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Two Families of Separation Axioms on Infra Soft Topological Spaces

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Abstract. Many generalizations of soft topology were studied in the literature; an infra soft topology is the recent one of these generalizations. In this paper, we put on view two classes of soft separation axioms in the frame of infra soft topologies, namely infra *pp*-soft T_j and infra *pt*-soft T_j-spaces ($j = 0, 1, 2, 3, 4$). Both of them are formulated with respect to distinct ordinary points such that the first class defined using partial belong and partial non-belong relations, and the second one defined using partial belong and total non-belong relations. Following systematic lines of this type of study, we first show the relationships between them with the aid of examples. We also establish main properties and explore their behaviour under some special types of infra soft topologies. Transmission of these classes between infra soft topology and its parametric infra topologies are amply studied. Moreover, we scrutinize their features in terms of hereditary and topological properties, and finite product of soft spaces.

1. Introduction

Soft set is a mathematical approach proposed by Molodtsov [32], in 1999, to cope with problems containing uncertainties. Molodtsov explained the potentiality of soft sets to handle many problems in different areas. Then, Maji et al. [31] successfully applied soft sets to deal with decision-making problems. Their methodology was later improved in [19]. Also, they [30] displayed some operations and operators on soft sets such as intersection and union of two soft sets, and the complement of a soft set. Despite the weakness of some concepts and results in this early reference, it forms the essential start point of soft set theory. Later on, Ali et al. [3] established new kinds of these operations and operators in a way that helps to preserve the main properties and results of crisp set theory. In these lines, Qin and Hong [34] introduced lower and upper soft equality, Abbas et al. [1] studied gf-soft union, and Al-shami [12] defined T-soft subset and T-soft equality. Since the advent of soft sets, many authors applied successfully to address problems in some disciplines such as computer science [18], decision-making [10, 22], and medical science [40]. These applications prove the adequacy of soft sets to treat and model a lot of real-life issues.

In 2011, topological notions have been hybridized with soft sets by Shabir and Naz [35]. Many researchers have explored the properties of soft topologies and compared their performance with the case of classical topologies; see, for example, [12, 16]. Zorlutuna et al. [42] came up with the idea of soft point

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which was independently reformulated by [20] and [33]. The soft points play a role of ordinary points (elements) in crisp setting. Enriched and extended soft topologies were investigated in [33] as special types of soft topologies; they produce some interesting properties between soft and crisp topological spaces as illustrated in [14]. Kočinac et al. [29] discussed the Menger selection principle in the context of soft sets and explored its properties in this context. Some studies on the frame of soft topologies were conducted in [25, 27].

It is always convenient to find the weakest conditions that preserve some topologically inspired properties, maybe under structures relaxing a topology. Supra soft topology [23] and infra soft topology [5] have been born with this goal. They have become two of the most interesting developments of soft topology in recent years. Supra topology is a class of subsets that extend the concept of topology by dispensing with the postulate that the class is closed under finite intersections, whereas infra topology is a class of subsets that extend the concept of topological space by dispensing with the postulate that the class is closed under arbitrary unions.

Another generalizations of a soft topology were given in the literature; for example, Thomas and John [39] formulated the concept of soft generalized topological spaces which defined as a family of soft sets which satisfies an arbitrary union condition of a soft topology, and Zakari et al. [41] originated the concepts of soft weak structures which defined as a family of soft sets which contains the null soft set. Ittanagi [26] established the structure of soft bitopology which can be regarded as a soft topology when the two soft topologies are identical. Lately, Al-shami et al. [13] have constructed soft topology on ordered setting as an extension of soft topology. Similarly, Al-shami and El-Shafei [9] studied supra soft topology on ordered setting.

Our contribution to this field concerns the analysis of what type of "separation axioms" are meaningful in the study of infra soft topology. As was the case of classical topology, soft separation axioms are among the most interesting and substantial concepts in soft topology. They form a tool to establish more restricted (and more wider) classes of well-behaved soft topological spaces. It should be noted that a large variety of separation axioms in soft topologies is attributed to two factors. One is the distinguished objects that we intend to separate: they can be either soft points or ordinary points. The other is the type of belongingness and non-belongingness relations that we require in the definitions: they can be either partial or total. For more details on the subject we refer to [17, 21, 22, 36–38].

In this article, we note that many properties of soft topological spaces are still valid on infra soft topological spaces, and examples that show some relationships between certain topological concepts are constructed easily on infra soft topological spaces. Therefore, we aim in this paper to perform an exhaustive analysis of infra soft topological spaces.

The organization of this article is as follows. In Section 2, we give some definitions and properties related to soft set theory and infra soft topology. In Section 3, we initiate infra *pp*-soft T_j-spaces ($j = 0, 1, 2, 3, 4$) and study basic properties. In Section 4, we define infra *pt*-soft T_j-spaces ($j = 0, 1, 2, 3, 4$) and discuss main properties. Then, we disclose some relationships among them with the help of elucidative examples. Finally, we give some conclusions and make a plan for future works in Section 5.

2. Preliminaries

The concepts and results that we need in this paper are recalled in this section.

The notation 2^X refers to the power set of X.

2.1. Soft set theory

Definition 2.1. ([32]) A map G from the set of parameters Λ to 2^X is called a soft set over X. It is denoted by G_Λ and identified as $G_\Lambda = \{(\lambda, G(\lambda)) : \lambda \in \Lambda \text{ and } G(\lambda) \in 2^X\}$.

The set of all soft sets over X under a set of parameters Λ is symbolized by $S(X_\Lambda)$.

Definition 2.2. ([3]) The relative complement of a soft set G_Λ, symbolized by G^c_Λ, is given by $G^c_\Lambda = (G^c)_\Lambda$, where $G^c : \Lambda \to 2^X$ is a map defined by $G^c(\lambda) = X \setminus G(\lambda)$ for each $\lambda \in \Lambda$.

Definition 2.3. ([31]) A soft set G_Λ over X is said to be the null soft set, symbolized by $\widetilde{\Phi}$, if $G(\lambda) = \emptyset$ for each $\lambda \in \Lambda$. Its relative complement is said to be the absolute soft set, symbolized by $\widetilde{X}$.

Definition 2.4. ([31]) The intersection of two soft sets G_Λ and F_Γ over X, symbolized by $G_\Lambda \widetilde{\bigcap} F_\Gamma$, is a soft set H_Ω, where $\Omega = \Lambda \cap \Gamma \neq \emptyset$, and a map $H : \Omega \to 2^X$ is given by $H(\omega) = G(\omega) \cap F(\omega)$ for each $\omega \in \Omega$.

Definition 2.5. ([3]) The union of two soft sets G_Λ and F_Γ over X, symbolized by $G_\Lambda \widetilde{\bigcup} F_\Gamma$, is a soft set H_Ω, where $\Omega = \Lambda \cup \Gamma$ and a map $H : \Omega \to 2^X$ is given as follows:

$$H(\omega) = \begin{cases} G(\omega) & : \ \omega \in \Lambda \setminus \Gamma \\ F(\omega) & : \ \omega \in \Gamma \setminus \Lambda \\ G(\omega) \cup F(\omega) & : \ \omega \in \Lambda \cap \Gamma \end{cases}$$

Definition 2.6. ([24]) A soft set G_Λ is a soft subset of a soft set F_Γ, symbolized by $G_\Lambda \widetilde{\subseteq} F_\Gamma$, if $\Lambda \subseteq \Gamma$ and for all $\lambda \in \Lambda$, we have $G(\lambda) \subseteq F(\lambda)$.
The soft sets G_Λ and G_Γ are called soft equal if each is a soft subset of the other.

Definition 2.7. ([20, 33]) A soft point P_Λ over X is a soft set such that $P(\lambda)$ is a singleton set, say x, and $P(\lambda')$ is the empty set for each $\lambda' \neq \lambda$. This soft point will be briefly symbolized by P^x_λ.

Definition 2.8. ([35]) A soft set x_Λ over X is defined by $x(\lambda) = \{x\}$ for each $\lambda \in \Lambda$.

