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Necessary and Sufficient Conditions for the Approximate Controllability of Fractional Linear Systems via $C-$Semigroups

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Abstract. The approximate controllability of fractional linear evolution systems is considered in this paper. Firstly, the definitions of the mild solution and the approximate controllability of fractional linear evolution systems are obtained by using the theory of $C-$semigroups. Secondly, a new set of necessary and sufficient conditions are established to examine that linear system is approximately controllable with the help of symmetric operator. Moreover, the restricted condition of the state space is weakened by means of the dual mapping. Finally, as applications, the approximate controllability of nonlinear evolution systems are derived under the assumption that the corresponding linear system is approximately controllable. Our work essentially improves and generalized the corresponding results which are based on strongly continuous semigroups.

1. Introduction

The concept of controllability in finite dimensional spaces has been proposed by Kalman in 1960, and aroused close attention among scholars. Some of them generalized the definition to infinite dimensional spaces. Controllability is the core problem in mathematical control theory. The significance of the controllability depends on the fact that it can steer a control system from an arbitrary initial state to arbitrary final state using the set of admissible controls. Recently, Hernández and O'Regan [9] indicated that in the cases the semigroup or the control function is compact, the controllability results would apply only to finite dimensional space. Hence, a relatively weaker concept of controllability, namely approximate controllability, has received a great deal of attention, and it is completely adequate in applications [1, 2, 5, 6, 10, 13, 16–18, 23, 26].

The necessary and sufficient conditions for linear evolution systems have been derived mostly based on strongly continuous semigroups in the sense of integer order. Zhou [28] investigated the inequality conditions of approximate controllability for linear system. Naito [24] considered the approximate controllability of the linear system under the range condition of the control operator. Mahmudov [20] derived the necessary and sufficient conditions in a resolvent form for the approximate controllability of the linear system. As a generation of strongly continuous semigroups, exponentially bounded C-semigroups which was introduced by Davies and Pang [7] and extended by Tanaka and Miyadera [27], have received a great

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deal of attention, see [3, 11, 15, 21, 22, 25, 30, 31]. In fact, there are many differential operators that generate $C-$semigroups rather than strongly continuous semigroups. A typical example is that the Schrödinger oprator $i\Delta$ in the space L^p ($p \neq 2$), see [8].

Fractional differential equations have been served as efficient mathematical models in applied areas, such as in biology, signal processing, control theory and so on. For more details, we refer to the book [12], and the reference therein. However, to the best of our knowledge, the study of the necessary and sufficient conditions for fractional linear systems based on $C-$semigroups is an untreated topic in the literature. In order to fill this gap, in this paper, we are devoted to investigating the approximate controllability of the following fractional linear evolution system:

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + Bu(t), \ t \in J = [0, b], \\ x(0) = x_0 \in X, \end{cases} \tag{1}$$

where $\frac{1}{2} < \alpha \leq 1$; the state x takes values in a reflexive Banach space X; $A : D(A) \subseteq X \to X$ is the generator of an exponentially bounded $C-$semigroup $\{S(t)\}_{t\geq 0}$; $C \in \mathfrak{B}(X)$ is injective. the control function u is given in $L^2(J, Y)$, Y is a Hilbert space; B is a bounded linear operator from Y to X. We establish four necessary and sufficient conditions of approximate controllability in the resolvent form for system (1). The proof is based on the characterization of the symmetric operator as well as the duality mapping. Moreover, the approximate controllability of the corresponding nonlinear system is obtained under the assumption that linear system (1) is approximately controllable. The results in our paper are new, and some of our results are also new ever in the case of strongly continuous semigroups, which improve the related results on this topic.

The rest of the paper is organized as follows. In section 2, we recall some definitions of Caputo fractional derivatives and $C-$semigroups. We also obtain the definitions of mild solutions and approximate controllability of system (1). The corresponding nonnegative and symmetric operator and duality mapping are also introduced. We establish the new set of necessary and sufficient conditions for the approximate controllability in a resolvent form of fractional linear system (1) in section 3. Section 4 solves the approximate controllability of a class of fractional nonlinear differential systems provided that the corresponding linear system is approximately controllable.

