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Palindromic p-Adic Continued Fractions

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Abstract. The aim of this paper is to establish new transcendence criteria of p-adic continued fractions. We prove that a p-adic number whose sequence of partial quotients is bounded in $\mathbb{Q}_p$ and begins with arbitrarily long palindromes is either quadratic or transcendental.

1. Introduction

Throughout the present work, $\mathcal{A}$ denotes a countable set. Recall that the length of a finite word W on the alphabet $\mathcal{A}$, that is, the number of letters composing W, is denoted by $|W|$. The reversal (or the mirror image) of $W = a_1, \dots, a_n$ is the word $\overline{W} = a_n, \dots, a_1$. In particular, W is a palindrome if and only if $W = \overline{W}$. From now on, we will identify any sequence $\mathbf{a} = (a_n)_{n\geq 1}$ of elements from $\mathcal{A}$ with the infinite word $a_1\, a_2 \dots a_n \dots$.

Continued fractions beginning with arbitrarily large palindromes appear in several works [1–4, 7] and afford other interesting transcendence criteria. As an example, we mention a result obtained by Adamczewski and Bugeaud [3] in the real case that is based on Schmith's theorem [15] on simultaneous approximations of two algebraic numbers by rationals.

Theorem 1.1. *[3] Let $\boldsymbol{a} = (a_n)_{n\geq 1}$ be a sequence of positive integers. If the word $\boldsymbol{a}$ begins with arbitrarily long palindromes, then the real number $\alpha = [0; a_1, a_2, \dots, a_n, \dots]$ is either quadratic or transcendental.*

Subsequently, they studied the case of quasi-palindromic continued fractions given by the following theorem:

Theorem 1.2. *[3] Let $\boldsymbol{a} = (a_n)_{n\geq 1}$ be a sequence of positive integers not eventually periodic and $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ two sequences of finite words such that:*

(i) For any $n \geq 1$, the word $U_n V_n \overline{U}_n$ is a prefix of the word $\boldsymbol{a}$;

(ii) The sequence $\left(\frac{|V_n|}{|U_n|}\right)_{n\geq 1}$ is bounded;

(iii) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

Let $\dfrac{p_n}{q_n}$ denote the sequence of convergents to the real number $\alpha = [0; a_1, a_2, \dots, a_n, \dots]$. Assume that the sequence $(q_n^{\frac{1}{n}})_{n\geq 1}$ is bounded. Then, α is transcendental.

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In the same context, Adamczewski and Bugeaud [4] were interested in studying pairs (α, α') of real numbers where the continued fractions of α and α' are dependent, and they proved that, under some conditions, at least one of them is transcendental. Their proof rests on the Schmidt Subspace Theorem [14].

Theorem 1.3. *[4] Let* $\boldsymbol{a} = (a_n)_{n\geq 1}$ *and* $\boldsymbol{a}' = (a'_n)_{n\geq 1}$ *be two sequences of positive integers. Set*

$$\alpha = [0; a_1, a_2, ...] \qquad \text{and} \qquad \alpha' = [0; a'_1, a'_2, ...].$$

If there exists a sequence of finite words $(V_n)_{n\geq 1}$ *such that:*
i) For every $n \geq 1$, *the word* V_n *is a prefix of the word* $\boldsymbol{a}$ *;*
ii) For every $n \geq 1$, *the word* $\overline{V_n}$ *is a prefix of the word* $\boldsymbol{a}'$*;*
iii) The sequence $(|V_n|)_{n\geq 1}$ *is increasing;*
then, either (at least) one of α *and* α' *is transcendental, or both are in the same real quadratic field.*

In the field of p-adic numbers $\mathbb{Q}_p$, there exist continued fraction expansions. In 1968, Schneider [13] proposed one of the first algorithms to compute a p-adic continued fraction expansion. Two years later, Ruban [11] introduced a simpler definition which is more similar to the real case. Since then, various authors studied properties of Ruban's continued fractions, motivated by the same type of questions studied in the real case. As an example, Laohakosol [8] and Wang [16] independently gave a characterization of rational numbers in terms of Ruban continued fractions. They proved that a p-adic number α is rational if and only if its Ruban continued fraction expansion is finite or ultimately periodic with all partial quotients in the period equal to $p - p^{-1} = (p-1) + \dfrac{(p-1)}{p}$. After that, Ubolsri, Laohakosol, Deze, and Wang [6, 9, 16, 17] studied the transcendence and algebraic independence of elements in $\mathbb{Q}_p$ which have certain Ruban continued fractions expansion. Recently, Ooto [10] proved that the analogue of Lagrange's theorem about the periodicity of real continued fractions does not hold for Ruban's continued fractions in $\mathbb{Q}_p$. In a recent paper Capuano, Veneziano and Zannier [5] gave an effective criterion to detect whether a p-adic number has periodic Ruban's continued fraction expansion.