Definition 2.9. A soft set G_Λ over X is said to be:

(i) a stable soft set [21] if all components are equal.

(ii) a full soft set [24] if $\bigcup\limits_{\lambda \in \Lambda} G(\lambda) = X$.

(iii) a partition soft set [24] if $\{G(\lambda) : \lambda \in \Lambda\}$ is a partition for X.

Definition 2.10. ([16]) The Cartesian product of G_Λ and H_Γ, symbolized by $(G \times H)_{\Lambda \times \Gamma}$, is defined as $(G \times H)(\lambda, \omega) = G(\lambda) \times H(\omega)$ for each $(\lambda, \omega) \in \Lambda \times \Gamma$.

The definition of soft maps given in [28] was reformulated in a way that reduces calculation burden and gives a logical explanation (justification) for some soft concepts such as why we determine that f_φ is injective, or surjective according to its two crisp mappings f and φ.

Definition 2.11. ([7]) Let $f : X \to Y$ and $\phi : \Lambda \to \Gamma$ be two crisp mappings. A soft mapping f_φ of $P(X_\Lambda)$ into $P(Y_\Gamma)$ is a relation such that each soft point in $P(X_\Lambda)$ is related to one and only one soft point in $P(Y_\Gamma)$ such that

$$f_\phi(P^x_\lambda) = P^{f(x)}_{\varphi(\lambda)} \text{ for each } P^x_\lambda \in P(X_\Lambda).$$

In addition, $f^{-1}_\phi(P^y_\gamma) = \bigsqcup\limits_{\substack{\lambda \in \varphi^{-1}(\gamma) \\ x \in f^{-1}(y)}} P^x_\lambda$ for each $P^y_\gamma \in P(Y_\Gamma)$.

Definition 2.12. ([28]) A soft map $f_\phi : S(X_\Lambda) \to S(Y_\Gamma)$ is said to be injective (resp. surjective, bijective) if both f and ϕ are injective (resp. surjective, bijective).

Definition 2.13. ([21, 35]) For a soft set G_Λ over X and $x \in X$, we say that:

(i) $x \in G_\Lambda$ (it reads as x totally belongs to G_Λ) if $x \in G(\lambda)$ for each $\lambda \in \Lambda$.

(ii) $x \notin G_\Lambda$(it reads as x does not partially belong to G_Λ) if $x \notin G(\lambda)$ for some $\lambda \in \Lambda$.

(iii) $x \Subset G_\Lambda$(it reads as x partially belongs to G_Λ) if $x \in G(\lambda)$ for some $\lambda \in \Lambda$.

(iv) $x \not\Subset G_\Lambda$(it reads as x does not totally belong to G_Λ) if $x \notin G(\lambda)$ for each $\lambda \in \Lambda$.

The following proposition connects the relations in Definition 2.13 with the study of the types of soft maps in Definition 2.12:

Proposition 2.14. ([21]) *Consider a soft map $g_\phi : S(X_\Lambda) \to S(Y_\Gamma)$ and let G_Λ and H_Γ be soft sets in $S(X_\Lambda)$ and $S(Y_\Gamma)$, respectively. Then the next results hold true:*

(i) *If $x \Subset G_\Lambda$, then $g(x) \Subset g_\phi(G_\Lambda)$.*

(ii) *If g is injective and $x \not\Subset G_\Lambda$, then $g(x) \not\Subset g_\phi(G_\Lambda)$.*

(iii) *If g_ϕ is injective and $x \notin G_\Lambda$, then $g(x) \notin g_\phi(G_\Lambda)$.*

(iv) *If ϕ is surjective and $y \Subset H_\Gamma$, then $x \Subset g_\phi^{-1}(H_\Gamma)$ for each $x \in g^{-1}(y)$.*

(v) *If $y \not\Subset H_\Gamma$, then $x \not\Subset g_\phi^{-1}(H_\Gamma)$ for each $x \in g^{-1}(y)$.*

(vi) *If ϕ is surjective and $y \notin H_\Gamma$, then $x \notin g_\phi^{-1}(H_\Gamma)$ for each $x \in g^{-1}(y)$.*

2.2. Infra oft topological spaces

The concepts of this subsection were introduced in [5, 7].

Definition 2.15. The collection ϑ of soft sets over X under a parameters set Ω is said to be an infra soft topology on X if it is closed under finite soft intersection and $\widetilde{\Phi} \in \vartheta$.

The triple (X, ϑ, Ω) is called an infra soft topological space. Every member of ϑ is called an infra soft open set and its relative complement is called an infra soft closed set.

Definition 2.16. We define the infra interior points and infra closure points of a soft subset H_Ω of (X, ϑ, Ω) which are respectively denoted by $Int_\vartheta(H_\Omega)$ and $Cl_\vartheta(H_\Omega)$ as follows.

(i) $Int_\vartheta(H_\Omega)$ is the union of all infra soft open sets contained in H_Ω.

(ii) $Cl_\vartheta(H_\Omega)$ is the intersection of all infra soft closed sets containing H_Ω.

Theorem 2.17. *Let H_Ω and F_Ω be two soft subsets of (X, ϑ, Ω). Then:*

(i) *If $H_\Omega \widetilde{\subseteq} F_\Omega$, then $Cl_\vartheta(H_\Omega) \widetilde{\subseteq} Cl_\vartheta(F_\Omega)$.*

(ii) *$P^x_\omega \in Cl_\vartheta(H_\Omega)$ if and only if $G_\Omega \widetilde{\bigcap} H_\Omega \neq \widetilde{\Phi}$ for each infra soft open set G_Ω containing P^x_ω.*

Proposition 2.18. *Let (X, ϑ, Ω) be an infra soft topological space. Then the collection $\vartheta_\omega = \{E(\omega) : E_\Omega \in \vartheta\}$ forms an infra topology on X for each $\omega \in \Omega$.*

We called ϑ_ω a parametric infra topology.

Definition 2.19. Let (X, ϑ, Ω) be an infra soft topological space and Y be a non-empty subset of X. Then $\vartheta_Y = \{\widetilde{Y} \widetilde{\bigcap} G_\Omega : G_\Omega \in \vartheta\}$ is called an infra soft relative topology on Y and (Y, ϑ_Y, Ω) is called an infra soft subspace of (X, ϑ, Ω).

Theorem 2.20. *Let (Y, ϑ_Y, Ω) be an infra soft subspace of (X, ϑ, Ω). Then H_Ω is an infra soft closed subset of (Y, ϑ_Y, Ω) if and only if there exists an infra soft closed subset F_Ω of (X, ϑ, Ω) such that $H_\Omega = \widetilde{Y} \widetilde{\bigcap} F_\Omega$.*

Definition 2.21. A soft mapping $g_\varphi : (X, \tau, \Omega) \to (Y, \theta, \Gamma)$ is said to be:

(i) infra soft continuous if the inverse image of each infra soft open set is an infra soft open set.

(ii) infra soft open (resp. infra soft closed) if the image of each infra soft open (resp. infra soft closed) set is an infra soft open (resp. infra soft closed) set.

(iii) an infra soft homeomorphism if it is bijective, infra soft continuous and infra soft open.

3. Infra *pp*-soft T_j-spaces ($j = 0, 1, 2, 3, 4$)

This section is devoted to introducing the concepts of infra *pp*-soft T_j-spaces and studying fundamental properties. Illustrative counterexamples are supplied to validate the obtained results and relationships.

Definition 3.1. (X, ϑ, Λ) is said to be:

(i) an infra *pp*-soft T_0-space if there exists an infra soft open set G_Λ for every $a \neq b \in X$ satisfies $a \Subset G_\Lambda, b \notin G_\Lambda$, or $b \Subset G_\Lambda, a \notin G_\Lambda$

(ii) an infra *pp*-soft T_1-space if there exist infra soft open sets G_Λ and F_Λ for every $a \neq b \in X$ satisfy $a \Subset G_\Lambda, b \notin G_\Lambda$, and $b \Subset F_\Lambda, a \notin F_\Lambda$.

