Filomat 36:5 (2022), 1685–1697
https://doi.org/10.2298/FIL2205685S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Numerical Solution of System of Fredholm-Volterra Integro-Differential Equations Using Legendre Polynomials

D. Shirani[a], M. Tavassoli Kajani[a], S. Salahshour[b]

[a]*Department of Mathematics, Isfahan (Khorasgan) Branch, Islamic Azad University, Isfahan, Iran.*
[b]*Faculty of Engineering and Natural Sciences, Bahcesehir University, Istanbul, Turkey.*

Abstract. In this paper, two collocation methods based on the shifted Legendre polynomials are proposed for solving system of nonlinear Fredholm-Volterra integro-differential equations. The equation considered in this paper involves the derivative of unknown functions in the integral term, which makes its numerical solution more complicated. We first introduce a single-step Legendre collocation method on the interval $[0, 1]$. Next, a multi-step version of the proposed method is derived on the arbitrary interval $[0, T]$ that is based on the domain decomposition strategy and specially suited for large domain calculations. The first scheme converts the problem to a system of algebraic equations whereas the later solves the problem step by step in subintervals and produces a sequence of systems of algebraic equations. Some error estimates for the proposed methods are investigated. Numerical examples are given and comparisons with other methods available in the literature are done to demonstrate the high accuracy and efficiency of the proposed methods.

1. Introduction

Integro-differential equation systems (IDESs) are encountered in lots of model problems in science and engineering [1]. As it is usually difficult or even impossible to find analytical solutions of IDESs, the development of novel numerical schemes is likely to seems essential. In recent years, many authors have worked on numerical methods for integral and integro-differential equations. For instance, Semper [2] used a fourth order integro-differential equation to model static deflection in suspension bridges and presented Galerkin approximation methods. Tavassoli-Kajani et al. [3] proposed a Galerkin method based on rational second kind Chebyshev functions for system of Integro-differential equations on semi-infinite intervals. Maleknejad et al. [4] have employed the operational matrices of Bernstein polynomials for a system of high order linear Volterra-Fredholm-integro-differential equations. Ghasemi et al. [5] compared wavelet-Galerkin method and homotopy perturbation method for the numerical solution of nonlinear integro-differential equations. Tavassoli-Kajani and Hadi-Vencheh [6] utilized the continuous Legendre wavelets on the interval $[0, 1]$ to solve linear second kind integro-differential equations. Abbasbandy and Taati [7] have solved the system of nonlinear Volterra-integro-differential equations using the operational

2020 *Mathematics Subject Classification*. 65-xx; 65Lxx; 47Gxx
Keywords. Shifted Legendre polynomials; Fredholm-Volterra integro-differential equations; Collocation method.
Received: 24 May 2021; Revised: 31 October 2021; Accepted: 26 December 2021
Communicated by Hari M. Srivastava
Corresponding author: M. Tavassoli Kajani
Email addresses: sshirani333@gmail.com (D. Shirani), tavassoli_k@yahoo.com (M. Tavassoli Kajani), soheilsalahshour@yahoo.com (S. Salahshour)