2. Preliminaries

Throughout this paper, we assume that the state space X is a reflexive Banach space, and the control space Y is a Hilbert space. Let $b > 0$ be fixed. $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of positive integer, real number, and nonnegative real number, respectively. We denote by $C(J, X)$ the space of X-valued continuous functions on J with the norm $\|x\| = \sup\{\|x(t)\|, t \in J\}$, and denote by $L^p(J, X)$ the space of X-valued Bochner integrable functions on J with the norm $\|f\|_{L^p} = (\int_0^b \|f(t)\|^p \mathrm{d}t)^{1/p}$, where $1 \leq p < \infty$. We also denote by $\mathfrak{B}(X)$ the space of all bounded linear operators from X to X endowed with the operator norm $\|\cdot\|$. Let C be an injective operator in $\mathfrak{B}(X)$. Let T be a bounded linear operator in $\mathfrak{B}(X)$. We denote by $R(T)$ be the range of the operator T. In this paper, we always suppose that A is a closed and densely defined linear operator on X.

First let us recall the following basic definitions and results about fractional derivative and resolvent.

Definition 2.1. *[12] The Riemann-Liouville fractional integral of order $\alpha > 0$ with the lower limit zero for a function $f(\cdot) \in L^1([0, \infty), \mathbb{R})$ is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \ t > 0,$$

provided the right side is point-wisely defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *[12] The Riemann-Liouville fractional derivative of order $\alpha > 0$ with the lower limit zero for a function $f(\cdot) \in L^1([0, \infty), \mathbb{R})$ is defined as*

$${}^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\mathrm{d}s, \ t > 0, \quad n-1 < \alpha < n.$$

Definition 2.3. *[12] The Caputo fractional derivative of order $\alpha > 0$ with the lower limit zero for a function $f(\cdot) \in L^1([0,\infty),\mathbb{R})$ is defined as*

$$^{C}D^{\alpha} f(t) = {}^{L}D^{\alpha}\Big(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^k(0)\Big), \ \ t > 0, \quad n-1 < \alpha < n.$$

If $f(\cdot) \in C^n[0,\infty)$, then

$$^{C}D^{\alpha} f(t) = I^{n-q} f^{(n)}(s) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) \mathrm{d}s, \ \ t > 0, \quad n-1 < \alpha < n.$$

If f is an abstract function with values in X, the integrals which appear in the above three definitions are taken in Bochner's sence.

Definition 2.4. *[27] Let $C \in \mathfrak{B}(X)$ be injective. A strongly continuous family $\{S(t)\}_{t\geq 0} \subseteq \mathfrak{B}(X)$ is said to be a $C-$semigroup, if it fufills*

1. *$S(t+s)C = S(t)S(s), t, s \geq 0$;*
2. *$S(0) = C$;*
3. *$\|S(t)\| \leq Me^{\omega t}$ for some constants M, ω and all $t \geq 0$.*

The generator $A : D(A) \subseteq X \to X$ is the operator
$Ax = C^{-1} \lim\limits_{t\to 0^+} \frac{S(t)x - Cx}{t}, x \in D(A)$,
$D(A) = \{x \in X | \lim\limits_{t\to 0^+} \frac{S(t)x-Cx}{t}$ exists, and contains in $R(C)\}$.

Lemma 2.5. *[14] Let $M \geq 0$, $\omega \in \mathbb{R}$ and A be a linear operator on X. Then the following conditions are equivalent:*

1. *$\{S(t)\}_{t\geq 0}$ is a $C-$semigroup generated by A, and $\|S(t)\| \leq Me^{\omega t}$, $t \geq 0$.*
2. *$(\omega, \infty) \subset \rho_C(A)$, $C^{-1}AC = A$. There exists a family of strongly continuous operators $\{S(t)\}_{t\geq 0} \subseteq \mathfrak{B}(X)$ such that $\|S(t)\| \leq Me^{\omega t}$, $t \geq 0$, and the $C-$resolvent of A satisfies*

$$R_C(\lambda; A)x = (\lambda I - A)^{-1}C = \int_0^{\infty} e^{-\lambda t} S(t)x \mathrm{d}t, \ \lambda > \omega, \ x \in X, \tag{2}$$

where $\rho_C(A) := \{r : R(C) \subseteq R(r - A)$ and $r - A$ is injective$\}$ is the $C-$resolvent of A. $D(C^{-1}AC) = \{x \in X : Cx \in D(A),\ ACx \in R(C)\}$. Furthermore, if condition 2 holds, then A is the infinitesimal generator of $C-$semigroup $\{S(t)\}_{t\geq 0}$.

Now, we give the mild solution of system (1) by using the Laplace transformation, some proper density function as well as the definition and properties of $C-$semigroup.