In this work, we study the p-adic analogous of the results of Adamczewski and Bugeaud mentioned above for Ruban continued fractions by using the p-adic version of the Schmidt Subspace Theorem, due to Schlickewei [12]. The rest of this paper is organized as follows: In Section 2, we start with introducing the p-adic absolute value $|.|_p$, the field of p-adic numbers $\mathbb{Q}_p$ and the Ruban continued fraction and we review some basic properties necessary in our work. In Section 3, we state our transcendence criteria in $\mathbb{Q}_p$, after that, we present some lemmas and notations needed to prove our results and we close this section by giving the proofs of our theorems and an example to illustrate our results.

2. Field of p-adic numbers $\mathbb{Q}_p$

Let p be a prime number. The field of p-adic numbers, $\mathbb{Q}_p$, is the completion of $\mathbb{Q}$ with respect to the metric induced by the valuation $|.|_p$. It is equivalent to the fraction field of the p-adic integers $\mathbb{Z}_p$ defined by

$$\mathbb{Z}_p = \left\{ \alpha = \sum_{i=0}^{+\infty} c_i p^i;\ c_i \in \{0, \ldots, p-1\} \right\},$$

and

$$\mathbb{Q}_p = \left\{ \alpha = \sum_{i=k}^{+\infty} c_i p^i;\ c_i \in \{0, \ldots, p-1\};\ k \in \mathbb{Z} \right\}.$$

The ultrametric absolute value over $\mathbb{Q}$ is defined by

$$|\alpha|_p = \begin{cases} 0 & \text{for } \alpha = 0; \\ p^{-v_p(\alpha)} & \text{for } \alpha \neq 0. \end{cases}$$

where ν_p is the p-adic valuation such that $\nu_p : \mathbb{Q} \to \mathbb{Z} \bigcup \{+\infty\}$ is defined as follows:

$$\text{for all } \alpha \in \mathbb{Q},\ \nu_p(\alpha) = \begin{cases} +\infty & \text{if } \alpha = 0, \\ \inf\{i | \ c_i \neq 0\} & \text{otherwise.} \end{cases}$$

Then, every element $\alpha \in \mathbb{Q}_p$ can be written as:

$$\alpha = c_{-d}p^{-d} + c_{-d+1}p^{-d+1} + \ldots + c_{-1}p^{-1} + c_0 + c_1 p + c_2 p^2 + \ldots$$

where $d \in \mathbb{Z}$, $c_{-d} \neq 0$ and $c_i \in \{0, \ldots, p-1\}$. We define the p-adic floor part of α by

$$[\alpha]_p = c_{-d}p^{-d} + c_{-d+1}p^{-d+1} + \ldots + c_{-1}p^{-1} + c_0.$$

Set $a_0 = [\alpha]_p$, if $[\alpha]_p \neq \alpha$, then α can be written in the form

$$\alpha = \alpha_0 = a_0 + \frac{1}{\alpha_1},$$

with $\alpha_1 \in \mathbb{Q}_p$. Note that $|\alpha_1|_p \geq p$ and $[\alpha_1]_p \neq 0$. Similarly, if $\alpha_1 \neq [\alpha_1]_p$, then, we have

$$\alpha_1 = [\alpha_1]_p + \frac{1}{\alpha_2},$$

with $\alpha_2 \in \mathbb{Q}_p$. We continue the process as soon as $\alpha_n \neq [\alpha_n]_p$. In this way, we obtain

$$\alpha = [\alpha_0]_p + \cfrac{1}{[\alpha_1]_p + \cfrac{1}{\ddots + \cfrac{1}{[\alpha_{n-1}]_p + \cfrac{1}{\alpha_n}}}} = [a_0, a_1, a_2, \ldots, \alpha_n]_p.$$

where $a_k = [\alpha_k]_p$ is called a partial quotient of α and α_n is called the n^{th} complete quotient of α. Then, a_k is a rational number such that $0 \leq a_k < p$ and, if $a_k \neq 0$, we have that $|a_k|_p \geq p$ for all $k \geq 1$.
If the above process stops at a certain step, then

$$[a_0, a_1, a_2, \ldots, a_n]_p$$

is called a finite Ruban continued fraction.
Otherwise, we have

$$[a_0, a_1, a_2, \ldots, a_n, \ldots]_p$$

which is called an infinite Ruban continued fraction.
Now, for an infinite Ruban continued fraction $\alpha = [a_0, a_1, \ldots]_p$, we define non-negative rational numbers p_n, q_n by using recurrence equations:

$$p_{-1} = 1,\ \ p_0 = a_0,\ \ q_{-1} = 0,\ \ q_0 = 1$$

and

$$p_n = a_n p_{n-1} + p_{n-2},\ \ q_n = a_n q_{n-1} + q_{n-2},\ \text{for any } n \geq 1.$$

We can easily check that the Ruban continued fraction has the following properties which are the same properties as the continued fraction of real numbers, for all $n \geq 0$:

$$\frac{p_n}{q_n} = [a_0,\ a_1, \ldots, a_n]_p, \tag{1}$$

$$\alpha = [a_0, a_1, \ldots, a_{n-1}, \alpha_n]_p = \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}}, \tag{2}$$

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n. \tag{3}$$

$\frac{p_n}{q_n}$ is called the n^{th} convergent of α and in $\mathbb{Q}_p$ the convergents satisfy $\lim_{n\to\infty} \frac{p_n}{q_n} = \alpha = [a_0, a_1, \ldots, a_n, \ldots]_p$.
Since $|a_{n+1}|_p > 1$ for all $n \geq 0$, we have the following equality:

$$\left|\alpha - \frac{p_n}{q_n}\right|_p = |a_{n+1}|_p^{-1} |q_n|_p^{-2} < |q_n|_p^{-2}.$$

For more properties of Ruban continued fraction see [16].