Tau method. Srivastava [8] investigated the existence theory for nonlinear third-order ordinary differential equations with nonlocal multi-point and multi-strip boundary conditions. Javidi [9] presented a modified homotopy perturbation method for solving system of linear Fredholm integral equations. Asady et al. [10] solved the linear integro-differential equations with the use of combination of Fourier and block-pulse functions. Zarebnia and Ali-Abadi [11] utilized the sinc-collocation method for solving nonlinear Fredholm-Volterra integro-differential equations of the second order with boundary conditions of the Fredholm and Volterra types. Maleki and Tavassoli-Kajani [12] presented a multi-domain Legendre-Gauss pseudospectral method for the approximate solution of the fractional Volterra's population model which is a Volterra integro-differential equation. Izadi and Srivastava [13] obtained the approximate solutions of the nonlinear Logistic equation of fractional order by developing a collocation approach based on the fractional-order Bessel and Legendre functions. They also developed and investigated the local discontinuous Galerkin method for the numerical solution of the fractional logistic differential equation [14]. Zhuang and Ren [15] have developed an spectral method for a nonlinear fourth-order integro-differential equation. Yang et al. [16] developed a new factorization technique for nonlinear PDEs involving local fractional derivatives by making use of the traveling-wave transformation. Ordokhani and Dehestani [17] have applied Bessel function for solving nonlinear Fredholm-Volterra-Hammerstein integro-differential equations under mixed conditions. Srivastava and Saxena [18] investigated the solutions of several Volterra-type fractional integro-differential equations with a multivariable confluent hypergeometric function as their kernel. A Chebyshev wavelet method has been introduced by Aminikhah and Hosseini [19] for nonlinear system of integro-differential equations. Yuzbasi [20] introduced a Bessel collocation method for system of linear Fredholm-Volterra integro-differential equations. Berenguer et al.[21] utilized a numerical treatment of fixed point for solving systems of integro-differential equations. Homotopy-perturbation method is employed by Roul and Meyer [22] to derive numerical solutions of systems of nonlinear integro-differential equations. Jafarzadeh and Keramati [23] have presented a Taylor collocation solution for system of integro-differential equations. Pour-Mahmoud et.al [24] extended the Tau method for the numerical solution of system of Fredholm integro-differential equations. Gao et al. [25] used ordinary differential equations (ODE) and partial differential equations (PDE) to describe thermal problems in engineering sciences for linear and nonlinear heat transfer equations. The spline collection method has been employed by Ebrahimi and Rashidinia [26] to solve system of Fredholm and Volterra integro-differential equations. The variational iteration method was proposed by Abbasbandy and Shivanian [27] to solve system of nonlinear Volterra integro-differential equations. Fazeli and Hojjati [28] solved two types of nonlinear Volterra integro-differential equations system including non-stiff and stiff problems using superimplicit multistep collocation methods. Loh and Phang [29] applied Genocchi polynomials to solve numerically a system of Volterra integro-differential equations. Mahdavi and Tavassoli-Kajani [30] solved nonlinear Fredholm integro-differential equations by the continuous Legendre wavelets constructed on the interval $[0, 1]$. Differential transform method has been applied by Arikoglu and Ozkol [31] to both integral and integro-differential equations systems. A numerical scheme based on the Tau method with arbitrary polynomial bases has been developed by Hosseini and Shahmorad [32] to find the numerical solution of Fredholm integro-differential equations. An extension of the spectral Lanczos' Tau method for systems of nonlinear integro-differential equations was proposed by Vasconcelos et.al [33]. Nemati [34] presented a numerical solution of Volterra-Fredholm integral equations using Legendre collocation method. Hadhoud et al. [35] proposed the non-polynomial B-spline method and the shifted Jacobi spectral collocation method to solve time-fractional nonlinear coupled Burgers' equations numerically. Some dynamical models involving fractional-order derivatives with the Mittag-Leffler type kernels and their applications based upon the Legendre spectral collocation method have been investigated by Srivastava et al. [36]. Kumar et al. [37] presented a convergent collocation method based on Jacobi poly-fractonomials with which to find the numerical solution of a generalized fractional integro-differential equation. Izadi and Srivastava [38] investigated a novel set of orthogonal basis functions combined with a matrix technique for treating a class of multi-order fractional pantograph differential equations computationally. Singh et al. [39] constructed approximate solution to multi-dimensional Fredholm integral equations of second kind using n-dimensional Legendre scaling functions. Solution of fractional Volterra-Fredholm integro-differential equations under mixed boundary conditions by using the HOBW method has been studied by Ali et al. [40]. A new Neumann series method to solve a family of local fractional Fredholm and Volterra integral equations was proposed by Ma et al. [41].

We consider the following system of Fredholm-Volterra integro-differential equations,

$$\sum_{n=0}^{m}\sum_{j=1}^{k} p_{ij}^{n}(x)y_j^{(n)}(x) = g_i(x) + \int_a^b \sum_{n=0}^{m}\sum_{j=1}^{k} k_{i,j}^{n,f}(x,t)\, y_j^{(n)}(t)\,dt + \int_a^x \sum_{n=0}^{m}\sum_{j=1}^{k} k_{i,j}^{n,v}(x,t)\, y_j^{(n)}(t)\,dt, \tag{1}$$

$$0 \le a \le x, \quad t \le b, \quad i = 1,2,\dots,k$$

with the initial conditions $y_j^{(n)}(a) = \lambda_j^n$, $0 \leqslant n \leqslant m-1$, $y_j(x)$ are unknown functions, and $p_{i,j}^n(x)$, $k_{i,j}^{n,v}(x,t)$, $k_{i,j}^{n,f}(x,t)$ and $g_i(x)$ are known functions defined on the interval $[a,b]$.

