Filomat 36:4 (2022), 1403–1413
https://doi.org/10.2298/FIL2204403B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Riemann Solitons on Almost Co-Kähler Manifolds

Gour Gopal Biswas[a], Xiaomin Chen[b], Uday Chand De[c]

[a]*Department of Mathematics, University of Kalyani, Kalyani-741235, West Bengal, India.*
[b]*College of Science, China University of Petroleum-Beijing, Beijing, 102249, China.*
[c]*Department of Pure Mathematics, University of Calcutta, 35, Ballygaunge Circular Road, Kolkata -700019, West Bengal, India.*

Abstract. The aim of the present paper is to characterize almost co-Kähler manifolds whose metrics are the Riemann solitons. At first we provide a necessary and sufficient condition for the metric of a 3-dimensional manifold to be Riemann soliton. Next it is proved that if the metric of an almost co-Kähler manifold is a Riemann soliton with the soliton vector field ξ, then the manifold is flat. It is also shown that if the metric of a (κ, μ)-almost co-Kähler manifold with $\kappa < 0$ is a Riemann soliton, then the soliton is expanding and κ, μ, λ satisfies a relation. We also prove that there does not exist gradient almost Riemann solitons on (κ, μ)-almost co-Kähler manifolds with $\kappa < 0$. Finally, the existence of a Riemann soliton on a three dimensional almost co-Kähler manifold is ensured by a proper example.

1. Introduction

Udrişte ([24], [25]) introduced the notion of Riemann flow. The Riemann flow is defined by

$$\frac{\partial}{\partial t}G(t) = -2R(g(t)), \tag{1}$$

where $G = \frac{1}{2}g \odot g$, R is the Riemann curvature tensor of type $(0,4)$ corresponding to the metric g and $\odot$ denotes the Kulkarni-Nomizu product given by

$$\begin{aligned}(P \odot Q)(E, F, W, X) = \quad & P(E, X)Q(F, W) + P(F, W)Q(E, X) \\ & -P(E, W)Q(F, X) - P(F, X)Q(E, W).\end{aligned}$$

In the same way as Ricci solitons, Riemann solitons were introduced by Hirică and Udrişte [16] which are the self-similar solution of Riemann flow. A Riemannian metric g on a smooth manifold M is said to be a Riemann soliton if there exists a smooth vector field Z and a real constant λ such that

$$2R + \lambda g \odot g + g \odot \pounds_Z g = 0, \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 53C25; Secondary 53C15.
Keywords. Riemann flow, Riemann solitons, Almost co-Kähler manifolds, (κ, μ)-almost co-Kähler manifolds.
Received: 24 March 2021; Accepted: 01 November 2021
Communicated by Ljubica Velimirović
The first author is financially supported by UGC, Ref. ID. 423044. The second author is the corresponding author and supported by Science Foundation of China University of Petroleum-Beijing (No.2462020XKJS02, No.2462020YXZZ004).
Email addresses: ggbiswas6@gmail.com (Gour Gopal Biswas), xmchen@cup.edu.cn (Xiaomin Chen), uc_de@yahoo.com (Uday Chand De)

where $£_Z$ is the Lie derivative along the vector field Z. The vector field Z is known as potential vector field. We denote a Riemann soliton by (g, Z, λ). When $\lambda \in C^\infty(M)$, then g is said to be an *almost Riemann soliton*. If Z is a Killing vector field, then M is a manifold of constant sectional curvature. Thus the Riemann soliton is the generalization of the space of constant curvature. The soliton will be called *expanding, steady or shrinking* according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. When the vector field Z is a gradient of some smooth function u, then the Riemann soliton is called *a gradient Riemann soliton* and the equation (2) takes the form

$$R + \frac{\lambda}{2} g \odot g + g \odot \nabla^2 u = 0, \tag{3}$$

where $\nabla^2 u$ is the Hessian of the function u. If λ is a smooth function in (3), then the metric g is called a *gradient almost Riemann soliton*. Using Kulkarni-Nomizu product, the equation (2) can be written as

$$\begin{aligned} &2R(E, F, W, X) + 2\lambda\{g(E, X)g(F, W) - g(E, W)g(F, X)\} \\ &\quad +g(E, X)(£_Z g)(F, W) + g(F, W)(£_Z g)(E, X) \\ &\quad -g(E, W)(£_Z g)(F, X) - g(F, X)(£_Z g)(E, W) = 0 \end{aligned} \tag{4}$$

for all vector fields E, F, W, X on M. Contracting the equation (4), we lead

$$2S(F, W) + 2\{(m-1)\lambda + \mathrm{div}Z\}g(F, W) + (m-2)(£_Z g)(F, W) = 0, \tag{5}$$

where S is the Ricci tensor, $m \geq 3$ is the dimension of the manifold M and div denotes the divergence operator. Contracting again the equation (5), we have

$$r + m(m-1)\lambda + (2m-2)\,\mathrm{div}Z = 0, \tag{6}$$

where r is the scalar curvature. From the foregoing equation, we can easily see that div$Z$ is constant if and only if r is constant.

