follows:*

$\mathcal{B}^\tau_k((f,E)) = \bigvee\{com_\tau(k,(g,E)) \mid (f,E) \sqsubseteq (g,E),\ (g,E) \in SS(X,E)\}$

satisfies the following properties.

(1) $\mathcal{B}^\tau_k(P^x_e) = 1_M$, for all $P^x_e \in SP(X)$ and for all $k \in K$.

(2) If $(f,E) \sqsubseteq (g,E)$, then $\mathcal{B}^\tau_k((f,E)) \geq \mathcal{B}^\tau_k((g,E))$, for all $(f,E),(g,E) \in SS(X,E)$ and for all $k \in K$.

(3) $\mathcal{B}^\tau_k((f,E) \sqcup (g,E)) \geq \mathcal{B}^\tau_k((f,E)) \wedge \mathcal{B}^\tau_k((g,E))$, for all $(f,E),(g,E) \in SS(X,E)$ and for all $k \in K$.

Here $SS(X,E) = \{(f,E) \mid f : E \to 2^X\}$ denotes the set of all soft sets over X with the set of parameters E and $SP(X) = \{P^x_e : E \to 2^X \mid P^x_e(e)(x) = \{x\}$ and otherwise $P^x_e(.)(.) = \emptyset, \forall e \in E, x \in X\}$ denotes the set of all soft points over X.

Proof. (1) By Definition 4.1, for a fixed parameter $k \in K$, the compactness degree $com_\tau(k,P^x_e) = 1_M$ for any soft point $P^x_e \in SP(X)$. So, it is obvious that $\mathcal{B}^\tau_k(P^x_e) \geq com_\tau(k,P^x_e) = 1_M$ for any $P^x_e \in SP(X)$ and for any $k \in K$.
(2) It is obvious by the construction.
(3) It is easily obtained by Theorem 4.2 and the construction of the mapping $\mathcal{B}^\tau$.

In the light of the above theorem, we may conclude that the mapping $\mathcal{B}^\tau : K \to M^{SS(X,E)}$ defined as a way of above, is a $(2,M)$-fuzzy (E,K)-soft bornology on X. □

Proposition 4.5. *If $\varphi_{\psi,\eta} : (X_1,\tau^1) \to (X_2,\tau^2)$ is a continuous fuzzy soft mapping between $(2,M)$-fuzzy (E_1,K_1)-soft and $(2,M)$-fuzzy (E_2,K_2)-soft topological spaces, then $\varphi_{\psi,\eta} : (X_1,\mathcal{B}^{\tau_1}) \to (X_2,\mathcal{B}^{\tau_2})$ is fuzzy soft bounded.*

Proof. Let $k \in K_1$ and $(f,E) \in SS(X_1,E_1)$ be given arbitrary. Then by Theorem 4.3, we have

$$\begin{aligned}
\mathcal{B}^{\tau^1}_k((f,E_1)) &= \bigvee\{com_{\tau^1}(k,(g,E_1)) \mid (f,E_1) \sqsubseteq (g,E_1),(g,E_1) \in SS(X_1,E_1)\}\\
&\leq \bigvee\{com_{\tau^2}(\eta(k),\varphi_\psi((g,E_1))) \mid (f,E_1) \sqsubseteq (g,E_1),(g,E_1) \in SS(X_1,E_1)\}\\
&\leq \bigvee\{com_{\tau^2}(\eta(k),(h,E_2)) \mid \varphi_\psi((f,E_1)) \sqsubseteq (h,E_2),(h,E_2) \in SS(X_2,E_2)\}\\
&= \mathcal{B}^{\tau_2}_{\eta(k)}(\varphi_\psi((f,E_1))).
\end{aligned}$$

This witnesses the boundedness of the fuzzy soft mapping $\varphi_{\psi,\eta}$. □

Hence we get a functor $\mathfrak{F} : \mathbf{STOP}(2,M,E,K) \to \mathbf{SBOR}(2,M,E,K)$ from the category of $(2,M)$-fuzzy (E,K)-soft topological spaces to $(2,M)$-fuzzy (E,K)-soft bornological spaces by defining $\mathfrak{F}(X,\tau) = (X,\mathcal{B}^\tau)$ and $\mathfrak{F}(\varphi_{\psi,\eta}) = \varphi_{\psi,\eta}$.

5. Conclusion

General bornological spaces play a key role in recent research of convergence structures on hyperspaces, in optimization theory and in the study of topologies on function spaces. On the other hand, most of the fundamental classical structures are extended now to the soft and the fuzzy soft universes by using the parametrization tool. In order to make a contribution to these investigations, we intended to develop counterparts of the theory of bornologies in the framework of soft and fuzzy soft sets. To achieve this goal, we provided the concept of an (L, M)-fuzzy (E, K)-soft bornology and by this way, we described the "parameterized degree of boundedness" for L-fuzzy soft sets. Furthermore, we studied some elementary properties of the proposed concept.

In conclude, we hope that the results presented in this research will open a new perspective for applied sciences. For further research, we plan to apply the notion of soft boundedness to the selection principles theory.

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