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(m,∞)-Expansive and (m,∞)-Contractive Commuting Tuple of Operators on a Banach Space

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Abstract. For a d-tuple of commuting operators $\mathbf{S} := (S_1, \cdots, S_d) \in \mathcal{B}[\mathcal{X}]^d$, $m \in \mathbb{N}$ and $p \in (0,\ \infty)$, we define

$$Q_m^{(p)}(\mathbf{S};\ u) := \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\Big(\sum_{\substack{\mu\ \in\ \mathbb{N}_0^d\\ |\mu|\ =\ k}}\frac{k!}{\mu}\|\mathbf{S}^\mu u\|^p\Big).$$

As a natural extension of the concepts of (m,p)-expansive and (m,p)-contractive for tuple of commuting operators, we introduce and study the concepts of (m,∞)-expansive tuple and (m,∞)-contractive tuple of commuting operators acting on a Banach space. We say that $\mathbf{S}$ is (m,∞)-expansive d-tuple $\Big($resp. (m,∞)-contractive d-tuple $\Big)$ of operators if $Q_m^{(p)}(\mathbf{S};\ u) \leq 0\ \ \forall\ u \in \mathcal{X}$ and $p \to \infty$ $\Big($resp. $Q_m^{(p)}(\mathbf{S};\ u) \geq 0\ \ \forall\ u \in \mathcal{X}$ and $p \to \infty\ \Big)$. These concepts extend the definition of (m,∞)-isometric tuple of bounded linear operators acting on Banach spaces was introduced and studied in [13].

1. Introduction

Let $\mathcal{X}$ be a complex normed space and $\mathcal{H}$ a complex Hilbert space. $\mathcal{B}[\mathcal{X}]$ (with respect to $\mathcal{B}[\mathcal{H}]$) be the set of bounded linear operator on $\mathcal{X}$ (resp. on $\mathcal{H}$). The authors J. Agler and M. Stankus introduced the class of m-isometry on Hilbert space [1–3]. An operator $S \in \mathcal{B}[\mathcal{H}]$ is said to be m-isometric operator for some integer $m \geq 1$ if it satisfies the operator equation

$$\beta_m(S) := \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}S^{*m-k}S^{m-k} = 0. \tag{1}$$

The class of ∞-isometry has been introduced by M. Chō et al. in [5]. An operator $S \in \mathcal{B}[\mathcal{H}]$ is an ∞-isometry if

$$\limsup_{m\to\infty}\Big\|\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}S^{*m-k}S^{m-k}\Big\|^{\frac{1}{m}} = 0. \tag{2}$$

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In [4, 12], the authors extended the concept of m-isometry on Hilbert space to general Banach space as follows: an operator $S \in \mathcal{B}[\mathcal{X}]$ is (m,p)-isometry if $\forall\ u \in \mathcal{X}$

$$\sum_{0\le k\le m} (-1)^k \binom{m}{k} \|S^{m-k}u\|^p = 0, \tag{3}$$

for some positive integer m and a $p \in (0,\infty)$. For $p = \infty$, extension of (3) has been introduced in [12]. For a positive integer m, an operator $S \in \mathcal{B}[\mathcal{H}]$ is an (m,∞)-isometry if $\forall\ u \in \mathcal{X}$

$$\max_{\substack{0\le k\le m\\ k\ \text{even}}} \|S^k u\| = \max_{\substack{0\le k\le m\\ k\ \text{odd}}} \|S^k u\|. \tag{4}$$

A generalization of (m,p)-isometries on Banach space to (m,p)-expansive and (m,p)-contractive, operators on a Banach space has been introduced in ([8, 16, 17]. We quote the definition given in [8]. An operator $S \in \mathcal{B}[\mathcal{X}]$ is (m,p)-expansive if $\forall\ u \in \mathcal{X}$

$$\sum_{0\le k\le m} (-1)^k \binom{m}{k} \|S^k u\|^p \le 0, \tag{5}$$

and it is (m,p)-contractive if for all $u \in \mathcal{X}$

$$\sum_{0\le k\le m} (-1)^k \binom{m}{k} \|S^k u\|^p \ge 0. \tag{6}$$

A natural extension of some concepts of single operator to tuple of operators on Hilbert and Banach spaces has attracted much attention of various authors. Several papers has been appeared on commuting tuples of operators (see [6, 7, 9, 11, 13–15]).