In [16], Hirică and Udrişte studied Sasaki-Riemann soliton. They proved that, if the metric g of a Sasakian manifold M is a gradient Riemann soliton with potential function u as harmonic or a Riemann soliton with potential vector field Z is pointwise collinear to Reeb vector field ξ, then M is a Sasaki-space form. In [14], Venkatesha et al. proved some interesting results on Riemann soliton within the framework of contact geometry. They also studied Riemann solitons and almost Riemann solitons on almost Kenmotsu manifolds (cf.[26]).

The present paper is organized as follows: After introduction, in Section 2 we recall the definition and basic properties of almost co-Kähler manifolds and (κ, μ)-almost co-Kähler manifolds. In the next section, we characterize a three-dimensional manifold whose metric is the Riemann soliton. In Sections 4 and 5, we prove some lemmas and theorems on Riemann soliton in almost co-Kähler manifolds and (κ, μ)-almost co-Kähler manifolds. In the Section 6, we consider gradient almost Riemann solitons on (κ, μ)-almost co-Kähler manifolds. Finally, we construct an example to verify our results.

2. Almost co-Kähler manifolds

A smooth manifold M^{2n+1} of dimension $(2n + 1)$ together with the triple (η, ξ, φ), where η is a 1-form, ξ is a global vector field and φ is a $(1, 1)$-tensor field, is said to be an almost contact manifold [2] if

$$\varphi^2 + \mathrm{id} = \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{7}$$

where id is the identity automorphism. From (7) we can obtain $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. An almost contact structure (η, ξ, φ) will be called normal if the almost complex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

$$J\left(E, \gamma\frac{d}{dt}\right) = \left(\varphi E - \gamma\xi, \eta(E)\frac{d}{dt}\right)$$

for all vector field E on M^{2n+1} and $\gamma \in C^\infty(M^{2n+1} \times \mathbb{R})$, is integrable. According to Blair [2], $[\varphi, \varphi] = -2d\eta \otimes \xi$ is the condition for normality of the almost contact structure (η, ξ, φ) and conversely, where $[\varphi, \varphi]$ denotes the Nijenhuis tensor of φ defined by

$$[\varphi, \varphi](E, F) = \varphi^2[E, F] + [\varphi E, \varphi F] - \varphi[\varphi E, F] - \varphi[E, \varphi F]$$

for all vector fields E, F on M^{2n+1}. If a Riemannian metric g on M^{2n+1} satisfies

$$g(E, F) = g(\varphi E, \varphi F) + \eta(E)\eta(F) \tag{8}$$

for all vector fields E, F on M^{2n+1}, then the manifold together with (η, ξ, φ, g) is said to be an almost contact metric manifold and g is called compatible metric with respect to the almost contact structure. The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(E, F) = g(E, \varphi F)$ for all vector fields E, F on M^{2n+1}.

An almost contact metric manifold M^{2n+1} is said to be an almost co-Kähler manifold if both η and Φ are closed i.e., $d\eta = 0$ and $d\Phi = 0$, where d denotes exterior derivative. In addition, if M^{2n+} is normal, then the manifold M^{2n+1} is called co-Kähler manifold. An (almost) co-Kähler manifold is nothing but an (almost) cosymplectic manifold defined by Blair [3] and studied by several authors (see [1], [4]-[8], [11]-[13], [17, 18] [23], [27]-[32]).

On any almost co-Kähler manifold, we can define an $(1, 1)$-tensor field $h = \frac{1}{2}\pounds_\xi \varphi$. According to [19], [20] and [22], it is known that h and $h'(= h \circ \varphi)$ are symmetric tensors and satisfy

$$h\xi = 0, \quad h' = -\varphi h, \quad \operatorname{tr} h = \operatorname{tr} h' = 0, \tag{9}$$

$$\nabla_\xi \varphi = 0, \quad \nabla \xi = h', \tag{10}$$

$$\varphi \ell \varphi - \ell = 2h^2, \tag{11}$$

$$\nabla_\xi h = -h^2 \varphi - \varphi \ell, \tag{12}$$

$$S(\xi, \xi) + \operatorname{tr} h^2 = 0, \tag{13}$$

where $\ell = R(., \xi)\xi$ is the Jacobi operator along the Reeb vector field, tr denotes for trace and ∇ is the Riemannian connection with respect to the metric g. Using the second equation of (10), we see that $(\pounds_\xi g)(E, F) = 2g(h'E, F)$ for all vector fields E, F on M^{2n+1}. Thus, ξ is a Killing vector field if and only if $h = 0$.

