Filomat 36:5 (2022), 1675–1684
https://doi.org/10.2298/FIL2205675R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Note on Class p-$wA(s,t)$ Operators

M.H.M.Rashid[a]

[a]*Department of Mathematics-Faculty of Science P.O.Box(7)- Mu'tah university- Al-Karak -Jordan*

Abstract. Let A and B be positive operators and $0 < q \le 1$. In this paper, we shall show that if

$$A^{q\alpha_0} \ge (A^{\alpha_0/2}B^{\beta_0}A^{\alpha_0/2})^{\frac{q\alpha_0}{\alpha_0+\beta_0}}$$

and

$$(B^{\beta_0/2}A^{\alpha_0}B^{\beta_0/2})^{\frac{q\beta_0}{\alpha_0+\beta_0}} \ge B^{q\beta_0}$$

hold for fixed $\alpha_0 > 0$ and $\beta_0 > 0$. Then the following inequalities hold:

$$A^{q_1\alpha} \ge (A^{\alpha/2}B^{\beta}A^{\alpha/2})^{\frac{q_1\alpha}{\alpha+\beta}}$$

and

$$(B^{\beta/2}A^{\alpha}B^{\beta/2})^{\frac{q_1\beta}{\alpha+\beta}} \ge B^{q_1\beta}$$

for all $\alpha \ge \alpha_0$, $\beta \ge \beta_0$ and $0 < q_1 \le q$. Also, we shall show a normality of class p-$A(s,t)$ for $s > 0, t > 0$ and $0 < p \le 1$. Moreover, we shall show that if T or T^* belongs to class p-$wA(s,t)$ for some $s > 0, t > 0$ and $0 < p \le 1$ and S is an operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.

1. Introduction

In what follows, an operator means a bounded linear operator on a complex Hilbert space $\mathcal{H}$ and $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator T is said to be positive (denoted $T \ge 0$) if $\langle Tx, x\rangle \ge 0$ for all $x \in \mathcal{H}$, and also T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. As a recent development on order preserving operator inequalities, it is known the following Theorem.

Theorem 1.1 (Furuta's inequality[10]). *If $A \ge B \ge 0$, then for each $r \ge 0$,*

(i) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \ge B^{\frac{r+p}{q}}$ *and*

(ii) $A^{\frac{r+p}{q}} \ge (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{1}{q}}$

2020 *Mathematics Subject Classification*. 47B20, 47A10
Keywords. class p-$wA(s,t)$,class p-$A(s,t)$, Furuta inequality, Normality, generalized Aluthge transformation
Received: 23 May 2021; Revised: 18 July 2021; Accepted: 10 October 2021
Communicated by Fuad Kittaneh
Email address: malik_okasha@yahoo.com (M.H.M.Rashid)

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.

Theorem 1.1 yields the famous Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0,1]$" by putting $r = 0$ in (i) or (ii) of Theorem 1.1.

As an application of Theorem 1.1, in [8] and [11], it was shown the following: For positive invertible operators A and B, $\log A \geq \log B$ (this order is called chaotic order) if and only if $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ for all $p \geq 0$ and $r \geq 0$ if and only if $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ for all $p \geq 0$ and $r \geq 0$. We remark that this result is an extension of [2] in case $p = r$. Related to these operator inequalities, the following assertions are well-known: Let A and B be strictly positive operators. Then

(a) $A \geq B \Rightarrow \log A \geq \log B$.

(b) $\log A \geq \log B \Rightarrow (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ for all $p \geq 0$ and $r \geq 0$.

(c) For each $p \geq 0$ and $r \geq 0$, $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \Leftrightarrow A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ [11].

Related to these results, it is shown in [23] that the invertibility in (a) and (b) can be replaced with the condition $\ker(A) = \ker(B) = \{0\}$, that is, (a) and (b) hold for some non-invertible operators A and B. In [15], the authors studied relations between

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \quad \text{and} \quad A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$$

when A and B are not invertible.

Every operator $T \in \mathcal{B}(\mathcal{H})$ can be decomposed into $T = U|T|$ with a partial isometry U where $|T|$ is the square root of T^*T. If U is determined uniquely by the kernel condition $\ker U = \ker |T|$, then this decomposition is called the polar decomposition of T. In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $\ker U = \ker |T|$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be hyponormal if $T^*T \geq TT^*$. The Aluthge transformation introduced by Aluthge[1] is defined by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ where $T = U|T|$ is the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. The generalized Aluthge transformation $\tilde{T}_{s,t}$ with $0 < s,t$ is defined by $\tilde{T}_{s,t} = |T|^sU|T|^t$. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$, and class $wA(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ and $|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}$ ([14]). Furuta el al. [9] introduced class $A(k)$ for $k > 0$ as a class of operators including p-hyponormal and log-hyponormal operators, where $A(1)$ coincides with class A operator. We say that an operator T is class $A(k)$, $k > 0$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$.

Definition 1.2. *Let* $s > 0, t > 0$, $0 < p \leq 1$ *and* $T = U|T|$ *be the polar decomposition of* T.

(i) T *belongs to class* p-$A(s,t) \Leftrightarrow (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{pt}{s+t}} \geq |T^*|^{2tp}$ *[16].*

(ii) T *belongs to class* p-$wA(s,t)$

$$\begin{aligned} &\Leftrightarrow \quad (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{pt}{s+t}} \geq |T^*|^{2tp} \quad \textit{and} \quad |T|^{2sp} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \\ &\Leftrightarrow \quad |\tilde{T}_{s,t}|^{\frac{2tp}{t+s}} \geq |T|^{2tp} \quad \textit{and} \quad |T|^{2sp} \geq |(\tilde{T}_{s,t})^*|^{\frac{2sp}{s+t}}, \end{aligned}$$

where $\tilde{T}_{s,t} = |T|^sU|T|^t$ *is the generalized Aluthge transformation [16].*

(iii) T *belongs to class* p-$A \Leftrightarrow |T^2|^p \geq |T|^{2p}$, *that is,* T *belongs to class* p-$A(1,1)$*[16].*