In ([13]) the authors introduced and studied the concept of (m,p)-isometric tuples on normed space. A tuple of commuting linear operators $\mathbf{S} := (S_1,\cdots,S_d) \in \mathcal{B}[\mathcal{X}]^d$ is an (m,p)-isometric tuple if and only if for given $m \in \mathbb{N}$ and $p \in (0,\infty)$,

$$\sum_{0\le k\le m} (-1)^{m-k} \binom{m}{k} \sum_{|\mu|=k} \frac{k!}{\mu!} \|\mathbf{S}^\mu u\|^p = 0 \ \text{ for all } \ u \in \mathcal{X}. \tag{7}$$

where $\mu = (\mu_1,\cdots,\mu_d) \in \mathbb{N}_0^d,\ \mu! = \mu_1!.\cdots.\mu_d!,\ |\mu| = \mu_1+\cdots+\mu_d$ and $\mathbf{S}^\mu = S_1^{\mu_1}\cdots.S_d^{\mu_d}$.

An extension of (7) to include the case $p = \infty$ was introduced in [13] as the following; For a positive integer m, a tuple of commuting operators $\mathbf{S} := (S_1,\cdots,S_d) \in \mathcal{B}[\mathcal{X}]^d$ is an (m,∞)-isometric tuple if for all $u \in \mathcal{X}$

$$\max_{\substack{|\mu|\in\{0,\cdots,m\}\\ |\mu|\ \text{even}}} \|\mathbf{S}^\mu u\| = \max_{\substack{|\mu|\in\{0,\cdots,m\}\\ |\mu|\ \text{odd}}} \|\mathbf{S}^\beta u\|. \tag{8}$$

Very recently, the author in ([10]) has extended the concept of (m,p)-isometric tuple of commuting operators to the concepts of (m,p)-expansive and (m,p)-contractive tuple of operators on Banach spaces. A tuple of commuting linear operators $\mathbf{S} = (S_1,\cdots,S_d) \in \mathcal{B}[\mathcal{X}]^d$ is an (m,p)-expansive tuple if and only if for given $m \in \mathbb{N}$ and $p \in (0,\infty)$,

$$Q_m^{(p)}(\mathbf{S};\ u) := \sum_{0\le k\le m} (-1)^k \binom{m}{k} \Big(\sum_{\substack{\mu\in\mathbb{N}^d\\ |\mu|=k}} \frac{k!}{\mu!} \|\mathbf{S}^\mu u\|^p \Big) \le 0 \ \text{ for all } \ u \in \mathcal{X}, \tag{9}$$

and it is an (m, p)-contractive tuple if

$$Q_m^{(p)}(\mathbf{S};\ u) := \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\Big(\sum_{\substack{\mu\,\in\,\mathbb{N}^d\\ |\mu|\,=\,k}}\frac{k!}{\mu!}\|\mathbf{S}^\mu u\|^p\Big) \geq 0 \ \text{ for all } \ u\in\mathcal{X}. \tag{10}$$

Our aim in this paper is to study the concepts of (m, p)-expansive and (m, p)-contractive tuples of operators when $p \longrightarrow \infty$. The new classes of operators will be called (m,∞)-expansive and (m,∞)-contractive tuples of commutative operators on Banach space. We will discuss the most interesting results concerning these classes of tuples of operators which will be obtained form the idea of generalizing some results of recently published works on single operators.

2. (m, ∞)-expansive and (m, ∞)-contractive tuples of operators

Let $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}[\mathcal{X}]^d$ be a commuting tuple of operators on $\mathcal{X}$. $\mathbf{S}$ is (m, p)-expansive tuple if and only if $Q_m^{(p)}(\mathbf{S};\ u) \leq 0\ \ \forall\ u\in\mathcal{X}$. Then

$$\begin{aligned}
& Q_m^{(p)}(\mathbf{S};\ u)\leq 0\\
\Longleftrightarrow\ & \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\Big(\sum_{\substack{\mu\,\in\,\mathbb{N}^d\\ |\mu|\,=\,k}}\frac{k!}{\mu!}\|\mathbf{S}^\mu u\|^p\Big)\leq 0\\
\Longleftrightarrow\ & \sum_{0\leq k\leq m}\binom{m}{k}\Big(\sum_{\substack{|\mu|\,=\,k\\ (\text{k even})}}\frac{k!}{\mu!}\beta|\mathbf{S}^\mu u\|^p\Big)\leq \sum_{0\leq k\leq m}\binom{m}{k}\Big(\sum_{\substack{|\mu|\,=\,k\\ (k\ \text{odd})}}\frac{k!}{\mu!}\|\mathbf{S}^\mu u\|^p\Big)\\
\Longleftrightarrow\ & \Big(\sum_{0\leq k\leq m}\binom{m}{k}\sum_{\substack{|\mu|\,=\,k\\ (\text{ k even})}}\frac{k!}{\beta!}\|\mathbf{S}^\mu u\|^p\Big)^{\frac{1}{p}}\leq\Big(\sum_{0\leq k\leq m}\binom{m}{k}\sum_{\substack{|\mu|\,=\,k\\ (k\ \text{odd})}}\frac{k!}{\mu!}\|\mathbf{S}^{\mu\mu}u\|^p\Big)^{\frac{1}{p}}.
\end{aligned}$$