A (κ, μ)-*almost co-Kähler manifold* M^{2n+1}, introduced by Endo [15], is an almost co-Kähler manifold whose structure vector field ξ belongs to the (κ, μ)-nullity distribution, i.e. the curvature tensor R satisfies

$$R(E, F)\xi = \kappa(\eta(F)E - \eta(E)F) + \mu(\eta(F)hE - \eta(E)hF) \tag{14}$$

for all vector fields E, F on M^{2n+1} and $(\kappa, \mu) \in \mathbb{R}^2$. Taking ξ instead of F in (14), we have $\ell = -\kappa\varphi^2 + \mu h$. Using this value of ℓ in (11), it follows that

$$h^2 = \kappa\varphi^2. \tag{15}$$

From the above, it is easy to see that $\kappa \le 0$ and $\kappa = 0$ if and only if M^{2n+1} is a K-almost co-Kähler manifold. In particular, if $\mu = 0$ then the manifold is said to be $N(\kappa)$-almost co-Kähler manifold [9]. Any co-Kähler manifold satisfies (14) with $\kappa = \mu = 0$. Dacko and Olszak [10] defined almost co-Kähler (κ, μ, ν)-spaces. An almost co-Kähler manifold is said to be a (κ, μ, ν)-space if the curvature tensor R satisfies

$$\begin{aligned} R(E, F)\xi &= \kappa(\eta(F)E - \eta(E)F) + \mu(\eta(F)hE - \eta(E)hF) \\ &- \nu(\eta(F)h'E - \eta(E)h'F) \end{aligned}$$

for all vectors fields E, F on M^{2n+1}.

In a (κ, μ)-almost co-Kähler manifold the following relations hold [21] :

$$\nabla_\xi h = \mu h', \tag{16}$$

$$\nabla_\xi h^2 = 0, \tag{17}$$

$$\ell\varphi - \varphi\ell = 2\mu h'. \tag{18}$$

Lemma 2.1. *[4] The Ricci operator Q of a (κ, μ)-almost co-Kähler manifold M^{2n+1}, $n \ge 1$, is given by*

$$Q = \mu h + 2n\kappa\eta \otimes \xi. \tag{19}$$

3. Riemann solitons on 3-dimensional manifolds

Suppose a metric (g, Z, λ) of a 3-dimensional manifold M^3 is a Riemann soliton. Then from the equation (5), we have

$$2S(F, W) + (4\lambda + 2\,\mathrm{div}Z)g(F, W) + (\pounds_Z g)(F, W) = 0 \tag{20}$$

for all vector fields F, W on M^3. Tracing the previous equation, we have

$$\mathrm{div}Z = -\frac{r + 6\lambda}{4}. \tag{21}$$

Using this value of $\mathrm{div}Z$ in (20), we obtain

$$2S(F, W) + \left(\lambda - \frac{r}{2}\right) g(F, W) + (\pounds_Z g)(F, W) = 0 \tag{22}$$

for all vector fields F, W on M^3.

Conversely, suppose the equation (22) is satisfied. It is well known that the curvature tensor R of any 3-dimensional manifold is given by

$$\begin{aligned} R(E, F)W = \; & S(F, W)E - S(E, W)F + g(F, W)QE - g(E, W)QF \\ & -\tfrac{r}{2}\{g(F, W)E - g(E, W)F\}. \end{aligned} \tag{23}$$

Taking inner product of (23) with X, we have

$$\begin{aligned} 2R(E, F, W, X) &= 2S(F, W)g(E, X) - 2S(E, W)g(F, X) \\ &+ 2g(F, W)S(E, X) - 2g(E, W)S(F, X) \\ &- r\{g(F, W)g(E, X) - g(E, W)g(F, X)\}. \end{aligned}$$

Using (22) in the foregoing equation, we obtain the equation (4). Hence (g, Z, λ) is a Riemann soliton. Thus we obtain the following:

Theorem 3.1. *Let (M^3, g) be a three dimensional manifold. Then (g, Z, λ) is a Riemann soliton if and only if*

$$2S + \left(\lambda - \frac{r}{2}\right) g + \pounds_Z g = 0. \tag{24}$$

4. Riemann solitons on almost co-Kähler manifolds

Suppose the metric g of an almost co-Kähler manifold M is the Riemann soliton with the soliton vector field $Z = f\xi$ and $df \wedge \eta = 0$.

The condition $df \wedge \eta = 0$ implies $Ef = (\xi f)\eta(E)$ for all vector field E on M. Taking covariant derivative of $Z = f\xi$ along the vector field E and using the second equation of (10), we have

$$\nabla_E Z = (\xi f)\eta(E)\xi + fh'E,$$

which gives

$$(\pounds_Z g)(E, F) = g(\nabla_E Z, F) + g(E, \nabla_F Z) = 2(\xi f)\eta(E)\eta(F) + 2fg(h'E, F) \tag{25}$$

for all vector fields E, F on M.