(iv) T *is* p-w-*hyponormal* $\Leftrightarrow |\tilde{T}|^{\frac{p}{2}} \geq |T|^p \geq |(\tilde{T})^*|^{\frac{p}{2}}$, *that is,* T *belongs to class* p-$wA(\frac{1}{2},\frac{1}{2})$, *where* $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is the Aluthge transformation[3].*

(v) T *is* (s,p)-w-*hyponormal* $\Leftrightarrow |\tilde{T}_{s,s}|^{\frac{p}{2}} \geq |T|^{2sp} \geq |(\tilde{T}_{s,s})^*|^{\frac{p}{2}}$, *that is,* T *belongs to class* p-$wA(s,s)$, *where* $\tilde{T}_{s,s} = |T|^sU|T|^s$ *is the generalized Aluthge transformation [12].*

It is well known that class p-$wA(s,t)$ operators enjoy many interesting properties as hyponormal operators, for example, Fuglede-Putnam type theorem, Weyl type theorem, subscalarity and Putnam's inequality ([5],[6],[17], [18],[22]). We remark that Aluthge transformation has many interesting properties, and many authors study this transformation, for instance, [1], [5], [7] and [25]. These classes are included in normaloid (i.e., $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T) (see [17],[3] and [12]). It has been shown that for $s > 0, t > 0$ and $0 < p \leq 1$, class p-$A(s,t)$ includes class p-$wA(s,t)$ by the definition 1.2 (i) and (ii). and also for each $s > 0, t > 0$ and $0 < p \leq 1$, class p-$A(s,t)$ and class p-$wA(s,t)$ are invertible which was shown in [16]. More precise inclusion relations among class p-$wA(s,t)$ were already shown as follows:

Theorem 1.3. *[5] If $T \in B(\mathcal{H})$ is class p-$wA(s,t)$ and $0 < s \leq \alpha, 0 < t \leq \beta, 0 < p_1 \leq p \leq 1$, then T is class p_1-$wA(\alpha,\beta)$.*

In this paper, we shall show a normality of class p-$A(s,t)$ for $s > 0, t > 0$ and $0 < p \leq 1$. Moreover, we shall show that if T or T^* belongs to class p-$wA(s,t)$ for some $s > 0, t > 0$ and $0 < p \leq 1$ and S is an operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.

2. Main results

In order to give the proof of our results. We need the following lemmas.

Lemma 2.1. *[13, Löwner-Heinz inequality] $A \geq B \geq 0$ ensure $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$.*

Lemma 2.2. *[25] Let $A > 0$ and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{1/2}(A^{1/2}B^*BA^{1/2})^{\lambda-1}A^{1/2}B^*$$

holds for any real number λ.

Proposition 2.3. *Let A and B be positive operators. Then the following assertions hold:*

(i) If $(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{\beta_0 p}{\alpha_0+\beta_0}} \geq B^{\beta_0 p}$ holds for fixed $\alpha_0 > 0$, $\beta_0 > 0$ and $0 < p \leq 1$, then

$$(B^{\frac{\beta}{2}}A^{\alpha_0}B^{\frac{\beta}{2}})^{\frac{\beta p_1}{\alpha_0+\beta}} \geq B^{\beta p_1} \tag{1}$$

holds for any $\beta \geq \beta_0$ and $0 < p_1 \leq p \leq 1$. Moreover, for each fixed $\gamma \geq -\alpha_0$,

$$f_{\alpha_0,\gamma}(\beta) = (A^{\frac{\alpha_0}{2}}B^{\beta}A^{\frac{\alpha_0}{2}})^{\frac{(\alpha_0+\gamma)p_1}{\alpha_0+\beta}}$$

is a decreasing function for $\beta \geq \max\{\beta_0, \gamma\}$. Hence the inequality

$$(A^{\frac{\alpha_0}{2}}B^{\beta_1}A^{\frac{\alpha_0}{2}})^{p_1} \geq (A^{\frac{\alpha_0}{2}}B^{\beta_2}A^{\frac{\alpha_0}{2}})^{\frac{p_1(\alpha_0+\beta_1)}{\alpha_0+\beta_2}} \tag{2}$$

holds for any β_1 and β_2 such that $\beta_2 \geq \beta_1 \geq \beta_0$ and $0 < p_1 \leq p$.

(ii) If $A^{\alpha_0 p} \geq (A^{\frac{\alpha_0}{2}}B^{\beta_0}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0 p}{\alpha_0+\beta_0}}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$ and $0 < p \leq 1$, then

$$A^{\alpha p_1} \geq (A^{\frac{\alpha}{2}}B^{\beta_0}A^{\frac{\alpha}{2}})^{\frac{\alpha p_1}{\alpha+\beta_0}} \tag{3}$$

holds for any $\alpha \geq \alpha_0$ and $0 < p_1 \leq p \leq 1$. Moreover, for each fixed $\delta \geq -\beta_0$,

$$g_{\beta_0,\delta}(\alpha) = (B^{\frac{\beta_0}{2}}A^{\alpha}B^{\frac{\beta_0}{2}})^{\frac{(\delta+\beta_0)p_1}{\alpha+\beta_0}}$$

is an increasing function for $\alpha \geq \max\{\alpha_0, \delta\}$. Hence the inequality

$$(B^{\frac{\beta_0}{2}}A^{\alpha_2}B^{\frac{\beta_0}{2}})^{\frac{p_1(\alpha_1+\beta_0)}{\alpha_2+\beta_0}} \geq (B^{\frac{\beta_0}{2}}A^{\alpha_1}B^{\frac{\beta_0}{2}})^{p_1} \tag{4}$$

holds for any α_1 and α_2 such that $\alpha_2 \geq \alpha_1 \geq \alpha_0$ and $0 < p_1 \leq p$.

Proposition 2.3 can be obtained as an application of Furuta inequality 1.1. We actually use the following form which is the essential part of Furuta inequality 1.1.

Lemma 2.4. *If $A \geq B \geq 0$, then*

(i) $(B^{x/2}A^yB^{x/2})^{\frac{1+x}{x+y}} \geq B^{1+x}$ and

(ii) $A^{1+x} \geq (A^{x/2}B^yA^{x/2})^{\frac{1+x}{x+y}}$

hold for $x \geq 0$ *and* $y \geq 1$.