$\mathbf{S}$ is a (m, p)-contractive tuple if and only if $Q_m^{(p)}(\mathbf{S};\ u) \geq 0\ \ \forall\ u\in\mathcal{X}$. Then

$$\begin{aligned}
& \mathbf{Q}_m^{(p)}(\mathbf{S};\ u)\leq 0\\
\Longleftrightarrow\ & \Big(\sum_{0\leq k\leq m}\binom{m}{k}\sum_{\substack{|\mu|\,=\,k\\ (\text{ k even})}}\frac{k!}{\mu!}\|\mathbf{S}^\mu u\|^p\Big)^{\frac{1}{p}}\geq\Big(\sum_{0\leq k\leq m}\binom{m}{k}\sum_{\substack{|\mu|\,=\,k\\ (k\ \text{odd})}}\frac{k!}{\mu!}\|\mathbf{S}^\mu u\|^p\Big)^{\frac{1}{p}}.
\end{aligned}$$

By taking the limit as $p \to \infty$, we make the following definition of (m,∞)-expansive tuple and a (m,∞)-contractive tuple of commuting operators.

Definition 2.1. *Let $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}[\mathcal{X}]^d$ be a commuting d-tuple. $\mathbf{S}$ is said to be*
(i) (m,∞)-expansive tuple if for all $u\in\mathcal{X}$

$$\max_{\substack{|\mu|\,\in\,\{0,\cdots,m\}\\ |\mu|\ \text{even}}}\|\mathbf{S}^\mu u\|\leq\max_{\substack{|\mu|\,\in\,\{0,\cdots,m\}\\ |\mu|\ \text{odd}}}\|\mathbf{S}^\mu u\|$$

(ii) (m,∞)-contractive tuple if for all $u \in \mathcal{X}$

$$\max_{\substack{|\mu| \in \{0,\cdots,m\} \\ |\mu| \text{ even}}} \|\mathbf{S}^{\mu}u\| \geq \max_{\substack{|\mu| \in \{0,\cdots,m\} \\ |\mu| \text{ odd}}} \|\mathbf{S}^{\mu}u\|.$$

(iii) (m,∞)-hyperexpansive tuple if for all $u \in \mathcal{X}$

$$\max_{\substack{|\mu| \in \{0,\cdots,l\} \\ |\mu| \text{ even}}} \|\mathbf{S}^{\mu}u\| \leq \max_{\substack{|\mu| \in \{0,\cdots,l\} \\ |\mu| \text{ odd}}} \|\mathbf{S}^{\mu}u\|$$

for $l = 0, 1, \cdots, m$.

(iv) (m,∞)-hypercontractive tuple if for all $u \in \mathcal{X}$

$$\max_{\substack{|\mu| \in \{0,\cdots,l\} \\ |\mu| \text{ even}}} \|\mathbf{S}^{\mu}u\| \geq \max_{\substack{|\mu| \in \{0,\cdots,l\} \\ |\mu| \text{ odd}}} \|\mathbf{S}^{\mu}u\|$$

for $l = 0, 1, \cdots, m$.

Definition 2.2. *Let* $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}[\mathcal{X}]^d$. $\mathbf{S}$ *is said to be*

(i) completely ∞-hyperexpansive tuple of operators if and only if $\mathbf{S}$ is a (k,∞)-expansive tuple for all $k \in \mathbb{N}$,

(ii) completely ∞-hypercontractive tuple of operators if and only if $\mathbf{S}$ is a (k,∞)-contractive tuple for all $k \in \mathbb{N}$.

Example 2.3. *Every* (m,∞)*-isometric tuple is a* (m,∞)*-expansive and a* (m,∞)*-contractive tuple.*

Remark 2.4. *For* $d = 1$*, the statements of Definition 2.1 coincides with*

(a) *S is* (m,∞)*-expansive if for all* $u \in \mathcal{X}$

$$\max_{\substack{0 \leq k \leq m \\ k \ \text{even}}} \|S^k u\| \leq \max_{\substack{0 \leq k \leq m \\ k \ \text{odd}}} \|S^k u\|.$$

(b) *S is* (m,∞)*-contractive if for all* $u \in \mathcal{X}$

$$\max_{\substack{0 \leq k \leq m \\ k \ \text{even}}} \|S^k u\| \geq \max_{\substack{0 \leq k \leq m \\ k \ \text{odd}}} \|S^k u\|.$$

(c) S is (m,∞)*-hyperexpansive if S is* (k,∞)*-expansive for* $k = 1, \cdots, m$.