Applying Laplace transform to (1), we have

$$\widehat{^{C}D^{\alpha}x}(\lambda) = A\hat{x}(\lambda) + B\hat{u}(\lambda),$$

$$\lambda^{\alpha}\hat{x}(\lambda) - \lambda^{\alpha-1}x(0) = A\hat{x}(\lambda) + B\hat{u}(\lambda),$$

$$(\lambda^{\alpha}I - A)\hat{x}(\lambda) = \lambda^{\alpha-1}x_0 + B\hat{u}(\lambda).$$

That is

$$\hat{x}(\lambda) = \lambda^{\alpha-1}(\lambda^{\alpha}I - A)^{-1}x_0 + (\lambda^{\alpha}I - A)^{-1}B\hat{u}(\lambda).$$

It follows from the fact $S(t)C = CS(t)$ that

$$C\hat{x}(\lambda) = \lambda^{\alpha-1}(\lambda^{\alpha}I - A)^{-1}Cx_0 + (\lambda^{\alpha}I - A)^{-1}CB\hat{u}(\lambda). \tag{3}$$

(2) together with (3) lead to the conclusion that

$$C\hat{x}(\lambda) = \lambda^{\alpha-1}\int_0^{\infty} e^{-\lambda^{\alpha}t}S(t)x_0\mathrm{d}t + \int_0^{\infty} e^{-\lambda^{\alpha}t}S(t)B\hat{u}(\lambda)\mathrm{d}t. \tag{4}$$

If $\alpha = 1$, (4) indicates that

$$C\hat{x}(\lambda) = \hat{S}(\lambda)x_0 + \hat{S}(\lambda)B\hat{u}(\lambda).$$

Take advantage of the inversion of Laplace transform, one gets

$$Cx(t) = S(t)x_0 + \int_0^t S(t-s)Bu(s)\mathrm{d}s. \tag{5}$$

If $0 < \alpha < 1$, similar to construction in [29], consider the one-sided stable probability density [19]

$$\overline{\omega}_{\alpha}(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(\pi n\alpha),\ \theta \in (0,\infty),$$

whose Laplace transform is given by

$$\int_0^{\infty} e^{-\lambda\theta}\overline{\omega}_{\alpha}(\theta)\mathrm{d}\theta = e^{-\lambda^{\alpha}},\ \ \alpha \in (0,1). \tag{6}$$

Using (4) and (6), one has

$$\begin{aligned} C\hat{x}(\lambda) &= \int_0^{\infty} e^{-\lambda t}\Big(\int_0^{\infty}\overline{\omega}_{\alpha}(\theta)S\big(\frac{t^{\alpha}}{\theta^{\alpha}}\big)x_0\mathrm{d}\theta\Big)\mathrm{d}t \\ &+ \alpha\int_0^{\infty} e^{-\lambda t}\Big(\int_0^t\int_0^{\infty}\overline{\omega}_{\alpha}(\theta)S\big(\frac{(t-s)^{\alpha}}{\theta^{\alpha}}\big)\frac{(t-s)^{\alpha-1}}{\theta^{\alpha}}Bu(s)\mathrm{d}\theta\mathrm{d}s\Big)\mathrm{d}t, \end{aligned}$$

Employing the inversion of Laplace transform, one gets

$$\begin{aligned} Cx(t) &= \int_0^{\infty}\overline{\omega}_{\alpha}(\theta)S\big(\frac{t^{\alpha}}{\theta^{\alpha}}\big)x_0\mathrm{d}\theta + \alpha\int_0^t\int_0^{\infty}\overline{\omega}_{\alpha}(\theta)S\big(\frac{(t-s)^{\alpha}}{\theta^{\alpha}}\big)\frac{(t-s)^{\alpha-1}}{\theta^{\alpha}}Bu(s)\mathrm{d}\theta\mathrm{d}s \\ &= \int_0^{\infty} h_{\alpha}(\theta)S(t^{\alpha}\theta)x_0\mathrm{d}\theta \\ &\quad +\alpha\int_0^t\int_0^{\infty}\theta(t-s)^{\alpha-1}h_{\alpha}(\theta)S((t-s)^{\alpha-1}\theta)Bu(s)\mathrm{d}\theta\mathrm{d}s, \end{aligned}$$

where $h_{\alpha}(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\overline{\omega}_{\alpha}(\theta^{-\frac{1}{\alpha}}) \geq 0, \theta \in (0,\infty)$ is a probability density function satisfying

$$\int_0^{\infty} h_{\alpha}(\theta)\mathrm{d}\theta = 1, \qquad \int_0^{\infty}\theta^{v}h_{\alpha}(\theta)\mathrm{d}\theta = \frac{\Gamma(1+v)}{\Gamma(1+\alpha v)},\ v \in [0,1].$$

Due to the argument above, we give the following definition of the mild solution of (1).