3. Results

In this paper, we study p-adic numbers whose sequence of partial quotients of the p-adic continued fraction expansion begins with arbitrarily large palindromes.

Throughout this section, for a given sequence $(a_i)_{i\geq 1} \in \mathbb{Q}_+^{\infty}$, we denote by $A = \max\{a_i| \ i \geq 1\}$ and by $B = B(A) = \frac{A+\sqrt{A^2+4}}{2}$.

Let $\mathbf{a} = (a_i)_{i\geq 1}$ and $\mathbf{a}' = (a_i')_{i\geq 1}$ be two sequences of elements from an alphabet $\mathcal{A}$, that we identify with the infinite words $a_1 \ a_2 \ldots$ and $a_1' \ a_2', \ldots$ respectively. We say that the pair $(\mathbf{a}, \mathbf{a}')$ satisfies Condition $(*)$ if there exists a sequence of finite words $(V_n)_{n\geq 1}$ such that:

(i) For every $n \geq 1$, the word V_n is a prefix of the word $\mathbf{a}$;
(ii) For every $n \geq 1$, the word $\overline{V}_n$ is a prefix of the word $\mathbf{a}'$;
(iii) The sequence $(|V_n|)_{n\geq 1}$ is increasing.

Theorem 3.1. *Let p be a prime number. Let $\alpha = [0; a_1, a_2, \ldots, a_i, \ldots]_p$ and $\alpha' = [0; a_1', a_2', \ldots, a_i', \ldots]_p$ be two p-adic numbers such that $\mathbf{a} = (a_i)_{i\geq 1}$ and $\mathbf{a}' = (a_i')_{i\geq 1}$ are two sequences of rational numbers in $\mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p)$ not ultimately periodic and satisfy Condition $(*)$. Assume that $-v_p(a_i)$ is bounded. If*

$$\frac{\log p}{\log B} > 3,$$

then, either (at least) one of α and α' is transcendental, or both are in the same quadratic field.

We display the immediate consequences of Theorem 3.1.

Corollary 3.2. *Let p be a prime number. Let $\mathbf{a} = (a_i)_{i\geq 1}$ be a sequence of rational numbers in $\mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p)$ not ultimately periodic such that $-v_p(a_i)$ is bounded. Suppose that the word $\mathbf{a}$ begins with arbitrarily long palindromes. If*

$$\frac{\log p}{\log B} > 3,$$

then the p-adic number $\alpha = [0; a_1, a_2, \ldots, a_i, \ldots]_p$ is either quadratic or transcendental.

Corollary 3.3. *Let $p \geq 5$ be a prime number. Let $\mathbf{a} = (a_i)_{i\geq 1}$ be a sequence of rational numbers in $\mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p)$ not ultimately periodic such that $-v_p(a_i)$ is bounded. Suppose that the word $\mathbf{a}$ begins with arbitrarily long palindromes. If $A < 1$, then the p-adic number $\alpha = [0; a_1, a_2, \ldots, a_i, \ldots]_p$ is either quadratic or transcendental.*

The purpose of our next transcendence criterion is to investigate the case of quasi-palindromic p-adic continued fractions with bounded partial quotients in $\mathbb{Q}_p$.

Let $\mathbf{a} = (a_i)_{i\geq 1}$ be a sequence of elements from $\mathcal{A}$. We say that $\mathbf{a}$ satisfies Condition $(**)$ if there exist two sequences of finite words $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ such that:

(i) For every $n \geq 1$, the word $U_n V_n \overline{U}_n$ is a prefix of the word $\mathbf{a}$;
(ii) The sequence $(|V_n|/|U_n|)_{n\geq 1}$ is bounded from above by $\omega > 0$;
(iii) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

Theorem 3.4. *Let p be a prime number. Let $\alpha = [0; a_1, a_2, \ldots]_p$ be a p-adic number such that $\boldsymbol{a} = (a_i)_{i\geq 1}$ is a sequence of rational numbers in $\mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p)$ not ultimately periodic and satisfies Condition (∗∗). Assume that $-\nu_p(a_i)$ is bounded. If*

$$\frac{\log B}{\log p} < \frac{1}{2(2+\omega)},$$

then α is either quadratic or transcendental.

Corollary 3.5. *Let p be a prime number. Let $\alpha = [0; a_1, a_2, \ldots]_p$ be a p-adic number such that $\boldsymbol{a} = (a_i)_{i\geq 1}$ is a sequence of rational numbers in $\mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p)$ not ultimately periodic and satisfies Condition (∗∗) with $0 < \omega < 1$. Assume that $-\nu_p(a_i)$ is bounded. If*

$$\frac{\log B}{\log p} < \frac{1}{6},$$

then α is either quadratic or transcendental.