(d) S is (m,∞)*-hypercontractive if S is* (k,∞)*-contractive for* $k = 1, \cdots, m$.

Example 2.5. *Let* $S_0 \in \mathcal{B}[\mathcal{X}]$ *be an* (m,∞)*-expansive single operator, then* $\mathbf{S} = (S_0, \cdots, S_0) \in \mathcal{B}[\mathcal{X}]^d$ *is an* (m,∞)*-expansive tuple of commuting operators.*

In fact, we have

$$\begin{aligned}\max_{\substack{|\mu| \in \{0,\cdots,m\}\\ |\mu|\ even}} \|\mathbf{S}^{\mu}u\| &= \max_{\substack{|\mu| \in \{0,\cdots,m\}\\ |\mu|\ even}} \|S_0{}^{|\mu|}u\| \\ &\leq \max_{\substack{|\mu| \in \{0,\cdots,m\}\\ |\mu|\ odd}} \|S_0{}^{|\mu|}u\|. \\ &= \max_{\substack{|\mu| \in \{0,\cdots,m\}\\ |\mu|\ odd}} \|\mathbf{S}^{\mu}u\|.\end{aligned}$$

Similarly, we have the following example.

Example 2.6. *Let $S_0 \in \mathcal{B}[\mathcal{X}]$ be an (m,∞)-contractive single operator, then $\mathbf{S} = (S_0,\cdots,S_0) \in \mathcal{B}[\mathcal{X}]^d$ is an (m,∞)-contractive tuple of commuting operators.*

Remark 2.7. *We note the following:*

(1) *If $d = 2$ and $\mathbf{S} = (S_1, S_2) \in \mathcal{B}[\mathcal{X}]^2$ be a pair of commutative operators, then* $\mathbf{S}$ *is $(2,\infty)$-expansive tuple if*

$$\max\Big\{\|u\|, \|S_1^2u\|, \|S_2^2u\|, \|S_1S_2u\|\Big\} \leq \max\Big\{\|S_1u\|, \|S_2u\|\Big\}.$$

(2) *If $d = 2$ and $\mathbf{S} = (S_1, S_2) \in \mathcal{B}[\mathcal{X}]^2$ is a pair of commutative operators, then* $\mathbf{S}$ *is $(2,\infty)$-contractive tuple if*

$$\max\Big\{\|u\|, \|S_1^2u\|, \|S_2^2u\|, \|S_1S_2u\|\Big\} \geq \max\Big\{\|S_1u\|, \|S_2u\|\Big\}.$$

(3) *If $\mathbf{S} = (S_1,\cdots,S_d) \in \mathcal{B}[\mathcal{X}]^d$ be a commutative tuple of operators, then* $\mathbf{S}$ *is $(1,\infty)$-expansive tuple if for all $u \in \mathcal{X}$,*

$$\|u\| \leq \max_{|\alpha|=1}\Big\{\|\mathbf{S}^{\alpha}u\|\Big\} := \max\Big\{\|S_ju\|;\quad j = 1,\cdots,d\Big\}$$

and it is $(1,\infty)$- contractive tuple if for all $u \in \mathcal{X}$,

$$\|u\| \geq \max_{|\alpha|=1}\Big\{\|\mathbf{S}^{\alpha}u\|\Big\} := \max\Big\{\|S_ju\|;\quad j = 1,\cdots,d\Big\}$$

(4) *If $\mathbf{S} = (S_1,\cdots,S_d) \in \mathcal{B}[\mathcal{X}]^d$ be a commutative tuple of operators, then* $\mathbf{S}$ *is $(2,\infty)$-expansive tuple if for all $u \in \mathcal{X}$*

$$\max\Big\{\|u\|, \|S_iS_ju\|, \|S_i^2u\|,\ 1 \leq i,\ j \leq d\Big\} \leq \max\Big\{\|S_ju\|;\quad j = 1,\cdots,d\Big\}$$

and it is $(2,\infty)$-contractive tuple if for all $u \in \mathcal{X}$

$$\max\Big\{\|u\|, \|S_iS_ju\|, \|S_i^2u\|,\ 1 \leq i\ ,\ j \leq d\Big\} \geq \max\Big\{\|S_ju\|;\quad j = 1,\cdots,d\Big\}.$$