Definition 2.6. *A function $x \in C(J,X)$ is called a mild solution of* (1) *if it satisfies the following integral equation:*

$$Cx(t) = \mathcal{T}_{\alpha}(t)x_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_{\alpha}(t-s)Bu(s)\mathrm{d}s,$$

for $t \in J$, where

$$\mathcal{T}_{\alpha}(t) = \begin{cases} \int_0^{\infty} h_{\alpha}(\theta)S(t^{\alpha}\theta)\mathrm{d}\theta, & 0 < \alpha < 1, \\ S(t), & \alpha = 1, \end{cases}$$

$$\mathcal{S}_{\alpha}(t) = \begin{cases} \alpha\int_0^{\infty}\theta h_{\alpha}(\theta)S(t^{\alpha}\theta)\mathrm{d}\theta, & 0 < \alpha < 1, \\ S(t), & \alpha = 1. \end{cases}$$

Remark 2.7. *For $t \in J$, set*

$$v(t) = \mathcal{T}_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)Bu(s)\mathrm{d}s. \tag{7}$$

Then, Definition 2.6 implies that

$$Cx(t) = v(t).$$

Lemma 2.8. *[29] For any fixed $t \in J$, $\mathcal{T}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are linear and bounded operators on X. Moreover, for any $x \in X$,*

$$\|\mathcal{T}_\alpha(t)x\| \le M\|x\|, \qquad \|\mathcal{S}_\alpha(t)x\| \le \frac{M\alpha}{\Gamma(1+\alpha)}\|x\|, \tag{8}$$

where M is the constant such that $\sup_{t\in J}\|S(t)\| = M$.

Define the operator $Q_b : L^2(J,Y) \to X$ as

$$Q_b u = \int_0^b (b-s)^{\alpha-1}\mathcal{S}_\alpha(b-s)Bu(s)\mathrm{d}s.$$

Since $\frac{1}{2} < \alpha \le 1$, by means of the Hölder inequality, we can deduce that

$$\begin{aligned} \|Q_b u\| &\le \frac{M\alpha}{\Gamma(1+\alpha)}\|B\|\Big(\int_0^b |(b-s)^{2(\alpha-1)}|\mathrm{d}s\Big)^{\frac{1}{2}}\|u\|_{L^2} \\ &= \frac{M\alpha}{\Gamma(1+\alpha)}\|B\|\Big(\frac{b^{2(\alpha-1)+1}}{2(\alpha-1)+1}\Big)^{\frac{1}{2}}\|u\|_{L^2}, \end{aligned}$$

for each $u \in L^2(J,Y)$, which means that $Q_b \in \mathfrak{B}(L^2(J,Y),X)$. Denote by $Q_b^* : X^* \to L^2(J,Y)$ the dual operator of Q_b. Then

$$\begin{aligned} \langle Q_b u, x^*\rangle &= \int_0^b (b-s)^{\alpha-1}\langle \mathcal{S}_\alpha(b-s)Bu(s), x^*\rangle \mathrm{d}s \\ &= \int_0^b \langle u(s), (b-s)^{\alpha-1}B^*\mathcal{S}_\alpha^*(b-s)x^*\rangle \mathrm{d}s \\ &= \langle u, Q_b^* x^*\rangle, \end{aligned} \tag{9}$$

for each $u \in L^2(J,Y)$ and $x^* \in X^*$, that is

$$Q_b^* x^*(\cdot) = (b-\cdot)^{\alpha-1}B^*\mathcal{S}_\alpha^*(b-\cdot)x^*, \tag{10}$$

and

$$\|Q_b^* x^*\| = \Big(\int_0^b \|(b-s)^{\alpha-1}B^*\mathcal{S}_\alpha^*(b-s)x^*\|^2\mathrm{d}s\Big)^{\frac{1}{2}}, \tag{11}$$

for every $x^* \in X^*$.

Definition 2.9. *We say that fractional linear system* (1) *is*

1. *approximately controllable on J if for any $x_0, x_1 \in X$ and $\varepsilon > 0$, there exists a control $u \in L^2(J,Y)$ such that $\|v(b) - Cx_1\| < \varepsilon$, i.e., $\|\mathcal{T}_\alpha(b)x_0 + Q_b u - Cx_1\| < \varepsilon$.*
2. *approximately null controllable on J if for any $x_0 \in X$ and $\varepsilon > 0$, there exists a control $u \in L^2(J,Y)$ such that $\|v(b)\| < \varepsilon$, i.e., $\|\mathcal{T}_\alpha(b)x_0 + Q_b u\| < \varepsilon$.*

Remark 2.10. *Definition 2.9 implies the following conclusions:*

1. *fractional linear system* (1) *is approximately controllable on J if and only if* $\overline{R(C)} \cup \overline{R(\mathcal{T}_\alpha(b))} \subseteq \overline{R(Q_b)}$.
2. *fractional linear system* (1) *is approximately null controllable on J if and only if* $\overline{R(\mathcal{T}_\alpha(b))} \subseteq \overline{R(Q_b)}$.