The main tool for the proofs of our theorems is the p-adic version of the Schmidt Subspace Theorem, established by Schlickewei [12], which is recalled below.
Let $k \geq 2$ be an integer, $\mathbf{x} = (x_1, ..., x_k)$ a k-tuple of rational numbers. Put
$|\mathbf{x}|_\infty = \{\max |x_i|; 1 \leq i \leq k\}$ and $|\mathbf{x}|_p = \{\max |x_i|_p; 1 \leq i \leq k\}$.

Theorem 3.6. *[12] Let p be a prime number, $L_{1,\infty}, \ldots, L_{k,\infty}$ be k linearly independent forms with k-dimensional variable $\mathbf{x}$ and algebraic real coefficients, $L_{1,p}, \ldots, L_{k,p}$ be k linearly independent forms with algebraic p-adic coefficients and the same variables and $\varepsilon > 0$ be a real number. Then, the set of solutions $\mathbf{x} \in \mathbb{Z}^k$ of the inequality :*

$$\prod_{i=1}^{k} (|L_{i,\infty}(\mathbf{x})||L_{i,p}(\mathbf{x})|_p) \leq |\mathbf{x}|_\infty^{-\varepsilon}$$

is contained in the union of a finite number of proper subspaces of $\mathbb{Q}^k$.

Moreover, the proofs of our theorems rest on the following four lemmas:

Lemma 3.7. *[16] The convergents $\frac{p_n}{q_n}$ of $\alpha = [0; a_1, a_2, \ldots]_p$ satisfy*

$$|q_n|_p = |a_1 \ldots a_n|_p, \quad \forall n \geq 1 \tag{4}$$

$$\begin{cases} |p_n|_p = |a_0 \ldots a_n|_p \quad \forall n \geq 1, \text{ if } a_0 \neq 0 \\ |p_1|_p = 1, \quad |p_n|_p = |a_2 \ldots a_n|_p \quad \forall n \geq 2, \text{ if } a_0 = 0 \end{cases} \tag{5}$$

$$|q_n|_p < |q_{n+1}|_p \text{ and } |p_n|_p < |p_{n+1}|_p. \tag{6}$$

Lemma 3.8. *[10] Let $\alpha = [0; a_1, a_2, \ldots]_p$ be a p-adic number with n^{th}convergent $\frac{p_n}{q_n}$. Then*

$$\frac{q_{n-1}}{q_n} = [0; a_n, a_{n-1}, \ldots, a_1]_p.$$

Lemma 3.9. *[10] Let $\alpha = [0; a_1, a_2, \ldots]_p$ and $\alpha' = [0; a_1', a_2', \ldots]_p$ be two p-adic numbers having the same first $(n+1)$ partial quotients. Then*

$$|\alpha - \alpha'|_p < |q_n|_p^{-2} \ .$$

Lemma 3.10. *Suppose that $a_i \in \mathbb{Q}_+^*$ and that $\{a_i | \ i \in \mathbb{N}\}$ is bounded. Let $A = \max\{a_i | \ i \in \mathbb{N}\}$. Then, we have the following inequalities for all n:*

$$q_n \leq \left(\frac{A + \sqrt{A^2+4}}{2}\right)^n \tag{7}$$

and

$$p_n \leq \left(\frac{A + \sqrt{A^2+4}}{2}\right)^{n+1}. \tag{8}$$

Proof : Clearly (7) is true for $n = 0$ and $n = 1$. Let $k \geq 1$ and assuming that (7) is true for $n = 0, \dots, k$, we obtain

$$\begin{aligned} q_{k+1} &= a_{k+1}q_k + q_{k-1} \\ &\leq Aq_k + q_{k-1} \\ &\leq AB^k + B^{k-1} \quad \textit{where } B = \frac{A + \sqrt{A^2+4}}{2} \\ &= B^{k-1}(AB+1) \\ &= B^{k+1} \quad (\textit{because } AB + 1 = B^2) \end{aligned}$$

We prove (8) in the similar way. ■

Note that p_n and q_n are not integers. Therefore we introduce the following notations:

Notation 3.11. *Let $a_i \in \mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p)$. Set $a_i = \frac{b_i}{c_i}$ where $b_i \in \mathbb{N}^*$ and $c_i = p^{-\nu_p(a_i)} \in \mathbb{N}^*$. We take*

$$\boldsymbol{P}_n = \left(\prod_{j=0}^{n} c_j\right) p_n \quad \textit{and} \quad \boldsymbol{Q}_n = \left(\prod_{j=0}^{n} c_j\right) q_n.$$

It is clear from the recurrent formulae for p_n and q_n that $\boldsymbol{P}_n$ and $\boldsymbol{Q}_n$ are integers.