(5) *If $\mathbf{S} = (S_1,\cdots,S_d) \in \mathcal{B}[\mathcal{X}]^d$ be a commutative tuple of operators, then* $\mathbf{S}$ *is $(3,\infty)$-expansive tuple if for all $u \in \mathcal{X}$*

$$\max\Big\{\|u\|, \|S_iS_ju\|, \|S_i^2u\|,\ 1 \leq i,\ j \leq d\Big\} \leq \max\Big\{\|S_ju\|, \|S_iS_jS_ku\|,\ \|S_i^2S_ju\|;\quad i, j,\ k = 1,\cdots,d\Big\}$$

and it is $(3,\infty)$-contractive tuple if for all $u \in \mathcal{X}$

$$\max\Big\{\|u\|, \|S_iS_ju\|, \|S_i^2u\|,\ 1 \leq i\ ,\ j \leq d\Big\} \geq \max\Big\{\|S_ju\|,\ \|S_iS_jS_ku\|,\ \|S_i^2S_ju\|;\ ;\quad j = 1,\cdots,d\Big\}$$

Proposition 2.8. *Let* $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}[\mathcal{X}]^d$ *be a commutative tuple of operators. The following statements hold.*

(i) If $\mathbf{S}$ is $(2, \infty)$-expansive tuple, then $\mathbf{S}$ is $(1, \infty)$-expansive tuple.

(ii) $\mathbf{S}$ is $(2, \infty)$-expansive tuple if and only if $\mathbf{S}$ is $(2, \infty)$-hyperexpansive tuple.

(iii) If $\mathbf{S}$ is (m, ∞)-expansive tuple, then $\mathcal{N}(\mathbf{S}) := \bigcap\limits_{1 \le j \le d} \mathcal{N}(S_j) = \{0\}$.

Proof. (i) Since $\mathbf{S}$ is $(2, \infty)$-expansive tuple, it follows that

$$\max\Big\{\|u\|, \|S_iS_ju\|, \|S_i^2u\|,\ 1 \le i,\ j \le d\Big\} \le \max\Big\{\|S_ju\|; \quad j = 1, \cdots, d\Big\},\ \forall\ u \in \mathcal{X}.$$

Hence, $\|u\| \le \max\Big\{\|S_ju\|; \quad j = 1, \cdots, d\Big\}$, $\forall\ u \in \mathcal{X}$. Therefore $\mathbf{S}$ is $(1, \infty)$-expansive tuple.

(ii) Follows from the statement (i).

(iii) Let $u \in \mathcal{N}(\mathbf{S})$, then $\mathbf{S}^\mu u = 0$ for all $\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{N}_0^d$ such that $|\mu| \in \{1, \cdots, m\}$. The joint (m, ∞)-expansivity of $\mathbf{S}$ implies that

$$\|u\| = \max_{\substack{|\mu| \in \{0, \cdots, m\} \\ |\mu| \text{ even}}} \|\mathbf{S}^\mu u\| \le \max_{\substack{|\mu| \in \{0, \cdots, m\} \\ |\mu| \text{ odd}}} \|\mathbf{S}^\mu u\| = 0.$$

Consequently, $u = 0$. This completes the proof. □

Proposition 2.9. *Let* $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}[\mathcal{X}]^d$ *be a commutative tuple of operators. Then* $\mathbf{S}$ *is* (m, ∞)*-expansive tuple if and only if for all* $l \in \mathbb{N}$ *and for all* $u \in \mathcal{X}$,

$$\max_{\left(\substack{|\mu| \in \{l, \cdots, m+l\} \\ |\mu| \text{ even}}\right)} \Big\{\|\mathbf{S}^\mu u\|\Big\} \le \max_{\left(\substack{|\mu| \in \{l, \cdots, m+l\} \\ |\mu| \text{ odd}}\right)} \Big\{\|\mathbf{S}^\mu u\|\Big\}. \tag{11}$$

Proof. Assume that $\mathbf{S}$ is (m, ∞)-expansive tuple and $l \in \mathbb{N}$ is an even integer. Then we have