Throughout the paper, without loss of generality, we may assume that X and X^* are strictly convex. For each $x \in X$, define the duality mapping $F : X \to X^*$ as

$$F(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\},$$

By virtue of Barbu [4], F is bijective, demicontinuous and strictly monotonic. Moreover, $F^{-1} : X^* \to X$ is also a duality mapping.

Remark 2.11. *For each $\lambda > 0$, the operator $\lambda I + \Lambda_0^b F : X \to X$ is bijective, where $\Lambda_0^b = Q_b Q_b^*$ is a nonnegative symmetric operator. In fact, on one hand, if*

$$\lambda F^{-1}x_1^* + \Lambda_0^b x_1^* = \lambda F^{-1}x_2^* + \Lambda_0^b x_2^*,$$

for some $x_1^*,\ x_2^* \in X^*$, then

$$\lambda\langle F^{-1}x_1^* - F^{-1}x_2^*, x_1^* - x_2^*\rangle = \langle \Lambda_0^b(x_2^* - x_1^*), x_1^* - x_2^*\rangle.$$

By virtue of the fact that F^{-1} is a duality mapping and Λ_0^b is nonnegative and symmetric, we have

$$0 \le \lambda(\|x_1^*\| - \|x_2^*\|)^2 = -\|Q_b^*x_2^* - Q_b^*x_1^*\|^2,$$

which implies that $\|x_1^*\| = \|x_2^*\|$ and $Q_b^*x_2^* = Q_b^*x_1^*$, i.e., $x_1^* = x_2^*$, which implies that it is injective.

On the other hand, by Lemma 2.2 of [20], we have that for every $\lambda > 0$ and $x \in X$, the equation

$$\lambda z_\lambda + \Lambda_0^b F(z_\lambda) = \lambda x$$

has a unique solution $z_\lambda = \lambda(\lambda I + \Lambda_0^b F)^{-1}x$. That is, the operator $\lambda I + \Lambda_0^b F$ is surjective.

3. Approximate controllability and approximate null controllability for linear system (1)

In this section, we give the necessary and sufficient conditions in a resolvent form of approximate controllability and approximate null controllability for fractional linear system (1).

Lemma 3.1. *For every $\lambda > 0$ and $x \in X$, one has*

$$\lambda\|F(\lambda I + \Lambda_0^b F)^{-1}x\| \le \|x\|.$$

Proof. For every $\lambda > 0$ and $x \in X$, set

$$y_\lambda = F(\lambda I + \Lambda_0^b F)^{-1}x,$$

that is

$$x = \lambda F^{-1}y_\lambda + \Lambda_0^b y_\lambda.$$

Then

$$\begin{aligned}\langle x, y_\lambda\rangle &= \lambda\langle F^{-1}y_\lambda, y_\lambda\rangle + \langle \Lambda_0^b y_\lambda, y_\lambda\rangle \\ &= \lambda\|y_\lambda\|^2 + \|Q_b^* y_\lambda\|^2 \\ &\ge \lambda\|y_\lambda\|^2,\end{aligned}$$

which implies that $\lambda\|y_\lambda\|^2 \le \|y_\lambda\|\|x\|$, i.e., $\lambda\|y_\lambda\| \le \|x\|$. This completes the proof. □

Lemma 3.2. *For every $\lambda > 0$ and $x^* \in X^*$, we have*

$$\|F(\lambda I + \Lambda_0^b F)^{-1}\Lambda_0^b x^*\| \leq \|x^*\|.$$

Proof. For every $x^* \in X^*$, set

$$z_\lambda = F(\lambda I + \Lambda_0^b F)^{-1}\Lambda_0^b x^*,$$

that is

$$\Lambda_0^b x^* = \lambda F^{-1} z_\lambda + \Lambda_0^b z_\lambda.$$

Then

$$0 \leq \langle x^* - z_\lambda, \Lambda_0^b (x^* - z_\lambda)\rangle = \lambda\langle x^*, F^{-1}z_\lambda\rangle - \lambda\langle z_\lambda, F^{-1}z_\lambda\rangle,$$

which implies that

$$\|z_\lambda\|^2 \leq \|z_\lambda\|\|x^*\|,$$

that is,

$$\|(\lambda F^{-1} + \Lambda_0^b)^{-1}\Lambda_0^b x^*\| \leq \|x^*\|.$$

This completes the proof. □

Remark 3.3. *Exploiting Lemma 3.2, we can deduce that*

$$\lambda F(\lambda I + \Lambda_0^b F)^{-1} x \to 0, \tag{12}$$

as $\lambda \to 0^+$ for every $x \in R(Q_b Q_b^*)$. This together with Lemma 3.1 gives that (12) holds for every $x \in \overline{R(Q_b Q_b^*)}$. Taking into account the fact $\overline{R(Q_b Q_b^*)} = N(Q_b^*)^\perp = \overline{R(Q_b)}$, one gets that (12) holds for every $x \in \overline{R(Q_b)}$.