Proof of Theorem 3.1. Since $\dfrac{3 + 1/m^2}{1 - 1/m}$ decreases to 3 as m grows, we can take m large enough such that $-\nu_p(a_i) \leq m$ for all $i \geq 1$ and $\dfrac{3 + 1/m^2}{1 - 1/m} < \dfrac{\log p}{\log B}$.

Assume that α and α' are two p-adic algebraic numbers. Set $s_n = |V_n|$, for any $n \geq 1$. We denote by $\left(\dfrac{p_n}{q_n}\right)_{n\geq 1}$ the sequence of convergent of α'. By assumption, we have

$$\frac{p_{s_n}}{q_{s_n}} = [0; \overline{V}_n]_p,$$

and by using Lemma 3.8, we obtain

$$\frac{q_{s_n-1}}{q_{s_n}} = [0; V_n]_p.$$

Since α and $\dfrac{q_{s_n-1}}{q_{s_n}}$ have the same first $(s_n + 1)$ partial quotients, and according to Lemma 3.9, we have

$$|q_{s_n}\alpha - q_{s_n-1}|_p < |q_{s_n}|_p^{-1}. \tag{9}$$

By using that $s_n \to +\infty$ and the property that $|q_{s_n}|_p \to +\infty$, we obtain

$$\lim_{n\to+\infty} \frac{q_{s_n-1}}{q_{s_n}} = \alpha \quad (\text{in } \mathbb{Q}_p). \tag{10}$$

Moreover, we have

$$|q_{s_n}\alpha' - p_{s_n}|_p < |q_{s_n}|_p^{-1} \quad \text{and} \quad |q_{s_n-1}\alpha' - p_{s_n-1}|_p < |q_{s_n-1}|_p^{-1}. \tag{11}$$

Let us consider the following six independent linear forms with algebraic (real and p-adic) coefficients in variable $\mathbf{X} = (X_1, X_2, X_3)$:

$L_{i,\infty}(\mathbf{X}) = X_i$, for $1 \le i \le 3$,
$L_{1,p}(\mathbf{X}) = \alpha X_1 - X_2$,
$L_{2,p}(\mathbf{X}) = \alpha' X_1 - X_3$,
$L_{3,p}(\mathbf{X}) = X_2$.

Keeping Notations 3.11, we evaluate the product of these linear forms at the integer points

$$\mathbf{X} = (\boldsymbol{Q}_{s_n}, c_{s_n}\boldsymbol{Q}_{s_n-1}, \boldsymbol{P}_{s_n}),$$

and we infer from (9) and (11) that:

$$\prod_{i=1}^{3} |L_{i,p}(\mathbf{X})|_p < \frac{|c_{s_n}|_p |\boldsymbol{Q}_{s_n-1}|_p |\Pi_{s_n}|_p^2}{|q_{s_n}|_p^2},$$

where $\Pi_{s_n} = \prod_{j=0}^{s_n} c_j$.

Since $|a_k|_p \ge p$ for all $k \ge 1$ and $|q_{s_n}|_p = |a_1 \ldots a_{s_n}|_p$, then $|q_{s_n}|_p \ge p^{s_n}$. So, we obtain

$$\prod_{i=1}^{3} |L_{i,p}(\mathbf{X})|_p < |\Pi_{s_n}|_p^3 |q_{s_n}|_p^{-1} \le \frac{|\Pi_{s_n}|_p^3}{p^{s_n}}.$$

On the other hand, $\prod_{i=1}^{3} |L_{i,\infty}(\mathbf{X})|_\infty = |\boldsymbol{Q}_{s_n}|_\infty |c_{s_n}\boldsymbol{Q}_{s_n-1}|_\infty |\boldsymbol{P}_{s_n}|_\infty = |\Pi_{s_n}^3|_\infty |q_{s_n} q_{s_n-1} p_{s_n}|_\infty$.

Using the inequalities given in Lemma 3.10, we have

$$\prod_{i=1}^{3} |L_{i,\infty}(\mathbf{X})|_\infty \le |\Pi_{s_n}^3|_\infty B^{3s_n}$$

where $B = B(A) = \frac{A+\sqrt{A^2+4}}{2}$. This easily implies that:

$$|\mathbf{X}|_\infty^\varepsilon \prod_{i=1}^{3} |L_{i,\infty}(\mathbf{X})|_\infty \le |\Pi_{s_n}^{3+\varepsilon}|_\infty B^{s_n(3+\varepsilon)}.$$

Therefore, we obtain

$$|\mathbf{X}|_\infty^\varepsilon \prod_{i=1}^{3} (|L_{i,\infty}(\mathbf{X})|_\infty |L_{i,p}(\mathbf{X})|_p) \le \frac{|\Pi_{s_n}^\varepsilon|_\infty B^{s_n(3+\varepsilon)}}{p^{s_n}} \le \left(\frac{B^{3+\varepsilon}}{p^{1-m\varepsilon}}\right)^{s_n}.$$

Then, from the hypothesis of Theorem 3.1, we can choose $\varepsilon = \frac{1}{m^2}$ such that for n large enough, we get

$$\prod_{i=1}^{3} (|L_{i,\infty}(\mathbf{X})|_\infty |L_{i,p}(\mathbf{X})|_p) \le \frac{1}{|\mathbf{X}|_\infty^\varepsilon}.$$