By virtue of (25) and (4), we get

$$\begin{aligned}
2R(E,F,W,X) &+ 2\lambda\{g(E,X)g(F,W) - g(E,W)g(F,X)\} \\
&+g(E,X)\{2(\xi f)\eta(F)\eta(W) + 2fg(h'F,W)\} \\
&+g(F,W)\{2(\xi f)\eta(E)\eta(X) + 2fg(h'E,X)\} \\
&-g(E,W)\{2(\xi f)\eta(F)\eta(X) + 2fg(h'F,X)\} \\
&-g(F,X)\{2(\xi f)\eta(E)\eta(W) + 2fg(h'E,W)\} = 0
\end{aligned} \tag{26}$$

for all vector fields E,F,W,X on M. Taking ξ instead of W in (26), we obtain

$$\begin{aligned}
R(E,F,\xi,X) &+ (\lambda + \xi f)\{g(E,X)\eta(F) - g(F,X)\eta(E)\} \\
&+f\{g(h'E,X)\eta(F) - g(h'F,X)\eta(E)\} = 0.
\end{aligned}$$

Eliminating X in the previous equation, we infer

$$R(E,F)\xi = -(\lambda + \xi f)\{\eta(F)E - \eta(E)F\} + f\{\eta(F)\varphi hE - \eta(E)\varphi hF\}. \tag{27}$$

This implies that M is a (κ,μ,ν)-space with $\kappa = -(\lambda+\xi f)$, $\mu = 0$ and $\nu = f$. Thus we can write the following:

Lemma 4.1. *If the metric (g,Z,λ) of an almost co-Kähler manifold M is a Riemann soliton with the soliton vector field $Z = f\xi$ and $df \wedge \eta = 0$, then M is a (κ,μ,ν)-space with $\kappa = -(\lambda + \xi f)$, $\mu = 0$ and $\nu = f$.*

Putting $f = 1$ in (27), we have

$$R(E,F)\xi = -\lambda\{\eta(F)E - \eta(E)F\} + \{\eta(F)\varphi hE - \eta(E)\varphi hF\}, \tag{28}$$

which gives $\ell = \lambda\varphi^2 + \varphi h$. Using this value of ℓ in (11), it follows that

$$h^2 = -\lambda\varphi^2. \tag{29}$$

Taking covariant derivative of (29) and using first equation of (10), we lead

$$\nabla_\xi h^2 = 0. \tag{30}$$

Using $\ell = \lambda\varphi^2 + \varphi h$ and (29) in (12), we get

$$\nabla_\xi h = h. \tag{31}$$

Now

$$0 = (\nabla_\xi h^2)E = (\nabla_\xi h)hE + h((\nabla_\xi h)E) = 2h^2E = -2\lambda\varphi^2E$$

for all vector field E on M, which gives $\lambda = 0$. Since h is a symmetric tensor, from (29) we get $h = 0$. Consequently, $(\pounds_\xi g)(E,F) = 0$. From (4), we see that $R(E,F)W = 0$ for all vector fields E,F,W on M. From the above discussion, we can state the following:

Theorem 4.2. *If the metric g of an almost co-Kähler manifold M is a Riemann soliton with the soliton vector field ξ, then M is flat.*

5. Riemann solitons on (κ,μ)-almost co-Kähler manifols

Suppose the metric (g,Z,λ) of a (κ,μ)-almost co-Kähler manifold M^{2n+1} of dimension $(2n+1)$ is the Riemann soliton. The equation (5) can be written as

$$(\pounds_Z g)(F,W) = -\frac{2}{2n-1}S(F,W) - \frac{2}{2n-1}(2n\lambda + \mathrm{div}Z)g(F,W) \tag{32}$$

for all vector fields F, W on M^{2n+1}. From (19) we have $r = 2n\kappa =$ constant. Consequently, div$Z$ is a constant.

Taking covariant derivative of (32) along the arbitrary vector field E, we have

$$(\nabla_E \pounds_Z g)(F, W) = -\frac{2}{2n-1}(\nabla_E S)(F, W). \tag{33}$$

From Yano [33], we recall a well known formula

$$\begin{aligned} & (\pounds_Z \nabla_E g - \nabla_E \pounds_Z g - \nabla_{[Z,E]} g)(F, W) \\ = \; & -g((\pounds_Z \nabla)(E, F), W) - g((\pounds_Z \nabla)(E, W), F). \end{aligned}$$

Using the symmetry property of $\pounds_Z \nabla$ in the above formula, we have

$$\begin{aligned} 2g((\pounds_Z \nabla)(E, F), W) &= (\nabla_E \pounds_Z g)(F, W) + (\nabla_F \pounds_Z g)(W, E) \\ &- (\nabla_W \pounds_Z g)(E, F). \end{aligned} \tag{34}$$

Using (19) and (33) in (34), we lead

$$\begin{aligned} g((\pounds_Z \nabla)(E, F), W) &= \frac{1}{2n-1}[\mu g((\nabla_W h)E, F) - \mu g((\nabla_E h)F, W) \\ &- \mu g((\nabla_F h)W, E) - 4n\kappa\eta(W) g(h'E, F)]. \end{aligned} \tag{35}$$