$$\begin{aligned}
&\max_{\left(\substack{|\mu| \in \{l, \cdots, m+l\} \\ |\mu| \text{ even}}\right)} \|\mathbf{S}^\mu u\| \\
=\ & \max_{\left(\substack{|\mu| \in \{0, \cdots, m\} \\ |\gamma| = l \\ |\mu| + |\gamma| \text{ even}}\right)} \|\mathbf{S}^\mu \mathbf{S}^\gamma u\|\} \\
=\ & \max_{|\gamma|=l}\Bigg(\max_{\left(\substack{|\mu| \in \{0, \cdots, m\} \\ |\mu| \text{ even}}\right)} \|\mathbf{S}^\mu \mathbf{S}^\gamma u\|\Bigg) \\
\le\ & \max_{|\gamma|=l}\Bigg(\max_{\left(\substack{|\mu| \in \{0, \cdots, m\} \\ |\mu| \text{ odd}}\right)} \|\mathbf{S}^\mu \mathbf{S}^\gamma u\|\Bigg) \\
=\ & \max_{\left(\substack{|\mu| \in \{0, \cdots, m\} \\ |\gamma| = l \\ |\mu| + |\gamma| \text{ odd}}\right)} \|\mathbf{S}^\mu \mathbf{S}^\gamma u\| \\
=\ & \max_{\left(\substack{|\mu| \in \{l, \cdots, m+l\} \\ |\mu| \text{ odd}}\right)} \|\mathbf{S}^\alpha u\|.
\end{aligned}$$

If l is an odd integer, we can repeat quite similar arguments as those above to prove that

$$\max_{\substack{|\mu| \in \{l, \cdots, m+l\} \\ |\mu| \text{ even}}} \|\mathbf{S}^{\mu} u\| = \max_{\substack{|\mu| \in \{l, \cdots, m+l\} \\ |\mu| \text{ odd}}} \|\mathbf{S}^{\mu} u\|.$$

This implies that (11) holds for all $l \in \mathbb{N}$. □

Proposition 2.10. *Let* $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}[\mathcal{X}]^d$ *be a commutative tuple of operators. Then* $\mathbf{S}$ *is* (m, ∞)*-contractive tuple if and only if for all* $l \in \mathbb{N}$ *and for all* $x \in \mathcal{X}$

$$\max_{\substack{|\mu| \in \{l, \cdots, m+l\} \\ |\mu| \text{ even}}} \{\|\mathbf{S}^{\mu} u\|\} \geq \max_{\substack{|\mu| \in \{l, \cdots, m+l\} \\ |\mu| \text{ odd}}} \{\|\mathbf{S}^{\mu} u\|\}. \tag{12}$$

Proof. By the statement (ii) of Definition 2.1and similar proof as of Proposition 2.9 the result follows. □

Theorem 2.11. *Let* $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}[\mathcal{X}]^d$ *be a commuting tuple of operators such that* $S_k S_j$ *is an isometry for* $k, j \in \{1, \cdots, m\}$*. Then the following statements are equivalent.*
(1) $\mathbf{S}$ *is* (m, ∞)*-expansive tuple,*

(2) $\mathbf{S}$ *is* $(1, \infty)$ *-expansive tuple,*

(3) $\mathbf{S}$ *is* $(1, \infty)$*-isometric tuple,*

(4) $\mathbf{S}$ *is* $(1, \infty)$*-contractive tuple,*

(5) $\mathbf{S}$ *is* (m, ∞)*-contractive tuple.*

Proof. As $S_k S_j$ is an isometry, it follows that

$$\max_{\substack{|\mu| \in \{0, \cdots, m\} \\ |\mu| \text{ even}}} \{\|\mathbf{S}^{\mu} u\|\} = \|u\| \quad \forall\ u \in \mathcal{X}$$

and

$$\max_{\substack{|\mu| \in \{0, \cdots, m\} \\ |\mu| \text{ odd}}} \{\|\mathbf{S}^{\mu} u\|\} = \max\{\|S_j u\| \ \ j = 1, \cdots, d\ \} \quad \forall\ u \in \mathcal{X}$$

and this shows (1) $\iff$ (2) and (4) $\iff$ (5). The equivalence of (2), (3) and (4) follows on replacing u by $S_k u$ for $k = 1, \cdots, d$. □

In the remaining part of this section, we discuss several properties of (m, ∞)-expansivity and (m, ∞)-contractivity for single operator.

Proposition 2.12. *Let* $S \in \mathcal{B}[\mathcal{X}]$*. Then* S *is an* $(2, \infty)$*-expansive if and only if* S *is an* $(2, \infty)$*-isometric operator.*

Proof. Assume that S is an $(2, \infty)$-expansive operator. It follows that for all $u \in \mathcal{X}$

$$\|Su\| \geq \max\left\{\|S^2 u\|, \|u\|\right\}.$$

It holds

$$\|Su\| \geq \|S^2 u\| \quad \text{and} \quad \|Su\| \geq \|u\|, \ \forall\ u \in \mathcal{X}.$$

This immediately yields

$$\|Su\| = \|S^2 u\| \geq \|u\| \ \ \forall \ \ u \in \mathcal{X}.$$

Hence we conclude that S is an $(2, \infty)$-isometric operator. The converse is obvious. □