Proof. [Proof of Proposition 2.3] (i) Put $A_1 = (B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{\beta_0 p}{\alpha_0+\beta_0}}$ and $B_1 = B^{\beta_0 p}$, then $A_1 \geq B_1 \geq 0$ holds by the hypothesis. By applying (i) of Lemma 2.4 to A_1 and B_1, we have

$$(B_1^{x_1/2}A_1^{y_1}B_1^{x_1/2})^{\frac{1+x_1}{x_1+y_1}} \geq B_1^{1+x_1} \text{ for any } y_1 \geq 1 \text{ and } x_1 \geq 0. \tag{5}$$

Let $\beta \geq \beta_0$, $y_1 = \frac{\alpha_0+\beta_0}{\beta_0 p}$ and $x_1 = \frac{\beta-\beta_0}{\beta_0 p} \geq 0$. Then

$$(B^{\beta/2}A^{\alpha_0}B^{\beta/2})^{\frac{\beta_0 p+\beta-\beta_0}{\beta_0+\beta}} \geq B^{\beta_0 p+\beta-\beta_0} \text{ for any } \beta \geq \beta_0. \tag{6}$$

Since $\frac{p_1\beta}{\beta_0 p+\beta-\beta_0} \in (0,1]$, applying Löwner-Heinz theorem to (6), we have

$$(B^{\beta/2}A^{\alpha_0}B^{\beta/2})^{\frac{p_1\beta}{\alpha_0+\beta}} \geq B^{p_1\beta} \text{ for any } \beta \geq \beta_0 \text{ and } 0 < p_1 \leq p. \tag{7}$$

By applying Löwner-Heinz theorem to (7), we have

$$(B^{\beta/2}A^{\alpha_0}B^{\beta/2})^{\frac{w}{\alpha_0+\beta}} \geq B^{w} \text{ for any } 0 < w \leq p_1\beta. \tag{8}$$

For each $\gamma \geq -\alpha_0$, $\beta \geq \max\{\beta_0, \gamma\}$ and w such that $p_1\beta \geq w \geq 0$, we have

$$\begin{aligned}
f_{\alpha_0,\gamma}(\beta) &= (A^{\alpha_0/2}B^{\beta}A^{\alpha_0/2})^{\frac{(\gamma+\alpha_0)p_1}{\alpha_0+\beta}} \\
&= \{(A^{\alpha_0/2}B^{\beta}A^{\alpha_0/2})^{\frac{\alpha_0+\beta+w}{\alpha_0+\beta}}\}^{\frac{(\gamma+\alpha_0)p_1}{\alpha_0+\beta+w}} \\
&= \{A^{\alpha_0/2}B^{\beta/2}(B^{\beta/2}A^{\alpha_0}B^{\beta/2})^{\frac{w}{\alpha_0+\beta}}B^{\beta/2}A^{\alpha_0/2}\}^{\frac{(\gamma+\alpha_0)p_1}{\alpha_0+\beta+w}} \\
&\geq \{A^{\alpha_0/2}B^{\beta/2}B^{w}B^{\beta/2}A^{\alpha_0/2}\}^{\frac{(\gamma+\alpha_0)p_1}{\alpha_0+\beta+w}} \\
&= (A^{\alpha_0/2}B^{\beta+w}A^{\alpha_0/2})^{\frac{(\gamma+\alpha_0)p_1}{\alpha_0+\beta+w}} \\
&= f_{\alpha_0,\gamma}(\beta+w).
\end{aligned}$$

The above inequality holds by (8) and Löwner-Heinz theorem for $\frac{(\gamma+\alpha_0)p_1}{\alpha_0+\beta+w} \in [0,1]$. Hence $f_{\alpha_0,\gamma}(\beta)$ is decreasing for $\beta \geq \max\{\beta_0,\gamma\}$. Moreover, in case $\gamma \geq \beta_0$,

$$(A^{\alpha_0/2}B^{\gamma}A^{\alpha_0/2})^{p_1} = f_{\alpha_0,\gamma}(\gamma) \geq f_{\alpha_0,\gamma}(\beta) = (A^{\alpha_0/2}B^{\beta}A^{\alpha_0/2})^{\frac{(\gamma+\alpha_0)p_1}{\alpha_0+\beta}}$$

holds for any $\beta \geq \gamma$, so that we have (2) by replacing γ and β with β_1 and β_2, respectively.

(ii) Put $A_2 = A^{\alpha_0 p}$ and $B_2 = (A^{\alpha_0/2}B^{\beta_0}A^{\alpha_0/2})^{\frac{\alpha_0 p}{\alpha_0+\beta_0}}$, then $A_2 \geq B_2$ holds by hypothesis. By applying (ii) of Lemma 2.4 to A_2 and B_2, we have

$$A_2^{1+x_2} \geq (A_2^{x_2/2}B_2^{y_2}A_2^{x_2/2})^{\frac{1+x_2}{y_2+x_2}} \text{ for any } y_2 \geq 1 \text{ and } x_2 \geq 0. \tag{9}$$

Put $y_2 = \frac{\alpha_0+\beta_0}{\alpha_0 p}$ and $x_2 = \frac{\alpha-\alpha_0}{\alpha_0 p} \geq 0$ in (9). Then we have

$$A^{\alpha_0 p+\alpha-\alpha_0} \geq (A^{\alpha/2}B^{\beta_0}A^{\alpha/2})^{\frac{\alpha_0 p+\alpha-\alpha_0}{\alpha+\beta_0}} \text{ for any } \alpha \geq \alpha_0. \tag{10}$$

Since $\frac{p_1\alpha}{\alpha_0 p+\alpha-\alpha_0} \in (0,1]$, applying Löwner-Heinz theorem to (10), we have

$$A^{p_1\alpha} \geq (A^{\alpha/2}B^{\beta_0}A^{\alpha/2})^{\frac{p_1\alpha}{\alpha+\beta_0}} \text{ for any } \alpha \geq \alpha_0 \text{ and } 0 < p_1 \leq p. \tag{11}$$

By applying Löwner-Heinz theorem to (11), we have

$$A^u \geq (A^{\alpha/2}B^{\beta_0}A^{\alpha/2})^{\frac{u}{\alpha+\beta_0}} \text{ for any } 0 < u \leq p_1\alpha. \tag{12}$$