(iii) an infra *pp*-soft T_2-space (or an infra *pp*-soft Hausdorff space) if there exist disjoint supra soft open sets G_Λ and F_Λ for every $a \neq b \in X$ satisfy $a \Subset G_\Lambda, b \notin G_\Lambda$, and $b \Subset F_\Lambda, a \notin F_\Lambda$.

(iv) an infra *pp*-soft regular space if for every infra soft closed set H_Λ such that $a \notin H_\Lambda$, there exist disjoint infra soft open sets G_Λ and F_Λ such that $H_\Lambda \widetilde{\subseteq} G_\Lambda$ and $a \Subset F_\Lambda$.

(v) an infra soft normal space if each disjoint infra soft closed sets are separated by disjoint infra soft open sets.

(vi) an infra *pp*-soft T_3 (resp. infra *pp*-soft T_4)-space if it is infra *pp*-soft regular (resp. infra soft normal) and infra *pp*-soft T_1.

We begin by explaining the relationship between these separation axioms.

Proposition 3.2. *Every infra pp-soft T_j-space is infra pp-soft T_{j-1} for $j = 1, 2, 3$.*

Proof. The proofs of $j = 1, 2$ follow immediately from Definition 3.1.

When $j = 3$. Consider a, b be two distinct points in an infra *pp*-soft T_3-space (X, ϑ, Λ). Then there are infra soft open sets U_Λ and V_Λ satisfy that $a \Subset U_\Lambda, b \notin U_\Lambda$, and $b \Subset V_\Lambda, b \notin V_\Lambda$. Now, $a \notin U^c_\Lambda$. By hypothesis of infra *pp*-soft regular, there are two disjoint infra soft open sets G_Λ and H_Λ satisfy that $U^c_\Lambda \widetilde{\subseteq} G_\Lambda$ and $a \Subset H_\Lambda$. Obviously, $b \Subset G_\Lambda$. The disjointness of G_Λ and H_Λ implies that $b \notin H_\Lambda$ and $a \notin G_\Lambda$. Hence, (X, ϑ, Λ) is infra *pp*-soft T_2. □

To show that the converse of Proposition 3.2 fails, we give the following example.

Example 3.3. Let $\Lambda = \{\lambda_1, \lambda_2\}$ be a set of parameters. Consider the following soft sets over $X = \{a, b\}$.

$F_\Lambda = \{(\lambda_1, \{a\}), (\lambda_2, \emptyset)\}$;

$G_\Lambda = \{(\lambda_1, \emptyset), (\lambda_2, X)\}$.

Now, the families $\vartheta_1 = \{\widetilde{\Phi}, \widetilde{X}, F_\Lambda\}$, $\vartheta_2 = \{\widetilde{\Phi}, \widetilde{X}, G_\Lambda\}$ and $\vartheta_3 = \{\widetilde{\Phi}, \widetilde{X}, F_\Lambda, G_\Lambda\}$ form infra soft topologies on X.

It can be easily checked that $(X, \vartheta_1, \Lambda)$, $(X, \vartheta_2, \Lambda)$ and $(X, \vartheta_3, \Lambda)$ are respectively infra *pp*-soft T_0, infra *pp*-soft T_1 and infra *pp*-soft T_2. On the other hand, $(X, \vartheta_1, \Lambda)$ is not infra *pp*-soft T_1, $(X, \vartheta_2, \Lambda)$ is not infra *pp*-soft T_2 and $(X, \vartheta_3, \Lambda)$ is not infra *pp*-soft T_3. Note that $(X, \vartheta_3, \Lambda)$ is infra *pp*-soft T_4.

Remark 3.4. We know that soft topology and general topology are identical if a set of parameters is a singleton. Then we can say that there is a (soft) topology satisfies infra *pp*-soft T_3, but not infra *pp*-soft T_4. Hence, infra *pp*-soft T_3 and infra *pp*-soft T_4-spaces are independent of each other.

Proposition 3.5. *(X, ϑ, Λ) is an infra pp-soft T_1-space if P^a_λ is an infra soft closed set for all $a \in X$.*

Proof. Let $a \neq b$. By hypothesis, $(P^a_\lambda)^c$ and $(P^b_\lambda)^c$ are infra soft open sets such that $b \in (P^a_\lambda)^c$, $a \notin (P^a_\lambda)^c$, and $a \in (P^b_\lambda)^c$, $b \in (P^b_\lambda)^c$. Hence, (X, ϑ, Λ) is infra *pp*-soft T_1. □

One can easily prove the following result; therefore, we omit its proof.

Proposition 3.6. *Let* $|X| \geq 2$ *and* $|\Lambda| \geq 2$. *Then we have the following results.*

(i) *Every infra pp-soft T_1-space (infra pp-soft T_4-space) (X, ϑ, Λ) contains at least one non-null proper infra soft open set.*

(ii) *Every infra pp-soft T_2-space (infra pp-soft T_3-space) (X, ϑ, Λ) contains at least two non-null proper infra soft open sets.*

Proposition 3.7. *If U_Λ and its relative complement are full soft sets such that U_Λ is an infra soft open subset of (X, ϑ, Λ), then (X, ϑ, Λ) is infra pp-soft T_1.*

Proof. Let $a \neq b \in X$. Since U_Λ is a full soft set, then $a \Subset U_\Lambda$ and $b \Subset U_\Lambda$; and since U^c_Λ is a full soft set, then $a \notin U_\Lambda$ and $b \notin U_\Lambda$. By hypothesis, U_Λ is an infra soft open set; hence, (X, ϑ, Λ) is an infra *pp*-soft T_1-space. □

Corollary 3.8. *If U_Λ is an infra soft open set in (X, ϑ, Λ) such that U_Λ is a partition soft set, then (X, ϑ, Λ) is infra pp-soft T_1.*

Corollary 3.9. *If U_Λ and its relative complement are full infra soft clopen sets in (X, ϑ, Λ), then (X, ϑ, Λ) is infra pp-soft T_2.*

Corollary 3.10. *If U_Λ is an infra soft clopen subset of (X, ϑ, Λ) such that U_Λ is partition soft set, then (X, ϑ, Λ) is infra pp-soft T_2.*

Theorem 3.11. *If (X, ϑ, Λ) has a basis of infra soft clopen sets, then (X, ϑ, Λ) is infra pp-soft regular.*

Proof. Let H_Λ be an infra soft closed set such that $a \notin H_\Lambda$. Then H^c_Λ is an infra soft open set such that $a \Subset H^c_\Lambda$. By hypothesis, the basis contains an infra soft clopen set F_Λ such that $a \Subset F_\Lambda \widetilde{\subseteq} H^c_\Lambda$. Now, $H_\Lambda \widetilde{\subseteq} F^c_\Lambda$. Obviously, F_Λ and F^c_Λ are disjoint infra soft open sets; hence, (X, ϑ, Λ) is infra *pp*-soft regular. □

In the following findings, we investigate transmission property between infra *pp*-soft T_j-spaces and parametric infra T_j-spaces.

Theorem 3.12. *If (X, ϑ_λ) is infra T_j, then (X, ϑ, Λ) is an infra pp-soft T_j-space for $j = 0, 1$.*

Proof. When $j = 1$. Let $a \neq b \in X$. Since (X, ϑ_λ) is infra T_1, there are two infra open sets U, V in ϑ_λ such that $a \in U$, $b \notin U$ and $b \in V$, $a \notin V$. Therefore, there are two infra soft open sets G_Λ, H_Λ in ϑ such that $G(\lambda) = U$ and $H(\lambda) = V$. Now, $a \Subset G_\Lambda$, $b \notin G_\Lambda$, and $b \Subset H_\Lambda$, $a \notin H_\Lambda$. Hence, (X, ϑ, Λ) is infra *pp*-soft T_1.

Similarly, one can prove the theorem when $j = 0$. □

Remark 3.13. It can be seen from Example 3.3 that $(X, \vartheta_2, \Lambda)$ is infra *pp*-soft T_1, but the two parametric infra topological spaces $(X, \vartheta_{2_{\lambda_1}})$, $(X, \vartheta_{2_{\lambda_2}})$ are not infra T_0.