In the next section, we review some properties of Legendre and shifted Legendre polynomials. In Section 3, two new Legendre collocation methods are derived for solving Eq. (1). A One-domain scheme is proposed for problems defined on the interval $[0,1]$ and a multi-domain scheme is constructed for larger domains. Section 4 is devoted to some error estimates. In Section 5, numerical results are given and conclusions are stated in Section 6.

2. Preliminaries

In this section, we recall some basic properties of Legendre polynomials required for our subsequent development. Legendre polynomials are orthogonal in the interval $[-1,1]$ with respect to the weight function $w(x) = 1$. They make a complete set in the space $L^2[-1,1]$ and can be obtained using the following recursive relation:

$$L_0(t) = 1, \quad L_1(t) = t,$$

$$nL_n(t) - t(2n-1)L_{n-1}(t) - (n-1)L_{n-2}(t) = 0, \quad n \ge 2.$$

Using the transformation $t = \frac{2x-(a+b)}{b-a}$, the shifted Legendre polynomials on the interval $\xi = [a,b]$ are defined by,

$$\psi_i^{\xi}(x) = L_i\left(\frac{2x-(a+b)}{b-a}\right), \quad i = 0,1,2,\dots$$

We also introduce the following inner product and norm,

$$< f, g >= \int_a^b f(x)g(x)dx,$$

$$\| f \|_2 =< f, f >^{1/2} .$$

The shifted Legendre polynomials have the following orthogonality relation,

$$\int_a^b \psi_i^{\xi}(x)\psi_j^{\xi}(x)dx = \begin{cases} \frac{b-a}{2i+1}, & i = j \\ 0, & i \neq j. \end{cases} \tag{2}$$

The completeness of the set $\{\psi_i^{\xi}\}$ implies that, any function $y(x) \in L^2[a,b]$ can be approximated as

$$y(x) = \sum_{i=0}^{\infty} c_i \psi_i^{\xi}(x), \tag{3}$$

where

$$c_i = \frac{(2i+1)}{b-a}\int_a^b y(x)\psi_i^{\xi}(x)dx.$$

Therefore, the approximation of y can be derived as

$$y(x) \cong P_N y(x) = \sum_{n=0}^{N} c_n \psi_n^{\xi}(x) = C^T \Psi^{\xi}(x),$$

where N is a positive integer, $C = [c_0, c_1, \ldots, c_N]^T$ and $\Psi^{\xi}(x) = [\psi_0^{\xi}(x), \psi_1^{\xi}(x), \ldots, \psi_N^{\xi}(x)]^T$.

Furthermore, in bounding from above the approximation error on the interval $[-1, 1]$, we note that the orthogonal projection P_N in Eq. (3) is exact for all polynomials of degree at most N. Thus, it is convenient introduce the semi-norms

$$|u|_{H^{m;N}(-1,1)} = \left(\sum_{k=min(m,N+1)}^{m} \|u^{(k)}\|^2_{L^2(-1,1)} \right)^{1/2},$$

where H^m is the Sobolev space of integer order m. Note that whenever $N \geq m - 1$, one has

$$|u|_{H^{m;N}(-1,1)} = \|u^{(m)}\|_{L^2(-1,1)}.$$

Then, for all $u \in H^m(-1, 1)$ and $m \geqslant 0$, the truncation error $u - P_N u$ can be estimated as follows [42]:

$$\|u - P_N u\|_{L^2(-1,1)} \leq C N^{-m} |u|_{H^{m;N}(-1,1)}, \tag{4}$$

where the constant C depends only on m.

3. Legendre collocation methods for Fredholm-Volterra integro-differential equations

In this section, we derive two efficient methods based on one-domain and multi-domain Legendre collection schemes for solving (1).

3.1. Method 1: One-domain Legendre collocation method

Consider the Fredholm-Volterra integro-differential equation (1). For simplicity of statement, we assume that $\xi = [0, 1]$. Let

$$y_j(x) \cong y_{j,N}(x) = \sum_{p=0}^{N} c_{j,p} \psi_p^{\xi}(x), \quad j = 1, \ldots, k. \tag{5}$$