Theorem 3.4. *Fractional linear system* (1) *is approximately controllable if and only if one of the following conditions holds:*

1. $\lambda F(R(\lambda, -\Lambda_0^b F)C) \to 0$ *and* $\lambda F(R(\lambda, -\Lambda_0^b F)\mathcal{T}_\alpha(b)) \to 0$ *as* $\lambda \to 0^+$ *in the strong topology, where* $R(\lambda, -\Lambda_0^b F) = (\lambda I + \Lambda_0^b F)^{-1}$.
2. $\lambda R(\lambda, -\Lambda_0^b F)C \to 0$ *and* $\lambda R(\lambda, -\Lambda_0^b F)\mathcal{T}_\alpha(b) \to 0$ *as* $\lambda \to 0^+$ *in the strong topology.*
3. $\lambda F(R(\lambda, -\Lambda_0^b F)C) \to 0$ *and* $\lambda F(R(\lambda, -\Lambda_0^b F)\mathcal{T}_\alpha(b)) \to 0$ *as* $\lambda \to 0^+$ *in the weak topology.*
4. $\lambda R(\lambda, -\Lambda_0^b F)C \to 0$ *and* $\lambda R(\lambda, -\Lambda_0^b F)\mathcal{T}_\alpha(b) \to 0$ *as* $\lambda \to 0^+$ *in the weak topology.*

Proof. In view of Remark 2.11, linear system (1) is approximately controllable if and only if $\overline{R(C)} \bigcup \overline{R(\mathcal{T}_\alpha(b))} \subseteq \overline{R(Q_b)}$. This together with Remark 3.3 yields that condition 1 holds.

The definition of duality mapping F gives that condition 1 implies condition 2. Moreover, it is obvious that condition 1 implies condition 3 and condition 2 implies condition 4. Now, we prove that if condition 3 holds, then condition 2 also holds. Owing to condition 3 and the reflexivity of X, one has

$$\langle x, \lambda F(R(\lambda, -\Lambda_0^b F)Cx\rangle \to 0 \tag{13}$$

as $\lambda \to 0^+$ for every $x \in X$. Set $R(\lambda, -\Lambda_0^b F)Cx = x_\lambda$. Then, $Cx = \lambda x_\lambda + \Lambda_0^b F x_\lambda$, and

$$\begin{aligned} \langle Cx, Fx_\lambda\rangle &= \lambda\|x_\lambda\|^2 + \langle \Lambda_0^b F x_\lambda, Fx_\lambda\rangle \\ &\geq \lambda\|x_\lambda\|^2, \end{aligned}$$

which implies that

$$\lambda^2\|x_\lambda\|^2 \leq \langle Cx, \lambda F x_\lambda\rangle. \tag{14}$$

This yields $\lambda\|x_\lambda\| \to 0$ as $\lambda \to 0^+$ since (13) holds, in other words,

$$\lambda(\lambda I + \Lambda_0^b F)^{-1}Cx \to 0. \tag{15}$$

With analogous arguments as above, we can prove that $\lambda(\lambda I+\Lambda_0^b F)^{-1}\mathcal{T}_\alpha(b)x \to 0$ as $\lambda \to 0^+$ provided that $\lambda F(R(\lambda,-\Lambda_0^b F)\mathcal{T}_\alpha(b)) \to 0$ as $\lambda \to 0^+$ in the weak topology. This gives that condition 2 holds.