Corollary 2.13. *Every* $(2, \infty)$*-expansive mapping is an completely* ∞*-hyperexpansive.*

Proof. Let S be an $(2, \infty)$-expansive operator. Then, we have S is a $(1, \infty)$-expansive and a $(2, \infty)$-isometry. Consequently, S is an (k, ∞)-expansive operator for all $k \in \mathbb{N}$. □

Corollary 2.14. *A power of an* $(2, \infty)$*-expansive operator is again an* $(2, \infty)$*-expansive operator.*

Proof. The proof is an immediate consequence of Proposition 2.12 □

Proposition 2.15. *Let* $S \in \mathcal{B}[\mathcal{X}]$ *such that* $S^2 = S$. *Then the following statement hold.*

(i) S is an (m, ∞)-expansive if and only if S is an $(1, \infty)$-expansive.

(ii) S is an (m, ∞)-contractive if and only if S is an $(1, \infty)$-contractive.

Proof. From that assumption that $S^2 = S$ it follows immediately that for all $u \in \mathcal{X}$

$$\max_{\substack{0 \le k \le m \\ k \text{ even}}} \|S^k u\| = \max\{\|u\|, \|Su\|\} \text{ and } \max_{\substack{0 \le k \le m \\ k \text{ odd}}} \|S^k u\| = \|Su\|.$$

Consequently,

$$\Big(\max_{\substack{0 \le k \le m \\ k \text{ even}}} \|S^k u\| \le \max_{\substack{0 \le k \le m \\ k \text{ odd}}} \|S^k u\|\Big) \Leftrightarrow \Big(\|u\| \le \|Su\|\Big)$$

and

$$\Big(\max_{\substack{0 \le k \le m \\ k \text{ even}}} \|S^k u\|\Big) \ge \Big(\max_{\substack{0 \le k \le m \\ k \text{ odd}}} \|S^k u\|\Big) \Leftrightarrow \Big(\|u\| \ge \|Su\|\Big).$$

Hence, the statements (i) and (ii) hold. □

Theorem 2.16. *Let* $S \in \mathcal{B}(\mathcal{X})$ *be invertible. The following statements hold.*

(i) If S is an (m, ∞)-expansive, then S^{-1} is an (m, ∞)-expansive for m even and an (m, ∞)-contractive for m odd.

(ii) If S is an (m, ∞)-contractive, then S^{-1} is an (m, ∞)-contractive for m even and an (m, ∞)-expansive for m odd.

Proof. (i) Assume that S is a invertible an (m, ∞)-expansive operator. It follows that for all $x \in \mathcal{X}$

$$\max_{\substack{0 \le k \le m \\ k \text{ even}}} \|S^k u\| \le \max_{\substack{0 \le k \le m \\ k \text{ odd}}} \|S^k u\|. \tag{13}$$

Replacing u by $S^{-m}u$ in (13), we get for all $u \in \mathcal{X}$,

$$\max_{\substack{0 \le k \le m \\ k \text{ even}}} \|(S^{-1})^{m-k} u\| \le \max_{\substack{0 \le k \le m \\ k \text{ odd}}} \|(S^{-1})^{m-k} u\|. \tag{14}$$

We obtain the following conclusions:

If m is even then, by equation (14) we have for all $u \in \mathcal{X}$

$$\max_{\substack{0 \le j \le m \\ j \text{ even}}} \|\big(S^{-1})^j u\| \le \max_{\substack{0 \le j \le m \\ j \text{ odd}}} \|(S^{-1})^j u\|$$

and so that S^{-1} is an (m,∞)-expansive operator.

If m is odd we have for all $u \in X$

$$\max_{\substack{0\le j\le m\\ j \text{ even}}} \|(S^{-1})^j u\| \ge \max_{\substack{0\le j\le m\\ j \text{ odd}}} \|(S^{-1})^j u\|$$

and so that S^{-1} is an (m,∞)-contractive operator.

(ii) This statement is proved in the same way as in the statement (i). □

Corollary 2.17. *Let $S \in \mathcal{B}[\mathcal{X}]$ be an invertible. The following statements hold.*

(i) If S is an $(2,\infty)$-expansive operator, then S is an $(1,\infty)$-isometry.

(ii) If S is an $(2,\infty)$-contractive operator, then S is an $(1,\infty)$-isometry.