With substituting Eq. (5) into Eq. (1), we have

$$\sum_{n=0}^{m} \sum_{j=1}^{k} p_{ij}^{n}(x) y_{j,N}^{(n)}(x) = g_i(x) + \int_0^1 \sum_{n=0}^{m} \sum_{j=1}^{k} k_{i,j}^{n,f}(x,t)\, y_{j,N}^{(n)}(t) dt + \int_0^x \sum_{n=0}^{m} \sum_{j=1}^{k} k_{i,j}^{n,v}(x,t)\, y_{j,N}^{(n)}(t) dt, \quad 1 \leqslant i \leqslant k, \tag{6}$$

and the initial condition will be,

$$y_{j,N}^{(n)}(0) = \lambda_j^n, \quad j = 1, 2, \ldots, k, \quad n = 0, 1, \ldots, m - 1, \tag{7}$$

in which $y_{j,N}^{(0)}(0) = y_{j,N}(0)$. Now, let θ_l, $l = 1, 2, \ldots, N$ be the shifted Legendre-Gauss collocation points on the interval $[0, 1]$ which are the zeros of the shifted Legendre polynomial $\psi_N^{\xi}(x)$. Collocating Eq (6) at the points θ_l, results

$$\sum_{n=0}^{m} \sum_{j=1}^{k} p_{ij}^{n}(\theta_l) y_{j,N}^{(n)}(\theta_l) = g_i(\theta_l) + \int_0^1 \sum_{n=0}^{m} \sum_{j=1}^{k} k_{i,j}^{n,f}(\theta_l,t)\, y_{j,N}^{(n)}(t) dt + \int_0^{\theta_l} \sum_{n=0}^{m} \sum_{j=1}^{k} k_{i,j}^{n,v}(\theta_l,t)\, y_{j,N}^{(n)}(t) dt, \tag{8}$$

$$l = n+1, ..., N, \quad i = 1, 2, ..., k.$$

Employing the Legendre-Gauss quadrature rule for the integrals involved in Eq. (8), we produce the collocation equations as

$$\sum_{n=0}^{m}\sum_{j=1}^{k} p_{ij}^{n}(\theta_l) y_{j,N}^{(n)}(\theta_l) = g_i(\theta_l) + \sum_{q=1}^{N}\sum_{n=0}^{m}\sum_{j=1}^{k} k_{i,j}^{n,f}\left(\theta_l, \theta_q\right) y_{j,N}^{(n)}(\theta_q)\tilde{w}_q + \theta_l \sum_{q=1}^{N}\sum_{n=0}^{m}\sum_{j=1}^{k} k_{i,j}^{n,v}\left(\theta_l, \theta_q\right) y_{j,N}^{(n)}(\theta_q)\tilde{w}_q, \quad (9)$$

where $\tilde{w}_q = \frac{1}{2} w_q$ and

$$w_q = \frac{1}{(1-\theta_q^2)(L_{N+1}(\theta_q))^2}, \quad q = 1, \dots, N,$$

are the Legendre-Gauss quadrature weights. Equations (7) and (9) give an algebraic system of $k(N+1)$ equations and $k(N+1)$ unknowns $c_{j,p}$, $j = 1, 2, \dots, k$, $p = 0, 1, \dots, N$. Finally, using Eq. (5) the approximation $y_{j,N}(x)$ of the functions $y_j(x)$ can be obtained.

3.2. Method 2: multi-domain Legendre collocation method

Assume that $\xi = [0, T]$ for a positive real value T. By choosing a step-size h, we divide the interval ξ to M equidistant subintervals $\xi_s = [(s-1)h, sh]$, $s = 1, 2, \dots, M$. Let $y_{j,N}^{s}(x)$, $j = 1, 2, \dots, k$ be the Legendre series approximate solutions of the system (1) on the subinterval ξ_s, i.e.,

$$y_j^s(x) \cong y_{j,N}^s(x) = \sum_{p=0}^{N} c_{j,p}^{s} \psi_p^{\xi_s}(x). \quad (10)$$

By substituting $y_{i,N}^s(x)$ into (1), restricted to the subinterval ξ_s, we arrive at

$$\sum_{n=0}^{m}\sum_{j=1}^{k} p_{ij}^{n}(x)(y_{j,N}^{s})^{(n)}(x) = g_i(x) \quad (11)$$

$$+ \int_{\xi_s} \sum_{n=0}^{m}\sum_{j=1}^{k} k_{i,j}^{n,f}(x,t)\,(y_{j,N}^{s})^{(n)}(t)dt + \int_{(s-1)h}^{x} \sum_{n=0}^{m}\sum_{j=1}^{k} k_{i,j}^{n,v}(x,t)\,(y_{j,N}^{s})^{(n)}(t)dt,$$

where $x \leqslant sh$. Noteworthy, in the first subinterval the initial conditions are given in (1), whilst in the subsequent subintervals, we use the following initial conditions:

$$(y_{i,N}^{s})^{(n)}((s-1)h) = (y_{i,N}^{s-1})^{(n)}((s-1)h), \quad s = 2, ..., M. \quad (12)$$