Taking ξ instead of F in (35) and using (10) and (16), we obtain

$$g((\pounds_Z \nabla)(E, \xi), W) = \frac{1}{2n-1}[2\mu\kappa g(E, \varphi W) - \mu^2 g(h'E, W)],$$

which gives

$$(\pounds_Z \nabla)(E, \xi) = -\frac{1}{2n-1}(2\mu\kappa\varphi E + \mu^2 h'E). \tag{36}$$

Taking covariant derivative of (36) along the vector field F, we lead

$$\begin{aligned} (\nabla_F \pounds_Z \nabla)(E, \xi) &= -\frac{1}{2n-1}(2\mu\kappa(\nabla_F \varphi)E + \mu^2(\nabla_F h')E) \\ &- (\pounds_Z \nabla)(E, h'F). \end{aligned} \tag{37}$$

Using the above equation in the formula [33]

$$(\pounds_Z R)(E, F)W = (\nabla_E \pounds_Z \nabla)(F, W) - (\nabla_F \pounds_Z \nabla)(E, W),$$

we get

$$(\pounds_Z R)(E, \xi)\xi = \frac{2}{2n-1}(2\mu\kappa hE + \mu^2\kappa\varphi^2 E) - \frac{\mu^3}{2n-1}hE, \tag{38}$$

where we have used (10), (15) and (16).

On the other hand taking Lie-derivative of $R(E, \xi)\xi = \kappa\{E - \eta(E)\xi\} + \mu hE$ along Z, we have

$$\begin{aligned} (\pounds_Z R)(E, \xi)\xi &= -\kappa(\pounds_Z \eta)E\xi - \kappa\eta(E)\pounds_Z \xi + \mu(\pounds_Z h)E \\ &- R(E, \pounds_Z \xi)\xi - R(E, \xi)\pounds_Z \xi. \end{aligned} \tag{39}$$

By virtue of (38) and (39) we obtain

$$\begin{aligned} & \frac{2}{2n-1}(2n\kappa hE + \mu^2\kappa\varphi^2 E) - \frac{\mu^3}{2n-1}hE \\ = \; & -\kappa(\pounds_Z \eta)E\xi - \kappa\eta(E)\pounds_Z \xi + \mu(\pounds_Z h)E - R(E, \pounds_Z \xi)\xi - R(E, \xi)\pounds_Z \xi. \end{aligned}$$

Contracting the previous equation with respect to the orthonormal basis $\{e_1, e_2, \cdots, e_n, \varphi e_1, \varphi e_2, \cdots, \varphi e_n, \xi\}$, where $he_i = \sqrt{-\kappa}e_i$, we get

$$S(\pounds_Z\xi, \xi) = \frac{2n}{2n-1}\mu^2\kappa.$$

Utilizing (19) in the above equation, we lead

$$g(\pounds_Z\xi, \xi) = \frac{\mu^2}{2n-1}. \tag{40}$$

Putting $F = W = \xi$ in (32) and using (40), we infer

$$\mathrm{div}Z = \mu^2 - 2n(\kappa + \lambda). \tag{41}$$

Contraction the equation (32), it follows that

$$2\,\mathrm{div}Z = -\kappa - (2n+1)\lambda. \tag{42}$$

By virtue of (41) and (42), we get

$$\kappa = \frac{2\mu^2}{4n-1} - \frac{2n-1}{4n-1}\lambda.$$

Thus we are in a position to state the following:

Theorem 5.1. *If the metric (g, Z, λ) of a (κ, μ)-almost co-Kähler manifold M^{2n+1} with $\kappa < 0$ is a Riemann soliton, then κ, μ and λ satisfy the relation*

$$\kappa = \frac{2\mu^2}{4n-1} - \frac{2n-1}{4n-1}\lambda. \tag{43}$$

Since $\kappa < 0$, from (43) we easily see that $\lambda > 0$. Hence, we can state that

Corollary 5.2. *If the metric (g, Z, λ) of a (κ, μ)-almost co-Kähler manifold M^{2n+1} with $\kappa < 0$ is a Riemann soliton, then the soliton is expanding.*

In particular, for $\mu = 0$ we can state that the followings:

Corollary 5.3. *If the metric (g, Z, λ) of a $N(\kappa)$-almost co-Kähler manifold M^{2n+1} with $\kappa < 0$ is a Riemann soliton, then $(4n-1)\kappa = -(2n-1)\lambda$.*

Corollary 5.4. *If the metric (g, Z, λ) of a $N(\kappa)$-almost co-Kähler manifold M^{2n+1} with $\kappa < 0$ is a Riemann soliton, then the soliton is expanding.*

6. Gradient almost Riemann solitons on (κ, μ)-almost co-Kähler manifolds

In this section we consider a (κ, μ)-almost co-Kähler manifold M^{2n+1} whose metric g is a gradient almost Riemann soliton. We need the following lemma before proving the main results.