For each $\delta \geq -\beta_0$, $\alpha \geq \max\{\alpha_0, \delta\}$ and u such that $p_1\alpha \geq u \geq 0$, we have

$$\begin{aligned} g_{\beta_0,\delta}(\alpha) &= (B^{\frac{\beta_0}{2}}A^\alpha B^{\frac{\beta_0}{2}})^{\frac{(\delta+\beta_0)p_1}{\alpha+\beta_0}} \\ &= \{(B^{\frac{\beta_0}{2}}A^\alpha B^{\frac{\beta_0}{2}})^{\frac{\alpha+u+\beta_0}{\alpha+\beta_0}}\}^{\frac{(\delta+\beta_0)p_1}{\alpha+u+\beta_0}} \\ &= \{B^{\beta_0/2}A^{\alpha/2}(A^{\alpha/2}B^{\beta_0}A^{\alpha/2})^{\frac{u}{\alpha+\beta_0}}A^{\alpha/2}B^{\beta_0/2}\}^{\frac{(\delta+\beta_0)p_1}{\alpha+u+\beta_0}} \\ &\leq \{B^{\beta_0/2}A^{\alpha/2}A^uA^{\alpha/2}B^{\beta_0/2}\}^{\frac{(\delta+\beta_0)p_1}{\alpha+u+\beta_0}} \\ &= (B^{\beta_0/2}A^{u+\alpha}B^{\beta_0/2})^{\frac{(\delta+\beta_0)p_1}{\alpha+u+\beta_0}} \\ &= g_{\beta_0,\delta}(\alpha+u). \end{aligned}$$

The above inequality holds by (12) and Löwner-Heinz theorem for $\frac{(\delta+\beta_0)p_1}{\alpha+u+\beta_0} \in [0,1]$. Hence $g_{\beta_0,\delta}(\alpha)$ is increasing for $\alpha \geq \max\{\alpha_0, \delta\}$. Moreover, in case $\delta \geq \alpha_0$,

$$(B^{\beta_0/2}A^\alpha B^{\beta_0/2})^{\frac{(\delta+\beta_0)p_1}{\alpha+\beta_0}} = g_{\beta_0,\delta}(\alpha) \geq g_{\beta_0,\delta}(\delta) = (B^{\beta_0/2}A^\delta B^{\beta_0/2})^{p_1} \tag{13}$$

holds for any $\alpha \geq \delta$, so that we have (4) by replacing δ and α with α_1 and α_2, respectively. □

Theorem 2.5. *Let $0 < q \leq 1$ and let A and B be positive operators such that*

$$A^{q\alpha_0} \geq (A^{\alpha_0/2}B^{\beta_0}A^{\alpha_0/2})^{\frac{q\alpha_0}{\alpha_0+\beta_0}} \tag{14}$$

and

$$(B^{\beta_0/2}A^{\alpha_0}B^{\beta_0/2})^{\frac{q\beta_0}{\alpha_0+\beta_0}} \geq B^{q\beta_0} \tag{15}$$

hold for fixed $\alpha_0 > 0$ and $\beta_0 > 0$. Then the following inequalities hold:

$$A^{q_1\alpha} \geq (A^{\alpha/2}B^\beta A^{\alpha/2})^{\frac{q_1\alpha}{\alpha+\beta}} \tag{16}$$

and

$$(B^{\beta/2}A^\alpha B^{\beta/2})^{\frac{q_1\beta}{\alpha+\beta}} \geq B^{q_1\beta} \tag{17}$$

for all $\alpha \geq \alpha_0$, $\beta \geq \beta_0$ and $0 < q_1 \leq q$.

Proof. [Proof of (16)] Applying Lemma 2.4 to (15), we have

$$\{B^{\frac{q\beta_0 r_1}{2}}(B^{\beta_0/2}A^{\alpha_0}B^{\beta_0/2})^{\frac{p_1 q\beta_0}{\alpha_0+\beta_0}}B^{\frac{q\beta_0 r_1}{2}}\}^{\frac{1+r_1}{p_1+r_1}} \geq B^{q\beta_0(1+r_1)} \tag{18}$$

for any $p_1 \geq 1$ and $r_1 \geq 0$. Putting $p_1 = \frac{\alpha_0+\beta_0}{q\beta_0}$ in (18), we have

$$(B^{\frac{\beta_0(1+qr_1)}{2}}A^{\alpha_0}B^{\frac{\beta_0(1+qr_1)}{2}})^{\frac{q\beta_0(1+r_1)}{\alpha_0+\beta_0+r_1 q\beta_0}} \geq B^{q\beta_0(1+r_1)} \tag{19}$$

for any $r_1 \geq 0$. Put $\beta = \beta_0(1+qr_1) \geq \beta_0$ in (19). Then we have

$$(B^{\frac{\beta}{2}}A^{\alpha_0}B^{\frac{\beta}{2}})^{\frac{\beta-(1-q)\beta_0}{\alpha_0+\beta}} \geq B^{\beta-(1-q)\beta_0}. \tag{20}$$

Hence we have

$$(B^{\frac{\beta}{2}}A^{\alpha_0}B^{\frac{\beta}{2}})^{\frac{w}{\alpha_0+\beta}} \geq B^w \text{ for } 0 < w \leq \beta - (1-q)\beta_0. \tag{21}$$

Next we show $f(\beta) = (A^{\alpha_0/2}B^{\beta}A^{\alpha_0/2})^{\frac{q\alpha_0}{\alpha_0+\beta}}$ is decreasing for $\beta \geq \beta_0$. By Löwner-Heinz theorem, (21) ensures the following (22)

$$(B^{\frac{\beta}{2}}A^{\alpha_0}B^{\frac{\beta}{2}})^{\frac{w}{\alpha_0+\beta}} \geq B^w \text{ for } 0 < w \leq \beta - (1-q)\beta_0. \tag{22}$$