Applying Schlickewei's theorem, the points $(\boldsymbol{Q}_{s_n}, c_{s_n}\boldsymbol{Q}_{s_n-1}, \boldsymbol{P}_{s_n})$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$ and then it implies the existence of non-zero integer triple (y_1, y_2, y_3) and an infinite set of distinct positive integers $\mathcal{N}_1$ such that

$$y_1\boldsymbol{Q}_{s_n} + y_2c_{s_n}\boldsymbol{Q}_{s_n-1} + y_3\boldsymbol{P}_{s_n} = 0,$$

for every $n \in \mathcal{N}_1$. From this equation, we have that

$$y_1q_{s_n} + y_2q_{s_n-1} + y_3p_{s_n} = 0. \tag{12}$$

By dividing (12) by q_{s_n} and letting n tend to infinity along $\mathcal{N}_1$, we obtain from (10) that

$$y_1 + y_2\alpha + y_3\alpha' = 0. \tag{13}$$

Let us consider now the linearly independent forms with variable $\mathbf{X} = (X_1, X_2, X_3)$ and algebraic (real and p-adic) coefficients:
$L'_{i,\infty}(\mathbf{X}) = X_i$, for $1 \le i \le 3$,
$L'_{1,p}(\mathbf{X}) = \alpha' X_2 - X_3$,
$L'_{2,p}(\mathbf{X}) = \alpha X_1 - X_2$,
$L'_{3,p}(\mathbf{X}) = X_2$.
Evaluating them on the triple $(\boldsymbol{Q}_{s_n}, c_{s_n}\boldsymbol{Q}_{s_n-1}, c_{s_n}\boldsymbol{P}_{s_n-1})$, we get again from (9) and (11) that

$$\prod_{i=1}^{3}(|L'_{i,\infty}(\mathbf{X})|_\infty |L'_{i,p}(\mathbf{X})|_p) \le \frac{1}{|\mathbf{X}|_\infty^{\varepsilon}}$$

holds for the same positive real number $\varepsilon = \frac{1}{m^2}$ and for n large enough in $\mathcal{N}_1$.
Furthermore, it then follows from Theorem 3.6 that the points $(\boldsymbol{Q}_{s_n}, c_{s_n}\boldsymbol{Q}_{s_n-1}, c_{s_n}\boldsymbol{P}_{s_n-1})$ with $n \in \mathcal{N}_1$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Consequently, there exists a non-zero integer triple (z_1, z_2, z_3) and an infinite set of distinct positive integers $\mathcal{N}_2$ such that

$$z_1\boldsymbol{Q}_{s_n} + z_2c_{s_n}\boldsymbol{Q}_{s_n-1} + z_3c_{s_n}\boldsymbol{P}_{s_n-1} = 0,$$

for every $n \in \mathcal{N}_2$. From this equation, we obtain

$$z_1q_{s_n} + z_2q_{s_n-1} + z_3p_{s_n-1} = 0. \tag{14}$$

By dividing (14) by q_{s_n} and passing to the limit as $\mathcal{N}_2 \ni n \longmapsto +\infty$, it follows from (10) that

$$z_1 + z_2\alpha + z_3\alpha\alpha' = 0. \tag{15}$$

Note that y_3 is non-zero since α is not a rational. Therefore, we derive from (13) and (15) that

$$z_1 + z_2\alpha - z_3\alpha\left(\frac{y_1 + y_2\alpha}{y_3}\right) = 0. \tag{16}$$

Since y_2z_3 is non-zero, (16) implies that α is a p-adic quadratic number, and we deduct from (15) that α' lies in the same quadratic field as α. ■

Proof of Corollary 3.2. It suffices to check that the pair $(\mathbf{a}, \mathbf{a})$ satisfies Condition $(*)$. ■

Proof of Corollary 3.3. If $p \ge 5$, then $p > \phi^3$, where ϕ is the golden ratio and so $\dfrac{\log p}{\log \phi} > 3$. Moreover, if we have $A < 1$, then $B < \phi$.
Applying Corollary 3.2, this concludes the proof. ■

To illustrate our results, we give the following example:

Example 3.12. *Let $p = 5$, $a = \frac{1}{p}$, $b = \frac{2}{p^2} + \frac{1}{p}$ and $c = \frac{3}{p^3} + \frac{2}{p^2} + \frac{1}{p}$.*
Let $(U_n)_{n\geq 0}$ be the sequence of blocks defined as follows:

$$U_0 = ab \quad and \quad U_n = U_{n-1}, \underbrace{c, c, \ldots,}_{n \text{ times}} \overline{U_{n-1}}.$$

It is easy to check that $(U_n)_{n\geq 0}$ is a Cauchy sequence in $\mathbb{Q}_5$. Let $U = \lim_{n\mapsto+\infty} U_n$. According to Corollary 3.3, the 5-adic number $\alpha = [0; U]_5$ is either quadratic or transcendental.