There are no relationships between infra *pp*-soft T_j-spaces and parametric infra T_j-spaces for each $j = 2, 3, 4$. This fact is explained in the following examples.

Example 3.14. Let the next two soft sets over the set of natural numbers $\mathbb{N}$ with a parameters set $\Lambda = \{\lambda_1, \lambda_2\}$ defined as follows:

$G_\Lambda = \{(\lambda_1, \mathbb{N}), (\lambda_2, \emptyset)\}$;
$H_\Lambda = \{(\lambda_1, \emptyset), (\lambda_2, \mathbb{N})\}$.

Then $\vartheta = \{\widetilde{\Phi}, \widetilde{\mathbb{N}}, G_\Lambda, H_\Lambda\}$ is an infra soft topology on $\mathbb{N}$. Now, $(\mathbb{N}, \vartheta, \Lambda)$ is infra *pp*-soft T_4 and infra *pp*-soft T_3. However, $(\mathbb{N}, \vartheta_{\lambda_1})$ and $(\mathbb{N}, \vartheta_{\lambda_2})$ are not infra T_0.

Example 3.15. Consider the following two soft sets over $X = \{a, b, c\}$ with a parameters set $\Lambda = \{\lambda_1, \lambda_2\}$ as follows:

$U_{\Lambda_1} = \{(\lambda_1, \{a\}), (\lambda_2, X)\}$;
$U_{\Lambda_2} = \{(\lambda_1, \{b\}), (\lambda_2, X)\}$;
$U_{\Lambda_3} = \{(\lambda_1, \{c\}), (\lambda_2, X)\}$;
$U_{\Lambda_4} = \{(\lambda_1, \{a, b\}), (\lambda_2, X)\}$;
$U_{\Lambda_5} = \{(\lambda_1, \{a, c\}), (\lambda_2, X)\}$;
$U_{\Lambda_6} = \{(\lambda_1, \{b, c\}), (\lambda_2, X)\}$ and
$U_{\Lambda_7} = \{(\lambda_1, \emptyset), (\lambda_2, X)\}$.

Then $\vartheta = \{\widetilde{\Phi}, \widetilde{X}, U_{\Lambda_i} : i = 1, 2, ..., 7\}$ is an infra soft topology on X. It is clear that (X, ϑ, Λ) is not infra pp-soft T_j for $j = 2, 4$. However, $(X, \vartheta_{\lambda_1})$ is infra T_3 and infra T_4.

Proposition 3.16. *Every extended soft topological space* (X, ϑ, Λ) *is infra pp-soft* T_2.

Proof. Let $a \neq b \in X$ and let F_Λ and G_Λ be defined as follows

$$F(\lambda_i) = G(\lambda_j) = X, \text{ where } i \neq j$$

$$F(\lambda_{i'}) = G(\lambda_{j'}) = \emptyset \text{ for each } i' \neq i \text{ and } j' \neq j.$$

Since ϑ is extended, then F_Λ and G_Λ are disjoint infra soft open sets such that $a \Subset G_\Lambda, b \notin G_\Lambda$, and $b \Subset F_\Lambda, a \notin F_\Lambda$. Hence, we obtain the desired result. □

An infra soft topological space $(X, \vartheta_3, \Lambda)$ given in Example 3.3 elucidates that the converse of the above proposition fails.

Theorem 3.17. *Let* (X, ϑ, Λ) *be extended. If all* (X, ϑ_λ) *are infra* T_j*, then* (X, ϑ, Λ) *is infra pp-soft* T_j *for* $j = 0, 1, 3, 4$.

Proof. The cases of $j = 0, 1$ were proved in Theorem 3.12.

To prove the theorem in the case of $j = 3, 4$, it suffices to prove the axioms of infra pp-soft regularity and infra soft normality.

First, let H_Λ be an infra soft closed set such that $a \notin H_\Lambda$. Then there exists $\lambda \in \Lambda$ such that $a \notin H(\lambda)$. Since $H(\lambda)$ is an infra closed set, and (X, ϑ_λ) is infra regular, then there exist disjoint infra open sets U and V such that $H(\lambda) \subseteq U$ and $a \in V$. Since (X, ϑ, Λ) is extended, then there exist infra soft open sets G_Λ and F_Λ such that

$$G(\lambda) = U \text{ and } G(\lambda') = X \text{ for each } \lambda' \neq \lambda$$

$$F(\lambda) = V \text{ and } F(\lambda) = \emptyset \text{ for each } \lambda' \neq \lambda$$

It is clear that $H_\Lambda \widetilde{\subseteq} G_\Lambda$ and $a \Subset F_\Lambda$. The disjointness of G_Λ and F_Λ proves that (X, ϑ, Λ) is infra pp-soft regular.

Second, we prove that (X, ϑ, Λ) is infra soft normal. Let H_Λ, L_Λ be two disjoint infra soft closed sets. Then $H(\lambda)$ and $L(\lambda)$ are two disjoint infra closed sets for each $\lambda \in \Lambda$. Since (X, ϑ_λ) is infra normal, then there exist two disjoint infra open sets U_λ and V_λ such that $H(\lambda) \subseteq U_\lambda$ and $L(\lambda) \subseteq V_\lambda$. Since (X, ϑ, Λ) is extended, then there exist disjoint infra soft open subsets G_Λ, F_Λ of (X, ϑ, Λ) such that

$$G(\lambda) = U_\lambda \text{ for each } \lambda \in \Lambda$$

$$F(\lambda) = V_\lambda \text{ for each } \lambda \in \Lambda$$

Thus, (X, ϑ, Λ) is infra soft normal. Hence, it is infra pp-soft T_4. □

Converse of the above theorem need not true as illustrated in Example 3.14.

Theorem 3.18. *Let (X, ϑ, Λ) be stable. Then, (X, ϑ_λ) is infra T_j if and only if (X, ϑ, Λ) is infra pp-soft T_j for each $j = 0, 1, 2, 3, 4$.*

Proof. Since (X, ϑ, Λ) is stable, then U is an infra open subset of (X, ϑ_λ) if and only if $\{(\lambda, U) : \lambda \in \Lambda\}$ is an infra soft open subset of (X, ϑ, Λ). Hence, the desired result is proved. □

Proposition 3.19. *If (X, ϑ, Λ) is an infra pp-soft regular space, then for each $a \in X$ and infra soft open set F_Λ partially containing a there exists an infra soft open set V_Λ such that $a \Subset V_\Lambda \widetilde{\subseteq} Cl_\vartheta(V_\Lambda) \widetilde{\subseteq} F_\Lambda$.*

Proof. Let $a \in X$ and F_Λ be an infra soft open set partially containing a. Then F^c_Λ is an infra soft closed set and $a \notin F^c_\Lambda$. By hypothesis, there are disjoint infra soft open sets U_Λ and V_Λ such that $F^c_\Lambda \widetilde{\subseteq} U_E$ and $a \Subset V_\Lambda$. Obviously, $V_\Lambda \widetilde{\subseteq} U^c_\Lambda \widetilde{\subseteq} F_\Lambda$. Thus, $Cl_\vartheta(V_\Lambda) \widetilde{\subseteq} U^c_\Lambda \widetilde{\subseteq} F_\Lambda$. □

Theorem 3.20. *If (X, ϑ, Λ) is an infra pp-soft regular space, then the following concepts are identical.*

(i) *(X, ϑ, Λ) is infra pp-soft T_2.*

(ii) *(X, ϑ, Λ) is infra pp-soft T_1.*

(iii) *(X, ϑ, Λ) is infra pp-soft T_0.*

Proof. The directions $(i) \to (ii) \to (iii)$ follow from Proposition 3.2.