Then we have

$$\begin{aligned}
f(\beta) &= (A^{\alpha_0/2}B^{\beta}A^{\alpha_0/2})^{\frac{q\alpha_0}{\alpha_0+\beta}} \\
&= \{(A^{\alpha_0/2}B^{\beta}A^{\alpha_0/2})^{\frac{\alpha_0+\beta+w}{\alpha_0+\beta}}\}^{\frac{q\alpha_0}{\alpha_0+\beta+w}} \\
&= \{A^{\alpha_0/2}B^{\beta/2}(B^{\beta/2}A^{\alpha_0}B^{\beta/2})^{\frac{w}{\alpha_0+\beta}}B^{\beta/2}A^{\alpha_0/2}\}^{\frac{q\alpha_0}{\alpha_0+\beta+w}} \text{ (by Lemma 2.2)} \\
&\geq (A^{\alpha_0/2}B^{\beta+w}A^{\alpha_0/2})^{\frac{q\alpha_0}{\alpha_0+\beta+w}} \\
&= f(\beta+w).
\end{aligned}$$

Hence $f(\beta)$ is decreasing for $\beta \geq \beta_0$. Therefore

$$A^{q\alpha_0} \geq (A^{\alpha_0/2}B^{\beta}A^{\alpha_0/2})^{\frac{q\alpha_0}{\alpha_0+\beta}} \text{ for } \beta \geq \beta_0 \tag{23}$$

holds since

$$A^{q\alpha_0} \geq (A^{\alpha_0/2}B^{\beta_0}A^{\alpha_0/2})^{\frac{q\alpha_0}{\alpha_0+\beta_0}} = f(\beta_0) \geq f(\beta) = (A^{\alpha_0/2}B^{\beta}A^{\alpha_0/2})^{\frac{q\alpha_0}{\alpha_0+\beta}}.$$

Again applying Lemma 1.1 to (23), we have

$$A^{q\alpha_0(1+r_2)} \geq (A^{\frac{qr_2\alpha_0}{2}}(A^{qr_2\alpha_0/2}B^{\beta}A^{\alpha_0/2})^{\frac{p_2q\alpha_0}{\alpha_0+\beta}}A^{\frac{qr_2\alpha_0}{2}})^{\frac{1+r_2}{p_2+r_2}} \tag{24}$$

for any $p_2 \geq 1$ and $r_2 \geq 0$. Putting $p_2 = \frac{\alpha_0+\beta}{q\alpha_0} \geq 1$ in (24), we have

$$A^{q\alpha_0(1+r_2)} \geq (A^{\frac{\alpha_0(1+qr_2)}{2}}B^{\beta}A^{\frac{\alpha_0(1+qr_2)}{2}})^{\frac{q\alpha_0(1+r_2)}{\alpha_0+\beta+qr_2\alpha_0}} \tag{25}$$

for any $r_2 \geq 0$. Put $\alpha = \alpha_0(1+qr_2) \geq \alpha_0$ in (25). Then we have

$$A^{\alpha+\alpha_0(q-1)} \geq (A^{\frac{\alpha}{2}}B^{\beta}A^{\frac{\alpha}{2}})^{\frac{\alpha+\alpha_0(q-1)}{\beta+\alpha}} \tag{26}$$

for all $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$. Now, since $\frac{q_1\alpha}{\alpha+\alpha_0(q-1)} \in (0,1]$, applying Löwner-Heinz theorem to (26), we have

$$A^{q_1\alpha} \geq (A^{\frac{\alpha}{2}}B^{\beta}A^{\frac{\alpha}{2}})^{\frac{q_1\alpha}{\beta+\alpha}}$$

for all $\alpha \geq \alpha_0$, $\beta \geq \beta_0$ and $0 < q_1 \leq q$.
Proof of (17). Applying Lemma 2.4 to (14), we have

$$A^{q\alpha_0(1+r_3)} \geq (A^{\frac{qr_3\alpha_0}{2}}(A^{\alpha_0/2}B^{\beta_0}A^{\alpha_0/2})^{\frac{p_3q\alpha_0}{\alpha_0+\beta_0}}A^{\frac{qr_3\alpha_0}{2}})^{\frac{1+r_3}{p_3+r_3}} \tag{27}$$

for any $p_3 \geq 1$ and $r_3 \geq 0$. Putting $p_3 = \frac{\alpha_0+\beta_0}{q\alpha_0} \geq 1$ in (27), we have

$$A^{q\alpha_0(1+r_3)} \geq (A^{\frac{\alpha_0(1+qr_3)}{2}}B^{\beta_0}A^{\frac{\alpha_0(1+qr_3)}{2}})^{\frac{q\alpha_0(1+r_3)}{\alpha_0+\beta_0+qr_3\alpha_0}} \tag{28}$$

for any $r_3 \geq 0$. Put $\alpha = \alpha_0(1+qr_3) \geq \alpha_0$ in (28). Then we have

$$A^{\alpha+\alpha_0(q-1)} \geq (A^{\frac{\alpha}{2}}B^{\beta_0}A^{\frac{\alpha}{2}})^{\frac{\alpha+\alpha_0(q-1)}{\beta_0+\alpha}} \text{ for } \alpha \geq \alpha_0. \tag{29}$$

Next we show that $g(\alpha) = (B^{\beta_0/2}A^{\alpha}B^{\beta_0/2})^{\frac{q\beta_0}{\alpha_0+\beta_0}}$ is increasing for $\alpha \geq \alpha_0$. By Löwner-Heinz theorem, (29) ensures the following (30).

$$A^u \geq (A^{\frac{\alpha}{2}}B^{\beta_0}A^{\frac{\alpha}{2}})^{\frac{u}{\beta_0+\alpha}} \text{ for } 0 \leq u \leq \alpha + \alpha_0(q-1). \tag{30}$$

Then we have

$$\begin{aligned} g(\alpha) &= (B^{\beta_0/2}A^{\alpha}B^{\beta_0/2})^{\frac{q\beta_0}{\alpha+\beta_0}} \\ &= \{(B^{\beta_0/2}A^{\alpha}B^{\beta_0/2})^{\frac{\alpha+\beta_0+u}{\alpha+\beta_0}}\}^{\frac{q\beta_0}{u+\beta_0+\alpha}} \\ &= \{B^{\beta_0/2}A^{\alpha/2}(A^{\alpha/2}B^{\beta_0}A^{\alpha/2})^{\frac{u}{\alpha+\beta_0}}A^{\alpha/2}B^{\beta_0/2}\}^{\frac{q\beta_0}{u+\beta_0+\alpha}} \\ &\leq (B^{\beta_0/2}A^{\alpha+u}B^{\beta_0/2})^{\frac{q\beta_0}{u+\beta_0+\alpha}} \\ &= g(\alpha+u). \end{aligned}$$