Lemma 6.1. *(Lemma 3.8 of [14]) If the metric g of a Riemannian manifold M^{2n+1} is a gradient almost Riemann soliton, then for any vector fields E, F on M^{2n+1} the curvature tensor R satisfies*

$$R(E,F)Du = \frac{1}{2n-1}\{(\nabla_F Q)E - (\nabla_E Q)F + F(2n\lambda + \Delta u)E - E(2n\lambda + \Delta u)F\}, \tag{44}$$

where D denotes the gradient operator and $\Delta = divD$.

By virtue (10) and (19) the equation (44) can be written as

$$\begin{aligned} R(E,F)Du = &= \frac{1}{2n-1}\{\mu(\nabla_F h)E - \mu(\nabla_E h)F \\ &+ 2n\kappa\eta(E)h'F - 2n\kappa\eta(F)h'E \\ &+ F(2n\lambda + \Delta u)E - E(2n\lambda + \Delta u)F\}. \end{aligned} \tag{45}$$

Balkan et al. [1] proved that in a (κ,μ)-almost co-Kähler manifold the tensor field h satisfies

$$\begin{aligned} (\nabla_E h)F - (\nabla_F h)E &= \kappa(\eta(F)\varphi E - \eta(E)\varphi F + 2g(\varphi E, F)\xi) \\ &+ \mu(\eta(F)\varphi hE - \eta(E)\varphi hF). \end{aligned} \tag{46}$$

Using (46) in (45), we obtain

$$\begin{aligned} R(E,F)Du &= \frac{1}{2n-1}\{\mu\kappa(\eta(E)\varphi F - \eta(F)\varphi E + 2g(E,\varphi F)\xi) \\ &+ (\mu^2 - 2n\kappa)(\eta(F)h'E - \eta(E)h'F) \\ &+ F(2n\lambda + \Delta u)E - E(2n\lambda + \Delta u)F\}. \end{aligned} \tag{47}$$

Taking inner product of the foregoing equation with ξ and using (14), it follows that

$$\begin{aligned} &\kappa((Fu)\eta(E) - (Eu)\eta(F)) + \mu(g(hF,Du)\eta(E) - g(hE,Du)\eta(F)) \\ = &\frac{1}{2n-1}\{2\mu\kappa g(E,\varphi F) + F(2n\lambda + \Delta u)\eta(E) - E(2n\lambda + \Delta u)\eta(F)\}. \end{aligned} \tag{48}$$

Replacing E and F by φE and φF respectively in (48), we infer

$$0 = \frac{2\mu\kappa}{2n-1} g(E,\varphi F),$$

which gives $\mu = 0$, since $\kappa < 0$.

Now letting $E = \xi$ in (48) gives

$$\kappa((Fu) - (\xi u)\eta(F)) = \frac{1}{2n-1}\{F(2n\lambda + \Delta u) - \xi(2n\lambda + \Delta u)\eta(F)\},$$

that is,

$$\frac{1}{2n-1}D(2n\lambda + \Delta u) = \kappa Du - \kappa\xi(u)\xi + \frac{1}{2n-1}\xi(2n\lambda + \Delta u)\xi. \tag{49}$$

On the other hand, by (9), contracting (47) with respect to E we find

$$S(F,Du) = \frac{2n}{2n-1}F(2n\lambda + \Delta u).$$

Recalling (19), we obtain

$$\frac{2n}{2n-1}D(2n\lambda + \Delta u) = QDu = 2n\kappa\xi(u)\xi.$$

Thus it follows from (49) that

$$2\kappa\xi(u)\xi = \kappa Du + \frac{1}{2n-1}\xi(2n\lambda + \Delta u)\xi.$$

From this we see $Du = \xi(u)\xi$. Differentiating this along E and using the second term of (10), we have

$$\nabla_E Du = E(\xi(u))\xi + \xi(u)h'E. \tag{50}$$

Since Equation (5) with $Z = Du$ may be expressed as

$$\nabla_E Du = -\frac{1}{2n-1}QE - \frac{1}{2n-1}(2n\lambda + \Delta u)E,$$

inserting (50) into the previous relation yields

$$\frac{1}{2n-1}QE = -E(\xi(u))\xi - \xi(u)h'E - \frac{1}{2n-1}(2n\lambda + \Delta u)E.$$

Using (19) again in the above relation, we get

$$\frac{2n\kappa}{2n-1}\eta(E)\xi = -E(\xi(u))\xi - \xi(u)h'E - \frac{1}{2n-1}(2n\lambda + \Delta u)E. \tag{51}$$

Applying φ to act on this equation and taking the inner product with φE, we have

$$-\xi(u)g(h'E, E) - \frac{1}{2n-1}(2n\lambda + \Delta u)g(\varphi E, \varphi E) = 0.$$

Because $trh' = 0$, the above formula shows $2n\lambda + \Delta u = 0$ and $\xi(u) = 0$. Since $Du = \xi(u)\xi$, u is a constant. Further, it implies from (51) that $\kappa = 0$, which is contradictory with $\kappa < 0$.