To prove $(iii) \to (i)$, let $a \neq b \in X$. Since (X, ϑ, Λ) is infra *pp*-soft T_0, then we have an infra soft open set G_Λ such that $a \Subset G_\Lambda$ and $b \notin G_\Lambda$, or $b \Subset G_\Lambda$ and $a \notin G_\Lambda$. Say, $a \Subset G_\Lambda$ and $b \notin G_\Lambda$. It is clear that $a \notin G^c_\Lambda$ and $b \Subset G^c_\Lambda$. Since (X, ϑ, Λ) is infra *pp*-soft regular, then there exist two disjoint infra soft open sets U_Λ and V_Λ such that $a \Subset U_\Lambda$ and $b \Subset G^c_\Lambda \widetilde{\subseteq} V_\Lambda$. Since U_Λ and V_Λ are disjoint, then $b \notin U_\Lambda$ and $a \notin V_\Lambda$. This ends the proof that (X, ϑ, Λ) is infra *pp*-soft T_2. □

Theorem 3.21. *The next statements are equivalent:*

(i) *(X, ϑ, Λ) is infra soft normal;*

(ii) *For every infra soft open sets U_Λ and V_Λ such that $U_\Lambda \widetilde{\bigcup} V_\Lambda = \widetilde{X}$, there are two infra soft closed sets F_Λ and H_Λ such that $F_\Lambda \widetilde{\subseteq} U_\Lambda$, $H_\Lambda \widetilde{\subseteq} V_\Lambda$, and $F_\Lambda \widetilde{\bigcup} H_\Lambda = \widetilde{X}$.*

Proof. $(i) \to (ii)$: Let U_Λ and V_Λ be infra soft open sets such that $U_\Lambda \widetilde{\bigcup} V_\Lambda = \widetilde{X}$. Then U^c_Λ and V^c_Λ are disjoint infra soft closed sets. By (i), there are infra soft open sets E_Λ and G_Λ such that $U^c_\Lambda \widetilde{\subseteq} E_\Lambda$ and $V^c_\Lambda \widetilde{\subseteq} G_\Lambda$. Thus, E^c_Λ and G^c_Λ are infra soft closed sets such that $E^c_\Lambda \widetilde{\subseteq} U_\Lambda$, $G^c_\Lambda \widetilde{\subseteq} V_\Lambda$, and $E^c_\Lambda \widetilde{\bigcup} G^c_\Lambda = \widetilde{X}$.

$(ii) \to (i)$: Let F_Λ and H_Λ be disjoint infra soft closed sets. Since F^c_Λ and H^c_Λ are infra soft open sets such that $F^c_\Lambda \widetilde{\bigcup} H^c_\Lambda = \widetilde{X}$, there are two infra soft closed sets M_Λ and N_Λ such that $M_\Lambda \widetilde{\subseteq} F^c_\Lambda$, $N_\Lambda \widetilde{\subseteq} H^c_\Lambda$ and $M_\Lambda \widetilde{\bigcup} N_\Lambda = \widetilde{X}$. Thus, M^c_Λ and N^c_Λ are two disjoint infra soft open sets such that $F_\Lambda \widetilde{\subseteq} M^c_\Lambda$ and $H_\Lambda \widetilde{\subseteq} N^c_\Lambda$. Hence, (X, ϑ, Λ) is infra soft normal. □

Theorem 3.22. *The property of being an infra soft T_j-space is an infra soft hereditary property for $j = 0, 1, 2, 3$.*

Proof. When $j = 3$.
Let $(Y, \vartheta_Y, \Lambda)$ be a subspace of (X, ϑ, Λ) which is infra *pp*-soft T_3. We first prove that $(Y, \vartheta_Y, \Lambda)$ is infra *pp*-soft T_1. Let $a \neq b \in Y$. Then ϑ contains infra soft open sets G_Λ and F_Λ such that $a \Subset G_\Lambda, b \notin G_\Lambda$ and $b \Subset F_\Lambda, a \notin F_\Lambda$. Now, $U_\Lambda = \widetilde{Y} \widetilde{\bigcap} G_\Lambda$ and $V_\Lambda = \widetilde{Y} \widetilde{\bigcap} F_\Lambda$ are infra soft open sets in $(Y, \vartheta_Y, \Lambda)$. It is clear that $a \Subset U_\Lambda$ and $b \Subset V_\Lambda$, also, $b \notin U_\Lambda$ and $a \notin V_\Lambda$. Thus, $(Y, \vartheta_Y, \Lambda)$ is infra *pp*-soft T_1.

To prove the infra *pp*-soft regularity of $(Y, \vartheta_Y, \Lambda)$, let $a \in Y$ and L_Λ be an infra soft closed subset of $(Y, \vartheta_Y, \Lambda)$ such that $a \notin L_\Lambda$. Then there exists an infra soft closed subset H_Λ of (X, ϑ, Λ) such that $L_\Lambda = \widetilde{Y} \widetilde{\bigcap} H_\Lambda$. Since $a \notin H_\Lambda$, there exist disjoint infra soft open sets G_Λ and F_Λ such that $H_\Lambda \widetilde{\subseteq} G_\Lambda$ and $a \Subset F_\Lambda$. Now, we find that $L_\Lambda \widetilde{\subseteq} \widetilde{Y} \widetilde{\bigcap} G_\Lambda$, $a \Subset \widetilde{Y} \widetilde{\bigcap} F_\Lambda$ and $(\widetilde{Y} \widetilde{\bigcap} G_\Lambda) \widetilde{\bigcap} (\widetilde{Y} \widetilde{\bigcap} F_\Lambda) = \widetilde{\Phi}$. Thus, $(Y, \vartheta_Y, \Lambda)$ is infra *pp*-soft regular. Hence, $(Y, \vartheta_Y, \Lambda)$ is infra *pp*-soft T_3. □

Theorem 3.23. *The property of being an infra soft T_4-space is an infra soft closed hereditary property.*

Proof. One can prove it easily. □

Theorem 3.24. *The finite product of infra pp-soft T_j-spaces is infra pp-soft T_j for $j = 0, 1, 2$.*

Proof. We give a proof for the theorem when $j = 2$. The remaining two cases follow similar lines.

Without loss of generality, we consider two infra *pp*-soft T_2-spaces (X, ϑ, Λ) and (Y, υ, Γ). Suppose that $(a_1, b_1) \neq (a_2, b_2)$ in $X \times Y$. Then $a_1 \neq a_2$ or $b_1 \neq b_2$. Say, $a_1 \neq a_2$. Then ϑ contains two disjoint infra soft open sets U_Λ, V_Λ such that $a_1 \Subset U_\Lambda$ and $a_2 \notin U_\Lambda$; and $a_2 \Subset V_\Lambda$ and $a_1 \notin V_\Lambda$. Now, $U_\Lambda \widetilde{\times} \widetilde{Y}$ and $V_\Lambda \times \widetilde{Y}$ are infra soft open sets such that $(a_1, b_1) \Subset U_\Lambda \times \widetilde{Y}$ and $(a_2, b_2) \notin U_\Lambda \times \widetilde{Y}$, and $(a_2, b_2) \Subset V_\Lambda \times \widetilde{Y}$ and $(a_1, b_1) \notin V_\Lambda \times \widetilde{Y}$. Since $[U_\Lambda \times \widetilde{Y}] \widetilde{\bigcap} [V_\Lambda \times \widetilde{Y}] = \widetilde{\emptyset_{\Lambda\times\Gamma}}$, then $X \times Y$ is infra *pp*-soft T_2. □

Proposition 3.25. *Let $f_\phi : (X, \vartheta, \Lambda) \to (Y, \upsilon, \Gamma)$ be an infra soft continuous map such that f is injective and ϕ is surjective. If (Y, υ, Γ) is infra pp-soft T_j, then (X, ϑ, Λ) is infra pp-soft T_j for $j = 0, 1, 2$.*

Proof. When $j = 2$. Let $a \neq b \in X$. Since f is injective, there exists only two distinct points $x, y \in Y$ such that $f(a) = x$ and $f(b) = y$. Since (Y, υ, Γ) is infra *pp*-soft T_2, there are two disjoint infra soft open sets G_Γ and F_Γ such that $x \Subset G_\Gamma$, $y \notin G_\Gamma$ and $y \Subset F_\Gamma$, $x \notin F_\Gamma$. By infra soft continuity of f_ϕ, we obtain $f_\phi^{-1}(G_\Gamma)$ and $f_\phi^{-1}(F_\Gamma)$ are infra soft open sets.