Hence $g(\alpha)$ is increasing for $\alpha \geq \alpha_0$. Therefore

$$(B^{\beta_0/2}A^{\alpha}B^{\beta_0/2})^{\frac{q\beta_0}{\alpha+\beta_0}} \geq B^{q\beta_0} \text{ for } \alpha \geq \alpha_0 \tag{31}$$

holds since

$$(B^{\beta_0/2}A^{\alpha}B^{\beta_0/2})^{\frac{q\beta_0}{\alpha+\beta_0}} = g(\alpha) \geq g(\alpha_0) = (B^{\beta_0/2}A^{\alpha_0}B^{\beta_0/2})^{\frac{q\beta_0}{\alpha_0+\beta_0}} \geq B^{q\beta_0}.$$

Again applying Lemma 1.1 to (31), we have

$$\{B^{\frac{qr_4\beta_0}{2}}(B^{\beta_0/2}A^{\alpha}B^{\beta_0/2})^{\frac{p_4q\beta_0}{\alpha+\beta_0}}B^{\frac{qr_4\beta_0}{2}}\}^{\frac{1+r_4}{p_4+r_4}} \geq B^{q\beta_0(1+r_4)} \tag{32}$$

for any $p_4 \geq 1$ and $r_4 \geq 0$. Putting $p_4 = \frac{\alpha+\beta_0}{q\beta_0} \geq 1$ in (32), we have

$$(B^{\frac{\beta_0(1+qr_4)}{2}}A^{\alpha}B^{\frac{\beta_0(1+qr_4)}{2}})^{\frac{q\beta_0(1+r_4)}{\alpha+\beta_0+q\beta_0r_4}} \geq B^{q\beta_0(1+r_4)} \tag{33}$$

for any $r_4 \geq 0$. Put $\beta = \beta_0(1+qr_4) \geq \beta_0$ in (33). Then we have

$$(B^{\frac{\beta}{2}}A^{\alpha}B^{\frac{\beta}{2}})^{\frac{\beta+\beta_0(q-1)}{\alpha+\beta}} \geq B^{\beta+\beta_0(q-1)} \text{ for } \alpha \geq \alpha_0 \text{ and } \beta \geq \beta_0. \tag{34}$$

Now, since $\frac{q_1\beta}{\beta+\beta_0(q-1)} \in (0,1]$, applying Löwner-Heinz theorem to (34), we have

$$(B^{\frac{\beta}{2}}A^{\alpha}B^{\frac{\beta}{2}})^{\frac{q_1\beta}{\alpha+\beta}} \geq B^{q_1\beta}$$

for all $\alpha \geq \alpha_0$, $\beta \geq \beta_0$ and $0 < q_1 \leq q$, so the proof is complete. □

By using Theorem 2.5, We shall give simplified proof of Theorem 1.3.

Corollary 2.6. *If $T \in B(\mathcal{H})$ is class p-$wA(s,t)$ and $0 < s \leq \alpha, 0 < t \leq \beta, 0 < p_1 \leq p \leq 1$, then T is class p_1-$wA(\alpha,\beta)$.*

Proof. Suppose that T is class p-$wA(s,t)$ for $s > 0$, $t > 0$ and $0 < p \leq 1$, i.e., the following (35) and (36) hold.

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}. \tag{35}$$

$$|T|^{2sp} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}}. \tag{36}$$

By Theorem 2.5, we have

$$(|T^*|^{\beta}|T|^{2\alpha}|T^*|^{\beta})^{\frac{p_1\beta}{\alpha+\beta}} \geq |T^*|^{2p_1\beta} \text{ and } |T|^{2p_1\alpha} \geq (|T|^{\alpha}|T^*|^{2\beta}|T|^{\alpha})^{\frac{p_1\alpha}{\alpha+\beta}}$$

for any $\alpha \geq s$, $\beta \geq t$ and $0 < p_1 \leq p$. Therefore T is class p_1-$wA(\alpha,\beta)$ for any $\alpha \geq s$, $\beta \geq t$ and $0 < p_1 \leq p$. □

In this section, we shall show a normality of some non-normal operators. It is known that if T and T^* are class A, then T is normal. But in the case T and T^* belong to weaker class than class A, the assertion is not obvious. Many authors obtained many results on this problem, and the following result were known until now.

Theorem 2.7 ([19]). *Let $T \in \mathcal{B}(\mathcal{H})$. If T and T^* are (s,p)-w-hyponormal, then T is normal.*

Theorem 2.8. *Let $s_i, t_i > 0$ and $0 < p_i \leq 1$, where $i = 1, 2$. If T is a class p_1-$wA(s_1, t_1)$ operator and T^* is is a class p_2-$wA(s_2, t_2)$ operator, then T is normal.*

Theorem 2.9. *Let $p, r > 0$, $0 < q \leq 1$, $s \geq p$ and $t \geq r$. If T is a class q-$wA(p,r)$ operator and $\tilde{T}_{s,t}$ is normal, then T is normal.*

To prove Theorem 2.8 and Theorem 2.9, we need the following results.

Lemma 2.10 ([14]). *Let $A > 0$ and $T = U|T|$ be the polar decomposition of T. Then for each $\alpha > 0$ and $\beta > 0$, the following assertions hold:*

(i) $U^*U(|T|^\beta A|T|^\beta)^\alpha = (|T|^\beta A|T|^\beta)^\alpha$.
(ii) $UU^*(|T^*|^\beta A|T^*|^\beta)^\alpha = (|T^*|^\beta A|T^*|^\beta)^\alpha$.
(iii) $(U|T|^\beta A|T|^\beta U^*)^\alpha = U(|T|^\beta A|T|^\beta)^\alpha U^*$.
(iv) $(U^*|T^*|^\beta A|T^*|^\beta U)^\alpha = U^*(|T^*|^\beta A|T^*|^\beta)^\alpha U$.