Theorem 6.2. *There does not exist gradient almost Riemann solitons on a (κ, μ)-almost co-Kähler manifolds with $\kappa < 0$.*

7. Example

In this section we construct an example of an almost co-Kähler manifold whose metric is a Riemann soliton.

Let $M^3 = \mathbb{R}^3(x, y, z)$, where (x, y, z) are the standard coordinates of $\mathbb{R}^3$. Let g be the Riemannian metric on M^3 defined by

$$g = dx^2 + dy^2 + \frac{4(x^2 + y^2) + 1}{e^{2z}}dz^2 - 4ye^{-z}dx\,dz - 4xe^{-z}dy\,dz.$$

Let $e_1 = \frac{\partial}{\partial x}$, $\quad e_2 = \frac{\partial}{\partial y}$, $\quad e_3 = 2y\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y} + e^z\frac{\partial}{\partial z}$. Then $\{e_1, e_2, e_3\}$ is an orthonormal basis of (M^3, g). We have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = 2e_2, \quad [e_2, e_3] = 2e_1.$$

Let the 1-form η, the vector field ξ and $(1, 1)$-tensor field φ are defined by

$$\eta = e^{-z}dz, \quad \xi = e_3, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0.$$

The 2-form Φ is given by

$$\Phi = -2dx \wedge dy - 4ye^{-z}dy \wedge dz - 4xe^{-z}dz \wedge dx.$$

Since $d\eta = 0$ and $d\Phi = 0$, M^3 is an almost co-Kähler manifold.

The Riemannian connection ∇ is given by

$$\nabla_{e_1}e_1 = 0, \quad \nabla_{e_1}e_2 = -2e_3, \quad \nabla_{e_1}e_3 = 2e_2,$$

$$\nabla_{e_2}e_1 = -2e_3, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_3 = 2e_1,$$

$$\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = 0.$$

The components of the Riemannian curvature tensor R are

$$R(e_1, e_2)e_1 = -4e_2, \quad R(e_1, e_2)e_2 = 4e_1, \quad R(e_1, e_2)e_3 = 0,$$

$$R(e_1,e_3)e_1 = 4e_3, \quad R(e_1,e_3)e_2 = 0, \quad R(e_1,e_3)e_3 = -4e_1,$$

$$R(e_2,e_3)e_1 = 0, \quad R(e_2,e_3)e_2 = 4e_3, \quad R(e_2,e_3)e_3 = -4e_2.$$

Using the above expression of the curvature tensor R, it follows that

$$R(E,F)\xi = -4\{\eta(F)E - \eta(E)F\}$$

for all $E, F \in \chi(M^3)$. Hence M^3 is a $N(-4)$-almost co-Kähler manifold. The expression of the curvature tensor R is

$$\begin{aligned} R(E,F)W &= 4\{g(F,W)E - g(E,W)F\} \\ &- 8\{\eta(F)\eta(W)E - \eta(E)\eta(W)F + g(F,W)\eta(E)\xi \\ &- g(E,W)\eta(F)\xi\} \end{aligned} \tag{52}$$

for all $E, F, W \in \chi(M^3)$. The components of of the Ricci tensor S are

$$S(e_1,e_1) = S(e_2,e_2) = 0, \quad S(e_3,e_3) = -8,$$

$$S(e_i,e_j) = 0, \quad \text{where } i,j = 1,2,3 \text{ and } i \neq j,$$

which gives $r = -8$. Also we have

$$S(E,F) = -8\eta(E)\eta(F) \tag{53}$$

for all $E, F \in \chi(M^3)$.

Let $Z = -8xe_1 - 8ye_2$ and $\lambda = 12$. By direct computations we obtain

$$(\pounds_Z g)(E,F) = -16g(E,F) + 16\eta(E)\eta(F), \tag{54}$$

which gives $\text{div}Z = -16$. From (52), and (54) we see that the equation (4) is satisfied. Hence g is a Riemann soliton. From (53) and (54), we lead

$$2S(E,F) + \left(\lambda - \frac{r}{2}\right)g(E,F) + (\pounds_Z g)(E,F) = 0$$

for all $E, F \in \chi(M^3)$. Thus Theorem 3.1 is verified. Also Corollaries 5.3 and 5.4 are verified.

References

[1] Y. S. Balkan, S. Uddin, and A. H. Alkhaldi, A class of φ-recurrent almost cosymplectic space, Honam Mathematical Journal 40 (2018), 293–304.

[2] D. E. Blair, Riemannian geometry of contact and symplectic manifold, 203, Birkhäuser, 2010.

[3] D. E. Blair, The theory of quasi-Sasakian structure, Journal of Differential Geometry 1 (1967), 331–345.

[4] B. Cappelletti-Mantano, A. D. Nicola, and I. Yudin, A survey on cosymplectic geometry, Review in Mathematical Physics. 25 (2013) 1343002, 55pp.