Since ϕ is surjective, it follows from Proposition 2.14 that $a \Subset f_\phi^{-1}(G_\Gamma)$, $b \notin f_\phi^{-1}(G_\Gamma)$ and $b \Subset f_\phi^{-1}(F_\Gamma)$, $a \notin f_\phi^{-1}(F_\Gamma)$. The disjointness of them ends the proof that (X, ϑ, Λ) is infra *pp*-soft T_2. □

In a similar manner, one can prove the following three results.

Proposition 3.26. *Let $f_\phi : (X, \vartheta, \Lambda) \to (Y, \upsilon, \Gamma)$ be a bijective infra soft continuous map. If (Y, υ, Γ) is infra pp-soft T_j, then (X, ϑ, Λ) is infra pp-soft T_j for $j = 0, 1, 2, 3, 4$.*

Proposition 3.27. *Let $f_\phi : (X, \vartheta, \Lambda) \to (Y, \upsilon, \Gamma)$ be a bijective infra soft open map. If (X, ϑ, Λ) is infra pp-soft T_j, then (Y, υ, Γ) is infra pp-soft T_j for $j = 0, 1, 2, 3, 4$.*

Proposition 3.28. *The property of being an infra pp-soft T_j-space $(j = 0, 1, 2, 3, 4)$ is preserved under an infra soft homeomorphism map.*

4. Infra *pt*-soft T_j-spaces $(j = 0, 1, 2, 3, 4)$

This section is devoted to presenting the concepts of infra *pt*-soft T_j-spaces and discussing main features. The relationships between them as well as their relationships with infra *pp*-soft T_j-spaces are showed with the aid of some counterexamples.

Definition 4.1. (X, ϑ, Λ) is said to be:

(i) an infra *pt*-soft T_0-space if there exists an infra soft open set G_Λ for every $a \neq b \in X$ satisfies $a \Subset G_\Lambda, b \not\Subset G_\Lambda$, or $b \Subset G_\Lambda, x \not\Subset G_\Lambda$.

(ii) an infra *pt*-soft T_1-space if there exist infra soft open sets G_Λ and F_Λ for every $a \neq b \in X$ satisfy $a \Subset G_\Lambda, b \not\Subset G_\Lambda$, and $b \Subset F_\Lambda, a \not\Subset F_\Lambda$.

(iii) an infra *pt*-soft T_2-space (or an infra *pt*-soft Hausdorff space) if there exist two disjoint infra soft open sets G_Λ and F_Λ for every $a \neq b \in X$ satisfy $a \Subset G_\Lambda, b \not\Subset G_\Lambda$, and $b \Subset F_\Lambda, a \not\Subset F_\Lambda$.

(iv) an infra *pt*-soft regular space if for every infra soft closed set H_Λ such that $a \not\Subset H_\Lambda$, there exist disjoint infra soft open sets G_Λ and F_Λ such that $H_\Lambda \widetilde{\subseteq} G_\Lambda$ and $a \Subset F_\Lambda$.

(v) an infra *pt*-soft T_3 (resp. infra *pt*-soft T_4)-space if it is infra *pt*-soft regular (resp. infra soft normal) and infra *pt*-soft T_1.

Proposition 4.2. *The following properties hold true:*

(i) *Every infra pt-soft T_j-space is infra pt-soft T_{j-1} for $j = 1, 2$.*

(ii) *Every infra pt-soft T_j-space is infra pp-soft T_j for $j = 0, 1, 2, 4$.*

(iii) *Every infra pp-soft regular space is infra pt-soft regular.*

Proof. The proof of (i) follows directly from the above definition.
The proof of (ii) and (iii) follows from the fact that $\not\Subset$ implies $\notin$. □

To clarify that the converse of the above proposition is not always true, we provide the following examples.

Example 4.3. Assume that $(X, \vartheta_1, \Lambda)$ is the same as in Example 3.3. It can be easily checked that $(X, \vartheta_1, \Lambda)$ is an infra *pt*-soft T_0-space; however, it is not infra *pt*-soft T_1.

Example 4.4. Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ be a set of parameters and $X = \{a, b, c\}$ the universal set. Consider the following soft sets which defined over X

$F_{1\Lambda} = \{(\lambda_1, \{a\}), (\lambda_2, \{b\}), (\lambda_3, \emptyset)\}$;
$F_{2\Lambda} = \{(\lambda_1, \emptyset), (\lambda_2, \emptyset), (\lambda_3, \{c\})\}$;
$F_{3\Lambda} = \{(\lambda_1, \emptyset), (\lambda_2, \emptyset), (\lambda_3, \{a, c\})\}$;
$F_{4\Lambda} = \{(\lambda_1, \emptyset), (\lambda_2, \emptyset), (\lambda_3, \{b, c\})\}$.

Now, a family ϑ of the above four soft sets with $\widetilde{\Phi}$ and $\widetilde{X}$ consist an infra soft topology on X. On the one hand, (X, ϑ, Λ) is infra *pt*-soft T_1. On the other hand, ϑ does not contain disjoint infra soft open sets satisfying a condition of infra *pt*-soft T_2 for the distinct points a and b. Therefore, (X, ϑ, Λ) is not infra *pt*-soft T_2.

Example 4.5. Let $(\mathbb{N}, \vartheta, \Lambda)$ be the same as in Example 3.14. As we saw it is infra *pp*-soft T_4 and infra *pp*-soft T_3. Since ϑ does not contain an infra soft open set (except for the null soft set) does not totally contain a or b, then $(\mathbb{N}, \vartheta, \Lambda)$ is not infra *pt*-soft T_0.

Example 4.6. Let $(X, \vartheta_2, \Lambda)$ be the same as in Example 3.3. Obviously, $(X, \vartheta_2, \Lambda)$ infra *pt*-soft regular, but not infra *pp*-soft regular.

In the following remark and examples, we illustrate that:

(i) The concepts of infra *pt*-soft T_2, infra *pt*-soft T_3 and infra *pt*-soft T_4-spaces are independent of each other.

(ii) The concepts of infra *pp*-soft T_3 and infra *pt*-soft T_3-spaces are independent of each other.

Remark 4.7. According to Remark 3.4, we suffice with examples given in general topology which show there exists an infra *pt*-soft T_2-space, but not infra *pt*-soft T_3, and there exists an infra *pt*-soft T_3-space, but not infra *pt*-soft T_4.

Example 4.8. Let (X, ϑ, Λ) be the same as in Example 4.4. One can check that (X, ϑ, Λ) is infra *pt*-soft T_3. On the other hand, ϑ does not contain two disjoint infra soft open sets satisfy a condition of infra *pt*-soft T_2-space for the distinct points a and b. Hence, (X, ϑ, Λ) is not infra *pt*-soft T_2.

Example 4.9. Let $\Lambda = \{\lambda_1, \lambda_2\}$ be a set of parameters. Then $\vartheta = \{\widetilde{\mathbb{R}}, F_\Lambda \widetilde{\sqsubseteq} \widetilde{\mathbb{R}} : F_\Lambda$ is finite$\}$ is an infra soft topology on the set of real numbers $\mathbb{R}$. Note that the intersection of any two infra soft closed sets (except for the null and absolute soft sets) in $(\mathbb{R}, \vartheta, \Lambda)$ is non-null. Therefore, $(\mathbb{R}, \vartheta, \Lambda)$ is infra soft normal, also, it is infra *pt*-soft T_2. Hence, it is infra *pt*-soft T_4. On the other hand, $H_\Lambda = \{(\lambda_1, \mathbb{R} \setminus \{1\}), (\lambda_2, \mathbb{R} \setminus \{1\})\}$ is an infra soft closed subset of $\widetilde{\mathbb{R}}$ such that $1 \not\Subset H_\Lambda$. But there does not exist infra soft open set containing H_Λ except for the absolute soft set. Hence, $(\mathbb{R}, \vartheta, \Lambda)$ is not infra *pt*-soft T_3.