Proof of Theorem 3.4. Since $\dfrac{4 + 1/m^2}{\frac{2}{2+\omega} - 1/m}$ decreases to $2(2 + \omega)$ as m grows, we can take m large enough such that $-v_p(a_i) \leq m$ for all $i \geq 1$ and $\dfrac{4 + 1/m^2}{\frac{2}{2+\omega} - 1/m} < \dfrac{\log p}{\log B}$.

Assume contrary that α is algebraic of degree at least three. Let $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ be the sequences satisfying the conditions of Theorem 3.4. For $n \geq 1$, set $r_n = |U_n|$ and $s_n = |U_n V_n \overline{U}_n|$. We denote by $\left(\dfrac{p_n}{q_n}\right)_{n\geq 1}$ the sequence of convergent of α. By assumption, we have

$$\frac{p_{s_n}}{q_{s_n}} = [0; U_n V_n \overline{U}_n]_p,$$

and Lemma 3.8 ensures that

$$\frac{q_{s_n-1}}{q_{s_n}} = [0; U_n \overline{V}_n \overline{U}_n]_p.$$

Since α and $\dfrac{q_{s_n-1}}{q_{s_n}}$ have the same first $(r_n + 1)$ partial quotients, and by using Lemma 3.9, we obtain

$$|q_{s_n}\alpha - q_{s_n-1}|_p < |q_{s_n}|_p |q_{r_n}|_p^{-2}, \tag{17}$$

this yields that in $\mathbb{Q}_p$

$$\lim_{n\to+\infty} \frac{q_{s_n-1}}{q_{s_n}} = \alpha. \tag{18}$$

Furthermore, we have

$$|q_{s_n}\alpha - p_{s_n}|_p < |q_{s_n}|_p^{-1} \quad \text{and} \quad |q_{s_n-1}\alpha - p_{s_n-1}|_p < |q_{s_n-1}|_p^{-1}. \tag{19}$$

Now, let us consider now the following independent linear forms with algebraic coefficients in variable $\mathbf{X} = (X_1, X_2, X_3, X_4)$:

$L_{i,\infty}(\mathbf{X}) = X_i$, for $1 \leq i \leq 4$,
$L_{1,p}(\mathbf{X}) = \alpha X_1 - X_3$,
$L_{2,p}(\mathbf{X}) = \alpha X_2 - X_4$,
$L_{3,p}(\mathbf{X}) = \alpha X_1 - X_2$,
$L_{4,p}(\mathbf{X}) = X_2$.

Evaluating them on the quadruple $(\boldsymbol{Q}_{s_n}, c_{s_n}\boldsymbol{Q}_{s_n-1}, \boldsymbol{P}_{s_n}, c_{s_n}\boldsymbol{P}_{s_n-1})$, we can derive from (17) and (19) that

$$\prod_{i=1}^{4} |L_{i,p}(\mathbf{X})|_p < \frac{|c_{s_n}|_p |\Pi_{s_n}|_p^4}{|q_{r_n}|_p^2},$$

where $\Pi_{s_n} = \prod_{j=0}^{s_n} c_j$. This implies that:

$$\prod_{i=1}^{4} |L_{i,p}(\mathbf{X})|_p < \frac{|\Pi_{s_n}|_p^4}{|q_{r_n}|_p^2} \leq \frac{|\Pi_{s_n}|_p^4}{p^{2r_n}}.$$

On the other hand,

$$\prod_{i=1}^{4} |L_{i,\infty}(\mathbf{X})|_\infty = |\boldsymbol{Q}_{s_n}|_\infty |c_{s_n}\boldsymbol{Q}_{s_n-1}|_\infty |\boldsymbol{P}_{s_n}|_\infty |c_{s_n}\boldsymbol{P}_{s_n-1}|_\infty = |\Pi_{s_n}^4|_\infty |q_{s_n}q_{s_n-1}p_{s_n}p_{s_n-1}|_\infty$$

Using the inequalities (7) and (8), we obtain

$$\prod_{i=1}^{4} |L_{i,\infty}(\mathbf{X})|_\infty \leq |\Pi_{s_n}^4|_\infty B^{4s_n}$$

where $B = B(A) = \frac{A+\sqrt{A^2+4}}{2}$.
Hence, we have

$$|\mathbf{X}|_\infty^\varepsilon \prod_{i=1}^{4} |L_{i,\infty}(\mathbf{X})|_\infty \leq |\Pi_{s_n}^{4+\varepsilon}|_\infty B^{s_n(4+\varepsilon)}.$$

Therefore, we obtain the following inequality

$$|\mathbf{X}|_\infty^\varepsilon \prod_{i=1}^{4} (|L_{i,\infty}(\mathbf{X})|_\infty |L_{i,p}(\mathbf{X})|_p) \leq \frac{|\Pi_{s_n}^\varepsilon|_\infty B^{s_n(4+\varepsilon)}}{p^{2r_n}} \leq \left(\frac{B^{4+\varepsilon}}{p^{\frac{2}{2+\omega}-m\varepsilon}}\right)^{s_n}.$$