[5] X. M. Chen, Cotton solitons on almost co-Kähler 3-manifolds, Quaestiones Mathematicae 44 (2021), 1055–1075.

[6] X. M. Chen, Almost quasi-Yamabe solitons on almost cosymplectic manifolds, International Journal of Geometric Methods in Modern Physics 17 (2020) 2050070 (16 pages).

[7] X. M. Chen, Einstein-Wely structures on almost cosymplectic manifolds, Periodica Mathematica Hugarica 79 (2019), 191–203.

[8] X. M. Chen and U. C. De, The k-almost Yamabe solitons and almost coKähler manifolds, International Journal of Geometric Methods in Modern Physics, 18 (2021) 2150179.

[9] P. Dacko, On almost cosymplectic manifolds with the structure vector field ξ belonging to the κ-nullity distribution, Balkan Journal of Geometry and its Applications 5 (2000), 47–60.

[10] P. Dacko and Z. Olszak, On almost cosymplectic (κ, μ, ν)-spaces, Banach center publication 69 (2005), 211–220.

[11] U. C. De, S. K. Chaubey and Y. J. Suh, A note on almost co-Kähler manifolds, International Journal of Geometric Methods in Modern Physics 17(2020) 2050153.

[12] U. C. De, P. Majhi and Y. J. Suh, Semisymmetric properties of almost co-Kähler manifolds, Bulletin of the Korean Mathematical Society 56 (2019), 219-228.

[13] U. C. De, S. K. Chaubey and Y. J. Suh, Gradient Yamabe and Gradient m-Quasi Einstein Metrics on Three-dimensional Cosymplectic Manifolds, Mediterranean Journal of Mathematics 18:80 (2021).
[14] M. N. Devaraja, H. A. Kumara and V. Venkatesha, Riemannian soliton within the frame work of contact geometry, Quaestiones Mathematicae 44 (2021) 637-651.
[15] H. Endo, Non-existence of almost cosymplectic manifolds satisfying a certain condition, Tensor (N.S.) 63 (2002), 272–284.
[16] I. E. Hirică and C. Udrişte, Ricci and Riemannian solitons, Balkan Journal of Geometry and its Applications 21 (2016), 35–44.
[17] D. M. Naik, V. Venkatesha and H. A. Kumara, Ricci solitons and certain related metrics on almost co-Kaehler manifolds, Journal of Mathematical Physics, Analysis, Geometry 16 (2020), 402-417.
[18] D. M. Naik, V. Venkatesha and H. Aruna Kumara, Certain types of metrics on almost coKähler manifolds, Annales mathématiques du Québec (2021). https://doi.org/10.1007/s40316-021-00162-w.
[19] Z. Olszak, On almost cosymplectic manifolds, Kodai Mathematical Journal 4 (1981), 239–250.
[20] Z. Olszak, On almost cosymplectic manifolds with Kählerian leaves, Tensor (N.S.) 46 (1987), 117–124.
[21] H. Öztürk, N. Aktan, and C. Murathan, Almost α-cosymplectic (κ, μ, ν)-spaces, arXiv:1007.0527v1 (2010).
[22] D. Perrone, Classification of homogeneous almost cosymplectic three-manifolds, Differential Geometry and its Applications 30 (2012), 49–58.
[23] Y. J. Suh and U. C. De, Yamabe solitons and Ricci solitons on almost co-Kähler manifolds, Canadian Mathematical Bulletin 62 (2019), 653–661.
[24] C. Udrişte, Riemann flow and Riemannian wave, Ann. Univ. Vest, Timisoara, Ser. Mat.-Inf. 48 (2010), 265-274.
[25] C. Udrişte, Riemannian flow and Riemannian wave via bialternate product Riemannian metric, arXiv:1112.4279v4 (2012).
[26] V. Venkatesha, H. A. Kumara and M. N. Devaraja, Riemann solitons and almost Riemann solitons on almost Kenmotsu manifolds, International Journal of Geometric Methods in Modern Physics 17 (2020) 2050105.
[27] Y. Wang, A generalization of Goldberg conjecture for co-Kähler manifolds, Mediterranean Journal of Mathematics 13 (2016), 2679-2690.
[28] Y. Wang, Almost co-Kähler manifolds satisfying some symmetry conditions, Turkish Journal of Mathematics 40 (2016), 740-752.
[29] Y. Wang, Cotton tensors on almost co-Kähler manifolds, Annales Polonici Mathematici 120 (2017), 135-148.
[30] Y. Wang, Ricci solitons on 3-dimensional cosymplectic manifolds, Mathematica Slovaca 67 (2017), 979-984.
[31] Y. Wang, Ricci solitons on almost co-Kähler manifolds, Canadian Mathematical Bulletin 62 (2019), 912-922.
[32] Y. Wang, Ricci tensors on three-dimensional almost co-Kähler manifolds, Kodai Mathematical Journal 39 (2016), 469-483.
[33] K. Yano, Integral Formulas in Riemannian Geometry, Marcel Dekker, New York, 1970.