Example 4.10. Let (X, ϑ, Λ) be the same as in Example 3.14. We pointed out that (X, ϑ, Λ) is infra *pp*-soft T_3. On the other hand, it is not infra *pt*-soft T_3 because it is not infra *pt*-soft T_0.

Example 4.11. Let $\Lambda = \{\lambda_1, \lambda_2\}$ be a set of parameters. Then $\vartheta = \{\widetilde{X}, F_\Lambda \widetilde{\sqsubseteq} \widetilde{X}$ such that $1 \notin F_\Lambda$ or $F_\Lambda = \{(\lambda_1, \{1\}), (\lambda_2, \{1\})\}\}$ is an infra soft topology on $X = \{1, 2\}$. One can check that (X, ϑ, Λ) is infra *pt*-soft T_3, but it is not infra *pp*-soft T_3.

In what follows, we investigate the main properties of infra *pt*-soft T_j-spaces.

Proposition 4.12. *(X, ϑ, Λ) is an infra pt-soft T_1-space if a_Λ is an infra soft closed set for all $a \in X$.*

Proof. Let $a \neq b$. By hypothesis, $(a_\Lambda)^c$ and $(b_\Lambda)^c$ are infra soft open sets such that $a \in (b_\Lambda)^c$, $b \notin (b_\Lambda)^c$, and $b \in (a_\Lambda)^c$, $a \notin (a_\Lambda)^c$. Hence, (X, ϑ, Λ) is *pt*-soft T_1. □

Example 4.4 clarifies that the reversal of the above proposition fails.

Proposition 4.13. *If (X, ϑ, Λ) is an infra pt-soft regular space, then for each $a \in X$ and infra soft open set F_Λ totally containing a, there exists an infra soft open set V_Λ such that $a \Subset V_\Lambda \widetilde{\subseteq} Cl_\vartheta(V_\Lambda) \widetilde{\subseteq} F_\Lambda$.*

Proof. One can prove the proposition following similar technique given in the proof of Proposition 3.19. □

Theorem 4.14. *Let (X, ϑ, Λ) be a stable space. Then the following statements holds.*

(i) *(X, ϑ, Λ) is infra pt-soft $T_0 \Leftrightarrow (X, \vartheta, \Lambda)$ is infra pp-soft T_0.*

(ii) *(X, ϑ, Λ) is infra pt-soft $T_1 \Leftrightarrow (X, \vartheta, \Lambda)$ is infra pp-soft T_1.*

(iii) *(X, ϑ, Λ) is infra pt-soft $T_2 \Leftrightarrow (X, \vartheta, \Lambda)$ is infra pp-soft T_2.*

(iv) *(X, ϑ, Λ) is infra pt-soft $T_3 \Leftrightarrow (X, \vartheta, \Lambda)$ is infra pp-soft T_3.*

(v) *(X, ϑ, Λ) is infra pt-soft $T_4 \Leftrightarrow (X, \vartheta, \Lambda)$ is infra pp-soft T_4.*

Proof. It comes from the fact that partial non-belong and total non-belong relations are identical with respect to the stable soft sets. □

Now, we point out that infra *pt*-soft T_j-spaces do not keep on their parametric infra topological spaces and vice versa.

Example 4.15. Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ be a set of parameters and $X = \{a, b, c\}$ the universal set. Consider the following soft sets which defined over X

$F_{1\Lambda} = \{(\lambda_1, \{a\}), (\lambda_2, \emptyset), (\lambda_3, \emptyset)\}$;
$F_{2\Lambda} = \{(\lambda_1, \emptyset), (\lambda_2, \{b\}), (\lambda_3, \emptyset)\}$;
$F_{3\Lambda} = \{(\lambda_1, \emptyset), (\lambda_2, \emptyset), (\lambda_3, \{c\})\}$

Now, a family ϑ of the above three soft sets with $\widetilde{\Phi}$ and $\widetilde{X}$ consist an infra soft topology on X. On the one hand, (X, ϑ, Λ) is infra *pt*-soft T_2, infra *pt*-soft T_3 and infra *pt*-soft T_4. On the other hand, $(X, \vartheta_{\lambda_1})$, $(X, \vartheta_{\lambda_2})$ and $(X, \vartheta_{\lambda_3})$ are not infra T_0.

Example 4.16. Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ be a set of parameters and $X = \{a, b, c\}$ the universal set. Consider the following soft sets which defined over X

$F_{1\Lambda} = \{(\lambda_1, \{a\}), (\lambda_2, \{b\}), (\lambda_3, \{c\})\}$;
$F_{2\Lambda} = \{(\lambda_1, \{c\}), (\lambda_2, \{a\}), (\lambda_3, \{b\})\}$;
$F_{3\Lambda} = \{(\lambda_1, \{b\}), (\lambda_2, \{c\}), (\lambda_3, \{a\})\}$;
$F_{4\Lambda} = \{(\lambda_1, \{a, b\}), (\lambda_2, \{b, c\}), (\lambda_3, \{a, c\})\}$;
$F_{5\Lambda} = \{(\lambda_1, \{a, c\}), (\lambda_2, \{a, b\}), (\lambda_3, \{b, c\})\}$;
$F_{6\Lambda} = \{(\lambda_1, \{b, c\}), (\lambda_2, \{a, c\}), (\lambda_3, \{a, b\})\}$

Now, a family ϑ of the above six soft sets with $\widetilde{\Phi}$ and $\widetilde{X}$ consist an infra soft topology on X. On the one hand, (X, ϑ, Λ) is not infra *pt*-soft T_0 because every infra soft open set (except for the null soft set) partially contains every points in X. On the other hand, $(X, \vartheta_{\lambda_1})$, $(X, \vartheta_{\lambda_2})$ and $(X, \vartheta_{\lambda_3})$ are infra T_2, infra T_3 and infra T_4.

Theorem 4.17. *Let (X, ϑ, Λ) be an extended infra soft topological space. If there exists $\lambda \in \Lambda$ such that (X, ϑ_λ) is infra T_j, then (X, ϑ, Λ) is infra pt-soft T_j for each $j = 0, 1, 2$.*

Proof. We prove the theorem in the case of $j = 2, 3$. The other ones follow similar lines.

When $j = 2$. Let $a \neq b \in X$ and consider (X, ϑ_λ) is infra T_2. Then there are two disjoint infra open subsets U, V of (X, ϑ_λ) containing a and b, respectively. Since (X, ϑ, Λ) is extended, there are infra soft open subsets G_Λ, F_Λ of (X, ϑ, Λ) such that $G(\lambda) = U$, $F(\lambda) = V$ and $G(\lambda') = F(\lambda') = \emptyset$ for each $\lambda' \neq \lambda$. This implies that $a \Subset G_\Lambda$, $b \notin G_\Lambda$, and $b \Subset F_\Lambda$, $a \notin F_\Lambda$; hence, (X, ϑ, Λ) is infra *pt*-soft T_2.

When $j = 3$. It suffices to prove the property of infra *pt*-soft regular. To do that, let H_Λ be an infra soft closed set such that $a \notin H_\Lambda$. Then $a \notin H(\lambda)$ for each $\lambda \in \Lambda$. By hypothesis, there exists $\lambda \in \Lambda$ such that (X, ϑ_λ) is infra regular. Then there are disjoint infra open subsets U, V of (X, ϑ_λ) such that $a \in U$, $H(\lambda) \subseteq V$. Since (X, ϑ, Λ) is extended, there exist infra soft open subsets G_Λ, F_Λ of (X, ϑ, Λ) such that

$$G(\lambda) = U \text{ and } G(\lambda') = \emptyset \text{ for each } \lambda' \neq \lambda$$

$$F(\lambda) = V \text{ and } G(\lambda') = X \text{ for each } \lambda' \neq \lambda$$

This shows that $a \Subset G_\Lambda$, $H_\Lambda \widetilde{\subseteq} F_\Lambda$. The disjointness of G_Λ and F_Λ ends the proof that (X, ϑ, Λ) is infra *pt*-soft regular. Hence, it is infra *pt*-soft T_3. □

To clarify that the reversal of the above theorem fails, we give the following example.