Finally, we verify that condition 4 implies that (1) is approximately controllable on J. For every $\lambda > 0$ and $x \in X$, let $y_\lambda = (\lambda I+\Lambda_0^b F)^{-1}Cx$. Then, $Cx = \lambda y_\lambda + \Lambda_0^b F y_\lambda$. By virtue of condition 4,

$$\langle \lambda y_\lambda, x^* \rangle \to 0 \tag{16}$$

for every $x^* \in X^*$. Morover, for every $x^* \in N(Q_b^*)$,

$$\begin{aligned} \langle Cx, x^* \rangle &= \langle \lambda y_\lambda, x^* \rangle + \langle \Lambda_0^b F y_\lambda, x^* \rangle \\ &= \langle \lambda y_\lambda, x^* \rangle. \end{aligned} \tag{17}$$

(16) and (17) yields

$$\langle y, x^* \rangle = 0, \tag{18}$$

for every $x^* \in N(Q_b^*)$ and $y \in \overline{R(C)}$. Next, we prove that $\overline{R(C)} \subseteq \overline{R(Q_b)}$. If not, with the help of Hahn-Banach theorem, there exist an $x_0 \in \overline{R(C)}\backslash\overline{R(Q_b)}$ and some $y^* \in X^*$ such that

$$\langle x_0, y^* \rangle = 1, \tag{19}$$

and

$$\langle y, y^* \rangle = 0, \tag{20}$$

for every $y \in \overline{R(Q_b)}$. By virtue of the fact that $\overline{R(Q_b)} = \overline{R(Q_b Q_b^*)}$, together with (20) gives

$$\langle Q_b Q_b^* x^*, y^* \rangle = 0,$$

for $x^* \in X^*$. Especially,

$$\langle Q_b Q_b^* y^*, y^* \rangle = 0,$$

which implies that $Q_b^* y^* = 0$, that is, $y^* \in N(Q_b^*)$. We now turn back to (18) and (19), which implies that the assumption is not true, and $\overline{R(C)} \subseteq \overline{R(Q_b)}$. A similar manner utilized above gives that $\overline{R(\mathcal{T}_\alpha(b))} \subseteq \overline{R(Q_b)}$, which means that $\overline{R(C)} \bigcup \overline{R(\mathcal{T}_\alpha(b))} \subseteq \overline{R(Q_b)}$, that is (1) is approximately controllable on J. This completes the proof. □

Remark 3.5. *The restricted condition of state space is weakened by taking full advantage of the dual mapping. Here, the state space just satisfies the reflexivity condition. Therefore, the results in our paper essentially generalize those in [20, 24, 28], and the references therein, where the state space must be a Hilbert space.*

Corollary 3.6. *Fractional linear system* (1) *is approximately null controllable on J if and only if one of the following conditions holds:*

1. $\lambda F(R(\lambda,-\Lambda_0^b F)\mathcal{T}_\alpha(b)) \to 0$ *as* $\lambda \to 0^+$ *in the strong topology.*
2. $\lambda R(\lambda,-\Lambda_0^b F)\mathcal{T}_\alpha(b) \to 0$ *as* $\lambda \to 0^+$ *in the strong topology.*
3. $\lambda F(R(\lambda,-\Lambda_0^b F)\mathcal{T}_\alpha(b)) \to 0$ *as* $\lambda \to 0^+$ *in the weak topology.*
4. $\lambda R(\lambda,-\Lambda_0^b F)\mathcal{T}_\alpha(b) \to 0$ *as* $\lambda \to 0^+$ *in the weak topology.*

Corollary 3.7. *If $C = I$, then fractional linear system* (1) *is approximately controllable on J if and only if one of the following conditions holds:*

1. $\lambda F(R(\lambda,-\Lambda_0^b F)) \to 0$ *as* $\lambda \to 0^+$ *in the strong topology.*
2. $\lambda R(\lambda,-\Lambda_0^b F) \to 0$ *as* $\lambda \to 0^+$ *in the strong topology.*
3. $\lambda F(R(\lambda,-\Lambda_0^b F)) \to 0$ *as* $\lambda \to 0^+$ *in the weak topology.*
4. $\lambda R(\lambda,-\Lambda_0^b F) \to 0$ *as* $\lambda \to 0^+$ *in the weak topology.*

Remark 3.8. *Theorem 2.3 of Mahmudov[20] is conditions 2 and 3 of Corollary 18 in the case of $\alpha = 1$.*

4. Applications

Consider the approximate controllability of the following fractional nonautonomous differential system:

$$\begin{cases} {}^{C}D^{\alpha}x(t) = Ax(t) + Bu(t) + f(t), \ t \in J = [0, b], \\ x(0) = x_0 \in X, \end{cases} \tag{21}$$

where $\frac{1}{2} < \alpha \leq 1$. $x(\cdot) \in X$. X is a reflexive Banach space. $A : D(A) \subseteq X \to X$ is a infinitesimal generator of an exponentially bounded C−semigroup $\{S(t)\}_{t\geq 0}$. $C \in \mathfrak{B}(X)$ is injective. The control function $u \in L^2(J, \ Y)$. Y is a Hilbert space, $B \in \mathfrak{B}(X)$ and $f \in L^1(J, X)$.