Lemma 2.11 ([15]). *Let $A \geq 0$ and $B \geq 0$. If*

$$B^{\frac{1}{2}}AB^{\frac{1}{2}} \geq B^2 \text{ and } A^{\frac{1}{2}}BA^{\frac{1}{2}} \geq A^2,$$

then $A = B$.

Lemma 2.12 ([4]). *Let $A, B \geq 0$ and $s, t \geq 0$. If $B^sA^{2t}B^s = B^{2t+2s}$ and $A^tB^{2s}A^t = A^{2t+2s}$, then $A = B$.*

Lemma 2.13. *([26, Proposition 4.5]) Let $A, B \geq 0$; $p_i, r_i > 0$; $-r_i < \delta_i \leq p_i$, $0 \leq \bar{\delta}_i < p_i$; $i = 1, 2$. Then the following assertions are mutually equivalent.*

(i) $A = B$.
(ii) $B^{\frac{r_1}{2}}A^{p_1}B^{\frac{r_1}{2}} = B^{r_1+p_1}$ and $A^{\frac{r_2}{2}}B^{p_2}A^{\frac{r_2}{2}} = A^{r_2+p_2}$.
(iii)
$$\begin{cases} \left(B^{\frac{r_1}{2}}A^{p_1}B^{\frac{r_1}{2}}\right)^{\frac{r_1+\delta_1}{r_1+p_1}} \geq B^{r_1+p_1}, \quad A^{p_1-\bar{\delta}_1} \geq \left(A^{\frac{p_1}{2}}B^{r_1}A^{\frac{p_1}{2}}\right)^{\frac{p_1-\bar{\delta}_1}{p_1+r_1}} \\ \left(B^{\frac{r_2}{2}}A^{p_2}B^{\frac{r_2}{2}}\right)^{\frac{r_2+\delta_2}{r_2+p_2}} \geq B^{r_2+p_2}, \quad A^{p_2-\bar{\delta}_2} \geq \left(A^{\frac{p_2}{2}}B^{r_2}A^{\frac{p_2}{2}}\right)^{\frac{p_2-\bar{\delta}_1}{p_2+r_2}} \end{cases}$$

Proof. [Proof of Theorem 2.8] Let $s = \max\{s_1, t_1, s_2, t_2\}$ and $p = \min\{p_1, p_2\}$.
Firstly, if T belongs to class p_1-$wA(s_1, t_1)$, then T belongs to class p-$wA(s,s)$ by Theorem 1.3. Hence we have

$$(|T^*|^s|T|^{2s}|T^*|^s)^{\frac{p}{2}} \geq |T^*|^{2sp} \quad \text{and} \quad |T|^{2sp} \geq (|T|^s|T^*|^{2s}|T|^s)^{\frac{p}{2}}. \tag{37}$$

Secondly, if T^* belongs to class p_2-$wA(s_2, t_2)$, then T^* belongs to class p-$wA(s,s)$ by Theorem 1.3. Hence we have

$$(|T|^s|T^*|^{2s}|T|^s)^{\frac{p}{2}} \geq |T|^{2sp} \quad \text{and} \quad |T^*|^{2sp} \geq (|T^*|^s|T|^{2s}|T^*|^s)^{\frac{p}{2}}. \tag{38}$$

Therefore

$$|T|^s|T^*|^{2s}|T|^s = |T|^{4s} \quad \text{and} \quad |T^*|^s|T|^{2s}|T^*|^s = |T^*|^{4s}$$

hold by (37) and (38), and then $|T| = |T^*|$ by Lemma 2.12. □

Proof. [Proof of Theorem 2.9] By hypothesis T belongs to class q-$wA(s,t)$ by Theorem 1.3. Hence it follows by (ii) of Definition 1.2 that

$$|\tilde{T}_{s,t}|^{\frac{2tq}{t+s}} \geq |T|^{2tq} \quad \text{and} \quad |T|^{2sq} \geq |(\tilde{T}_{s,t})^*|^{\frac{2sq}{s+t}}.$$

Hence

$$|\tilde{T}_{s,t}|^{\frac{2rq}{s+t}} \geq |T|^{2rq} \geq |(\tilde{T}_{s,t})^*|^{\frac{2rq}{s+t}} \quad \text{for all } r \in (0, \min\{s,t\}].$$

On the other hand, $\tilde{T}_{s,t}$ is normal, i.e., $|\tilde{T}_{s,t}|^2 = |(\tilde{T}_{s,t})^*|^2$. It follows by Lemma 2.10 that

$$|T^*|^t|T|^{2s}|T|^t = |T^*|^{2(s+t)} \quad \text{and} \quad |T|^s|T^*|^{2t}|T|^s = |T|^{2(s+t)},$$

and then $|T| = |T^*|$ by Lemma 2.12. □

The numerical range of an operator T, denoted by $W(T)$, is the set defined by

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}.$$

In general, the condition $S^{-1}TS = T^*$ and $0 \notin \overline{W(T)}$ do not imply that T is normal. If $T = SB$, where S is positive and invertible, B is self-adjoint, and S and B do not commute, then $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$, but T is not normal. Therefore the following question arises naturally.

Question: Which operator T satisfying the condition $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$ is normal?

In 1966, Sheth [21] showed that if T is a hyponormal operator and $S^{-1}TS = T^*$ for some operator S, where $0 \notin \overline{W(S)}$, then T is self-adjoint. Recently, Rashid [20] extended the result of Sheth to the class $A(k), k > 0$ operators. In this paper, we extend the result of Sheth to the class p-$wA(s,t)$ as follows.

Theorem 2.14. *Let $T \in \mathcal{B}(\mathcal{H})$. If T or T^* belongs to class p-$wA(s,t)$ for some $s > 0, t > 0$ and $0 < p \leq 1$ and S is an operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.*

To prove Theorem 2.14 we need the following Lemmas.