Proof. Assume the S is an $(2,\infty)$-expansive. Then it follows that $\|Su\| \ge \|u\|$ for all $x \in \mathcal{X}$. On the other hand, by the fact that S is invertible $(2,\infty)$-expansive, we have by Theorem 2.16 that S^{-1} is a $(2,\infty)$-expansive and hence $\|S^{-1}u\| \ge \|u\|$ for all $u \in \mathcal{X}$. This means that $\|u\| \ge \|Su\|$ for all $u \in \mathcal{X}$. Consequently, $\|Su\| = \|u\|$ for all $u \in \mathcal{X}$, which shows that S is an $(1,\infty)$-isometry as required.

(ii) This statement is proved in the same way as in the statement (i). □

Theorem 2.18. *For $i = 1,2,\cdots,n$, let $(\mathcal{X}_i, \|\cdot\|_i)$ be a Banach space and let $S_i \in \mathcal{B}[\mathcal{X}_i]$, $m_i \ge 1$. Denote by $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times ... \times \mathcal{X}_n$ the product space endowed with the product distance $\|(u_1, x_2, \cdots, u_n)\| := \max\limits_{1\le i\le n} \|u_i\|$. Let $S := S_1 \times S_2 \times ... \times S_n \in \mathcal{B}[\prod\limits_{1\le i\le n} \mathcal{X}_i]$ defined by*

$$S(u_1,\cdots,u_n) := (S_1u_1, S_2u_2, \cdots, S_nu_n).$$

The following statements hold.

(i) If each S_i is an (m_i,∞)-hyperexpansive for $i = 1,2,\cdots,n$, then S is an (m,∞)-expansive, where $m = \min(m_1,\cdots,m_n)$.

(ii) If each S_i is an (m_i,∞)-hypercontractive for $i = 1,2,\cdots,n$, then S is an (m,p)-contractive,where $m = \min(m_1,\cdots,m_n)$.

(iii) If each S_i is an completely ∞-hyperexpansive for $i = 1,2,\cdots,n$, then so that S.

(iv) If each S_i is completely ∞-hypercontractive for $i = 1,2,\cdots,n$, then so that$S$.

Proof. (i) Let $m = \min(m_1, m_2, \cdots, m_n)$ and consider for all $u \in \mathcal{X}$

$$\begin{aligned}
\max_{\substack{0\le k\le m\\ k \text{ even}}} \|S^k u\| &= \max_{\substack{0\le k\le m\\ k \text{ even}}} \left(\max_{1\le i\le n}\left\{\|S_i^k u_i\|_i\right\}\right)\\
&= \max_{1\le i\le n}\left(\max_{\substack{0\le k\le m\\ k \text{ even}}} \left\{\|S_i^k u_i\|_i\right\}\right)
\end{aligned}$$

Since S_i is an (m_i,∞)-hyperexpansive for $i = 1,2,\cdots,n$, it follows that S_i is an (m,∞)-expansive for $i = 1,2,\cdots,n$ and hence

$$\begin{aligned}\max_{\substack{0\le k\le m\\ k\ \text{even}}} \|S^k u\| &\le \max_{1\le i\le n}\Big(\max_{\substack{0\le k\le m\\ k\ \text{odd}}}\Big\{\|S_i^k u_i\|_i\Big\}\Big)\\ &= \max_{\substack{0\le k\le m\\ k\ \text{odd}}}\Big(\max_{1\le i\le n}\Big\{\|S_i^k u_i\|_i\Big)\Big\}\Big).\end{aligned}$$

Thus, we have

$$\max_{\substack{0\le k\le m\\ k\ \text{even}}} \|S^k u\| \le \max_{\substack{0\le k\le m\\ k\ \text{odd}}} \|S^k u\|.$$

Consequently, S is an (m,∞)-expansive operator.

(ii) This statement follows from the statement in (i) by reversing the inequality above.

(iii) Suppose that each S_i is an completely ∞-hyperexpansive for each $i = 1,2,\cdots,n$, and hence each S_i is an (k,∞)-expansive for any $k \in \mathbb{N}$. As a consequence of this observation , one can deduce the following inequality for all $u \in X$

$$\begin{aligned}\max_{\substack{0\le j\le k\\ j\ \text{even}}} \|S^j u\| &= \max_{\substack{0\le j\le k\\ j\ \text{even}}}\Big(\max_{1\le i\le n}\|S_i^j u_i\|_i\Big)\\ &= \max_{1\le i\le n}\Big(\max_{\substack{0\le j\le k\\ j\ \text{even}}}\|S_i^j u_i\|_i\Big)\\ &\le \max_{\substack{0\le j\le k\\ j\ \text{odd}}} \|S^j u\| \quad \forall\ k\in\mathbb{N}.\end{aligned}$$

From which the statement in (iii) follows.

(iv) This statement is proved in the same way as in the statement (iii). □

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