Example 4.18. Let $\Lambda = \{\lambda_1, \lambda_2\}$ be a set of parameters and $X = \{a, b, c\}$ the universal set. Consider the following soft sets which defined over X

$F_{1\Lambda} = \{(\lambda_1, \{a\}), (\lambda_2, \emptyset)\}$;
$F_{2\Lambda} = \{(\lambda_1, \{a\}), (\lambda_2, \{c\})\}$;
$F_{3\Lambda} = \{(\lambda_1, \{a\}), (\lambda_2, \{a, b\})\}$;
$F_{4\Lambda} = \{(\lambda_1, \{a\}), (\lambda_2, X)\}$;
$F_{5\Lambda} = \{(\lambda_1, \{b, c\}), (\lambda_2, \emptyset)\}$;
$F_{6\Lambda} = \{(\lambda_1, \{b, c\}), (\lambda_2, \{c\})\}$;
$F_{7\Lambda} = \{(\lambda_1, \{b, c\}), (\lambda_2, \{a, b\})\}$;
$F_{8\Lambda} = \{(\lambda_1, \{b, c\}), (\lambda_2, X)\}$;
$F_{9\Lambda} = \{(\lambda_1, \emptyset), (\lambda_2, \{c\})\}$;
$F_{10\Lambda} = \{(\lambda_1, X), (\lambda_2, \{c\})\}$;
$F_{11\Lambda} = \{(\lambda_1, \emptyset), (\lambda_2, \{a, b\})\}$;
$F_{12\Lambda} = \{(\lambda_1, X), (\lambda_2, \{a, b\})\}$;
$F_{13\Lambda} = \{(\lambda_1, X), (\lambda_2, \emptyset)\}$;
$F_{14\Lambda} = \{(\lambda_1, \emptyset), (\lambda_2, X)\}$

Now, a family ϑ of the above fourteen soft sets with $\widetilde{\Phi}$ and $\widetilde{X}$ consist an extended infra soft topology on X. Obviously, (X, ϑ, Λ) is infra *pt*-soft T_2 and infra *pt*-soft T_3. However, $(X, \vartheta_{\lambda_1})$ and $(X, \vartheta_{\lambda_2})$ are not infra T_0.

Theorem 4.19. *Let (X, ϑ, Λ) be an extended infra soft topological space. If all (X, ϑ_λ) are infra T_4, then (X, ϑ, Λ) is infra pt-soft T_4.*

Proof. The property of an infra *pt*-soft T_1-space was proved in Theorem 4.17, and the property of an infra soft normal space was proved in Theorem 3.17. Hence, (X, ϑ, Λ) is infra *pt*-soft T_4. □

Remark 4.20. Example 4.16 confirms that the restriction to extended infra soft topological spaces in Theorems 4.17 and 4.19 is not redundant.

It can be proved the following findings in a similar way to their counterparts in the last section.

Theorem 4.21. *Suppose that* (X, ϑ, Λ) *is stable. Then* (X, ϑ_λ) *is infra* T_j *iff* (X, ϑ, Λ) *is infra pt-soft* T_j *for each* $j = 0, 1, 2, 3, 4$.

Theorem 4.22. *Every soft subspace of an infra pt-soft* T_j*-space is infra pt-soft* T_j *for* $j = 0, 1, 2, 3$.

Theorem 4.23. *Every infra soft closed subspace of an infra pt-soft* T_4*-space is infra pt-soft* T_4.

Theorem 4.24. *The finite product of infra pt-soft* T_j*-spaces is infra pt-soft* T_j *for* $j = 0, 1, 2$.

Proposition 4.25. *Let* $f_\phi : (X, \vartheta, \Lambda) \to (Y, \upsilon, \Gamma)$ *be an infra soft continuous map such that* f *is injective and* ϕ *is surjective. If* (Y, υ, Γ) *is infra pt-soft* T_j*, then* (X, ϑ, Λ) *is infra pt-soft* T_j *for* $j = 0, 1, 2$.

Proposition 4.26. *Let* $f_\phi : (X, \vartheta, \Lambda) \to (Y, \upsilon, \Gamma)$ *be a bijective infra soft continuous map. If* (Y, υ, Γ) *is infra pt-soft* T_j*, then* (X, ϑ, Λ) *is infra pt-soft* T_j *for* $j = 0, 1, 2, 3, 4$.

Proposition 4.27. *Let* $f_\phi : (X, \vartheta, \Lambda) \to (Y, \upsilon, \Gamma)$ *be a bijective infra soft open map. If* (X, ϑ, Λ) *is infra pt-soft* T_j*, then* (Y, υ, Γ) *is infra pt-soft* T_j *for* $j = 0, 1, 2, 3, 4$.

Proposition 4.28. *The property of being an infra pt-soft* T_j*-space* ($j = 0, 1, 2, 3, 4$) *is preserved under an infra soft homeomorphism map.*

We complete this section by the following diagram which illustrates the relationships between infra soft separation axioms introduced in this manuscript.

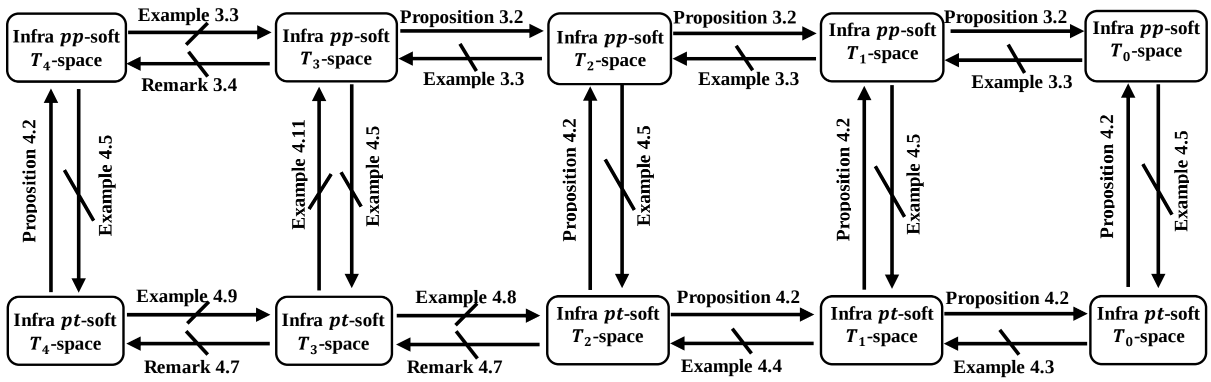

Figure 1: The relationships between infra soft separation axioms

5. Conclusion and future work

This paper is at the junction of two disciplines, namely, soft set theory and infra soft topology. In this paper, we perform an exhaustive analysis of separation axioms in the context of infra soft topological spaces. The motivations of this work are, first, to create new families of soft structures and discover which one of the topological properties is still valid in these structures. Second, these families of soft structures open a door to study new types of approximations and accuracy measures in the content of rough sets models as those given [2]. Third, they will allow us to investigate many results induced from their interaction with some soft topological concepts such as infra soft compactness and infra soft connectedness which were given in [6] and [8], respectively.

Through this article, we have formulated the concepts of infra *pp*-soft T_j and infra *pt*-soft T_j-spaces ($j = 0, 1, 2, 3, 4$). They have been defined with respect to the relationships between distinct ordinary points and soft sets using partial belong and partial non-belong relations in the first type, and partial belong and total non-belong relations in the second type. We have provided some examples to show the relationships between the spaces in each type of them as well as the interrelationships between the two types. Also, we have studied their transmission between infra soft topology and its parametric infra topologies. Furthermore, we have discussed their image and pre-image under some types of infra soft mappings.

In future works, we plan to complete this study by introducing new types of infra soft separation axioms using the two pairs of relations $(\in, \notin)$ and $(\Subset, \not\Subset)$. Also, we shall investigate further concepts in the context of infra soft topological spaces such as infra soft α-open, infra soft pre-open, infra soft semi-open, infra soft b-open, and infra soft β-open sets. Finally, we promote the researchers interested in the topology of soft sets and rough sets to carry out further investigations in the area of infra soft topologies.

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