Lemma 2.15 ([24]). *If $T \in \mathcal{B}(\mathcal{H})$ is any operator such that $S^{-1}TS = T^*$, where $0 \notin \overline{W(S)}$, then $\sigma(T) \subseteq \mathbb{R}$.*

Lemma 2.16 ([18]). *Let $T \in \mathcal{B}(\mathcal{H})$ and let T belongs to the class p-$wA(s,t)$ for some $s > 0$, $t > 0$ and $0 < p \leq 1$. If $m_2(\sigma(T)) = 0$, where m_2 means the planer Lebsegue measure, then T is normal .*

Proof. [Proof of Theorem 2.14] Suppose that T or T^* is a class p-$wA(s,t)$ for $s,t > 0$ and $0 < p \leq 1$. Since $\sigma(S) \subseteq \overline{W(S)}$, S is invertible and hence $ST = T^*S$ becomes $S^{-1}T^*S = T = (T^*)^*$. Apply Lemma 2.15 to T^* to get $\sigma(T^*) \subseteq \mathbb{R}$. Then $\sigma(T) = \overline{\sigma(T^*)} = \sigma(T^*) \subseteq \mathbb{R}$. Thus $m_2(\sigma(T)) = m_2(\sigma(T^*))) = 0$ for the planer Lebesgue measure m_2. It follows from Lemma 2.16 that T or T^* is normal. Since $\sigma(T) = \sigma(T^*) \subseteq \mathbb{R}$. Therefore, T is self-adjoint. □

Acknowledgement. The author would like to sincerely thank the referee for several useful suggestions and comments improving the paper.

References

[1] A. Aluthge, On p-hyponormal operators for $0 < p < 1$, Integral Equations Operator Theory 13(1990) 307–315.
[2] T. Ando, On some operator inequalities, Math. Ann. 279 (1987) 157–159.
[3] Y. Changsen and L. Haiying, On p-w-hyponormal operators, Chin Q J Math. 20(2005) 79–84.
[4] C. Yang and Y. Zhao, on class $wF(p,r,q)$ operators and quasisimilarity, J. Ineq. Pure Applied Math. Volume 8 (2007), Issue 3, Article 90, 7 pp.
[5] M. Chō, M.H.M. Rashid, K. Tanahashi and A. Uchiyama, Spectrum of class p-$wA(s,t)$ operators, Acta Sci. Math. (Szeged) 82 (2016) 641–649.
[6] M. Chō, T. Prasad, M.H.M. Rashid, K. Tanahashi and A. Uchiyama, Fuglede-Putnam theorem and quasisimilarity of class p-$wA(s,t)$ operators, Operator and Matrices 13 (1) (2019) 293–299.

[7] M. Chō and K. Tanahashi, Isolated point of spectrum of p-hyponormal, log-hyponormal operators, Integral Equations Operator Theory 43 (2002) 379–384.

[8] M.Fujii, T.Furuta and E.Kamei, Furuta's inequality and its application to Ando's theorem, Linear Algebra Appl. 179(1993) 161–169.

[9] T. Furuta, M. Ito and T. Yamazaki, A subclass of paranormal operators including class of *log*-hyponormal and several related classes, Sci. math. 1(1998) 389–403.

[10] T. Furuta, $A \geq B \geq O$ *assures* $(B^r A^p B^r)^{\frac{1}{q}} \geq B^{\frac{p+2r}{q}}$ *for* $r \geq 0,\ p \geq 0,\ q \geq 1$ *with* $(1+2r)q \geq (p+2r)$, Proc. Amer. Math. Soc. 101 (1987) 85–88.

[11] T. Furuta, Applications of order preserving operator inequalities, Operator Theory Adv. Appl. 59 (1992) 180–190.

[12] L. Haiying, *Powers of an invertible* (s,p)*-w-hyponormal operator*, Acta Math Scientia 20(2008)282–288.

[13] E. Heinz, Beiträge zur Störungstheorie der Spektralzerlegung, Math. Ann. 123(1951) 415–438.

[14] M. Ito, Some classes of operators with generalised Aluthege transformations, SUT J. Math. 35(1999) 149-165.

[15] M. Ito and T. Yamazaki, Relations between two inequalities $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{r}{p+r}}$ and their applications, Integral Equations Operator Theory 44 (2002) 442–450.

[16] T. Prasad and K. Tanahashi, On class p-$wA(s,t)$ operators, Functional Analysis, Approximation and Computation 6 (2) (2014) 39–42.

[17] T. Prasad, M.Chō, M.H.M Rashid, K. Tanahashi and A Uchiyama, On class p-$wA(s,t)$ operators and range kernal orthogonality, Sci. Math. Japon (in Editione Electronica , e-2017, Whole Number 30)

[18] M.H.M Rashid, M. Chō, T. Prasad, K. Tanahashi and A. Uchiyama, Weyls theorem and Putnam's inequality for p-$wA(s,t)$ operators, Acta. Sci. Math. (Szeged) 84 (2018) 573–589.

[19] M.H.M Rashid, Quasinormality and Fuglede-Putnam theorem for (s,p)-w-hyponormal operators, Linear and Multilinear Algebra 65 (8) (2017) 1600–1616.

[20] M.H.M.Rashid, Generalized Fuglede-Putnam Theorem And m-Quasi-Class $A(k)$ Operators, Facta Universitatis (Niš) Ser. Math. Inform. 34 (1) (2019) 73–84.

[21] I. H. Sheth, On hyponormal operators, Proc. Amer. Math. Soc. 17 (1966) 998-–1000.

[22] K. Tanahashi, T. Prasad and A Uchiyama, Quasinormality and subscalarity of class p-$wA(s,t)$ operators, Funct. Anal. Approx. Comput. 9 (1)(2017) 61–68.

[23] M. Uchiyama, Inequalities for semibounded operators and their applications to log-hyponormal operators, Operator Theory: Advances and Applications 127(2001) 599–611.

[24] J. P. Williams, Operators similar to their adjoints, Proc. Amer. Math. Soc. 20(1969) 121-–123.

[25] M. Yanagida, Powers of class $wA(s,t)$ operators with generalised Aluthge transformation, J. Inequal. Appl. 7 (2002) 143–168.

[26] C. Yang and J. Yuan, On Class $wF(p,q,r)$ operators, Acta Math. Sinica 27 A (5) (2007)769–780.