Definition 4.1. *A function $x \in C(J, X)$ is called a mild solution of fractional nonautonomous differential system* (21) *if*

$$\begin{cases} Cx(t) = y(t), \quad t \in J, \\ y(t) = \mathcal{T}_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)[Bu(s) + f(s)]\mathrm{d}s, \quad t \in J, \end{cases}$$

for each $t \in J$.

Definition 4.2. *System* (21) *is said to be approximately controllable on J if for each $x_0, \ x_1 \in X$ and $\varepsilon > 0$ small enough, there exists an $u \in L^2(J, \ Y)$ such that $\|y(b) - Cx_1\| < \varepsilon$.*

Theorem 4.3. *Let one of the conditions of Theorem 3.4 be satisfied. Assume that $x_0 \in R(C)$, and $f(t) \in R(C)$, $Bu(t) \in R(C)$ for $0 \leq t \leq b$. Then, fractional nonautonomous differential system* (21) *is approximately controllable on J.*

Proof. It follows from the facts $x_0 \in R(C)$, and $f(t) \in R(C)$, $Bu(t) \in R(C)$ for $0 \leq t \leq b$ that the function $y(t)$ defined in Definition 4.1 satisfies that $y(t) \in R(C)$ for $0 \leq t \leq b$, i.e., the mild of system (21) is well defined.

In view of Theorem 3.4, one can obtain that

$$\lambda F(R(\lambda, -\Lambda_0^b F)Cx) \to 0, \quad \lambda F(R(\lambda, -\Lambda_0^b F)\mathcal{T}_\alpha(b)x) \to 0,$$

as $\lambda \to 0^+$ for each $x \in X$. For each $x_0, \ x_1 \in X$, $\lambda > 0$ and *a.e.* $t \in J$, set

$$u_\lambda(t) \ = \ (b-t)^{\alpha-1}B^*\mathcal{S}_\alpha^*(b-t)[F(R(\lambda, -\Lambda_0^b F)Cy_0) - F(R(\lambda, -\Lambda_0^b F)\mathcal{T}_\alpha(b)x_0)], \tag{22}$$

where $y_0 = x_1 - C^{-1}(\int_0^b (b-s)^{\alpha-1}\mathcal{S}_\alpha(b-s)f(s)\mathrm{d}s) \in X$. Notice that $f(t) \in R(C)$ indicates $\int_0^b (b-s)^{\alpha-1}\mathcal{S}_\alpha(b-s)f(s)\mathrm{d}s) \in R(C)$, i.e., y_0 is well defined. We will verify that $\|y_\lambda(b) - Cx_1\| \to 0$ as $\lambda \to 0^+$ through the control function u_λ. In fact,

$$\begin{aligned} y_\lambda(b) &= \mathcal{T}_\alpha(b)x_0 + \Lambda_0^b[F(R(\lambda, -\Lambda_0^b F)Cy_0) - F(R(\lambda, -\Lambda_0^b F)\mathcal{T}_\alpha(b)x_0)] \\ &\quad + \int_0^b (b-s)^{\alpha-1}\mathcal{S}_\alpha(b-s)f(s)\mathrm{d}s) \\ &= \mathcal{T}_\alpha(b)x_0 + Cy_0 - \lambda(R(\lambda, -\Lambda_0^b F)Cy_0) - \mathcal{T}_\alpha(b)x_0 \\ &\quad + \lambda(R(\lambda, -\Lambda_0^b F)\mathcal{T}_\alpha(b)x_0) + \int_0^b (b-s)^{\alpha-1}\mathcal{S}_\alpha(b-s)f(s)\mathrm{d}s) \\ &= Cx_1 - \lambda R(\lambda, -\Lambda_0^b F)Cy_0 + \lambda R(\lambda, -\Lambda_0^b F)\mathcal{T}_\alpha(b)x_0. \end{aligned}$$

Moreover,

$$\begin{aligned} \|y_\lambda(b) - Cx_1\| &= \|\lambda R(\lambda, -\Lambda_0^b F)Cy_0 - \lambda R(\lambda, -\Lambda_0^b F)\mathcal{T}_\alpha(b)x_0)\| \\ &\leq \|\lambda R(\lambda, -\Lambda_0^b F)Cy_0\| + \|\lambda R(\lambda, -\Lambda_0^b F)\mathcal{T}_\alpha(b)x_0\|. \end{aligned}$$

It follows from Theorem 3.4 that,

$$\|y_\lambda(b) - Cx_1\| \to 0,$$

as $\lambda \to 0^+$. This completes the proof. □

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