Then, from the hypothesis of Theorem 3.4, we can choose $\varepsilon = \frac{1}{m^2}$ such that for n large enough, we get

$$\prod_{i=1}^{4} (|L_{i,\infty}(\mathbf{X})|_\infty |L_{i,p}(\mathbf{X})|_p) \leq \frac{1}{|\mathbf{X}|_\infty^\varepsilon}.$$

Schlickewei's theorem confirms that the points $(\boldsymbol{Q}_{s_n}, c_{s_n}\boldsymbol{Q}_{s_n-1}, \boldsymbol{P}_{s_n}, c_{s_n}\boldsymbol{P}_{s_n-1})$, lie in a finite number of proper subspaces of $\mathbb{Q}^4$ and then it implies the existence of non-zero integer quadruple (y_1, y_2, y_3, y_4) and an infinite set of distinct positive integers $\mathcal{N}_1$ such that

$$y_1\boldsymbol{Q}_{s_n} + y_2c_{s_n}\boldsymbol{Q}_{s_n-1} + y_3\boldsymbol{P}_{s_n} + y_4c_{s_n}\boldsymbol{P}_{s_n-1} = 0,$$

for every $n \in \mathcal{N}_1$. From this equation, it results that

$$y_1q_{s_n} + y_2q_{s_n-1} + y_3p_{s_n} + y_4p_{s_n-1} = 0. \tag{20}$$

Dividing (20) by q_{s_n} and letting n tend to infinity along $\mathcal{N}_1$, we get from (18) that

$$y_1 + (y_2 + y_3)\alpha + y_4\alpha^2 = 0. \tag{21}$$

Since, and by assumption, α is not a quadratic number, we have $y_1 = y_4 = 0$ and $y_2 = -y_3$. Therefore, (20) involve that:

$$q_{s_n-1} = p_{s_n} \tag{22}$$

Likewise, we consider the linearly independent linear forms with variable $\mathbf{X} = (X_1, X_2, X_3)$ and algebraic coefficients:

$L'_{i,\infty}(\mathbf{X}) = X_i$, for $1 \le i \le 3$,
$L'_{1,p}(\mathbf{X}) = \alpha X_1 - X_2$,
$L'_{2,p}(\mathbf{X}) = \alpha X_2 - X_3$,
$L'_{3,p}(\mathbf{X}) = X_1$.
Evaluating them on the triple $(\boldsymbol{Q}_{s_n}, \boldsymbol{P}_{s_n}, c_{s_n}\boldsymbol{P}_{s_n-1})$, we get again from (17) and (22) that

$$|\mathbf{X}|_\infty^\varepsilon \prod_{i=1}^{3} (|L'_{i,\infty}(\mathbf{X})|_\infty |L'_{i,p}(\mathbf{X})|_p) \le \frac{\Pi_{s_n}^\varepsilon B^{s_n(3+\varepsilon)}}{p^{s_n}} \le \left(\frac{B^{3+\varepsilon}}{p^{1-m\varepsilon}}\right)^{s_n} \le \left(\frac{B^{4+\varepsilon}}{p^{\frac{2}{2+\omega}-m\varepsilon}}\right)^{s_n}.$$

Then, from the hypothesis of Theorem 3.4, we can choose $\varepsilon = \frac{1}{m^2}$ such that for n large enough, we get

$$\prod_{i=1}^{3} (|L'_{i,\infty}(\mathbf{X})|_\infty |L'_{i,p}(\mathbf{X})|_p) \le \frac{1}{|\mathbf{X}|_\infty^\varepsilon}.$$

holds for the same positive real number $\varepsilon = \frac{1}{m^2}$ and for n large enough in $\mathcal{N}_1$.
Moreover, it then follows from Theorem 3.6 that the points $(\boldsymbol{Q}_{s_n}, \boldsymbol{P}_{s_n}, c_{s_n}\boldsymbol{P}_{s_n-1})$ with $n \in \mathcal{N}_1$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Thereby, there exists a non-zero integer triple (z_1, z_2, z_3) and an infinite set of distinct positive integers $\mathcal{N}_2$ such that

$$z_1\boldsymbol{Q}_{s_n} + z_2\boldsymbol{P}_{s_n} + z_3 c_{s_n}\boldsymbol{P}_{s_n-1} = 0,$$

for every $n \in \mathcal{N}_2$. From this equation, we obtain

$$z_1 q_{s_n} + z_2 q_{s_n-1} + z_3 p_{s_n-1} = 0. \tag{23}$$

By dividing (23) by q_{s_n} and passing to the limit as $\mathcal{N}_2 \ni n \longmapsto +\infty$, it follows from (18) that

$$z_1 + z_2\alpha + z_3\alpha^2 = 0. \tag{24}$$

Since (z_1, z_2, z_3) is a non-zero triple of integers, a contradiction is achieved.
Consequently, the p-adic number α is transcendental and finally the proof of our theorem is reached. ■

Proof of Corollary 3.5. It suffices to observe that $\inf_{\omega \in]0,1[} \left\{\frac{1}{2(2+\omega)}\right\} = \frac{1}{6}$. ■

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