Collocating the system (11) at the shifted Legendre-Gauss quadrature points of the subinterval ξ_s i.e., $\theta_{s,l}$, and utilizing the Legendre-Gauss quadrature rule, we get

$$\sum_{n=0}^{m}\sum_{j=1}^{k} p_{ij}^{n}(\theta_{s,l})(y_{j,N}^{s})^{(n)}(\theta_{s,l}) = g_i(\theta_{s,l}) \quad (13)$$

$$+ \sum_{q=1}^{N}\sum_{n=0}^{m}\sum_{j=1}^{k} k_{i,j}^{n,f}\left(\theta_{s,l}, \theta_q\right)(y_{j,N}^{s})^{(n)}(\theta_q)\tilde{w}_q + \theta_l \sum_{q=1}^{N}\sum_{n=0}^{m}\sum_{j=1}^{k} k_{i,j}^{n,v}\left(\theta_{s,l}, \theta_q\right)(y_{j,N}^{s})^{(n)}(\theta_q)\tilde{w}_q, \quad n+1 \leqslant l \leqslant N.$$

At step s, Eqs. (12)-(13) give a system of algebraic equations with $k(N+1)$ equations and $k(N+1)$ unknowns $c_{j,p}^{s}$. By solving this algebraic system and substituting the obtained unknowns into (10), the approximate solution of (1) on the subinterval ξ_s, $s = 1, 2, \dots, M$ is obtained.

4. Error estimate

In this section, we give a convergence analysis for the methods developed in Section 3. Let the k-dimensional vector function $Y(x) \in H^m(0,1)$ be the exact solution of the problem (1) and $\bar{Y}_N(x)$ be its corresponding Legendre expansion. According to (4), there exists a constant C independent of Y and N such that

$$\left\|Y - \bar{Y}_N\right\|_{L^2(0,1)} \leq CN^{-m}|Y|_{H^{m;N}(0,1)}. \tag{14}$$

The next theorem establishes the convergence of the one-domain Legendre collocation scheme.

Theorem 4.1. *Suppose that the known functions in (1) are real-valued functions in $H^m(0,1)$ and let $\tilde{Y}_N(x)$ be the approximate solution for (1) obtained using the Method 1. Then,*

$$\|Y - \tilde{Y}_N\|_{L^2(0,1)} \leq CN^{-m}|Y|_{H^{m;N}(0,1)} + \left\|\bar{C} - \tilde{C}\right\|_2 \Big(\sum_{i=0}^{k} \frac{1}{2i+1} \Big)^{1/2}, \tag{15}$$

where $\bar{C} = [\bar{c_0}, \bar{c}_1, \ldots, \bar{c_N}]^T$ and $\tilde{C} = [\tilde{c_0}, \tilde{c}_1, \ldots, \tilde{c_N}]^T$ are the coefficients vectors in the expansions of $\bar{Y}_N(x)$ and $\tilde{Y}_N(x)$, respectively.

Proof: Using the triangle inequality we have

$$\|Y - \tilde{Y}_N\|^2_{L^2(0,1)} \leq \|Y - \bar{Y}_N\|^2_{L^2(0,1)} + \|\bar{Y}_N - \tilde{Y}_N\|^2_{L^2(0,1)}.$$

On the other hand, Eq. (14) yields

$$\|Y - \bar{Y}_N\|_{L^2(0,1)} \leq CN^{-m}|Y|_{H^{m;N}(0,1)}.$$

In addition,

$$\|\bar{Y}_N - \tilde{Y}_N\|^2_{L^2(0,1)} = \int_0^1 \left|\bar{Y}_N(x) - \tilde{Y}_N(x)\right|^2 dx, \tag{16}$$

where,

$$\bar{Y}_N(x) - \tilde{Y}_N(x) = [\bar{y}_{1,N} - \tilde{y}_{1,N}, \bar{y}_{2,N} - \tilde{y}_{2,N}, ..., \bar{y}_{k,N} - \tilde{y}_{k,N}],$$

and

$$\begin{aligned}\left|\bar{Y}_N(x) - \tilde{Y}_N(x)\right|^2 &= \left(\bar{Y}_N(x) - \tilde{Y}_N(x)\right) \cdot \left(\bar{Y}_N(x) - \tilde{Y}_N(x)\right) \\ &= (\bar{y}_{1,N} - \tilde{y}_{1,N})^2 + (\bar{y}_{2,N} - \tilde{y}_{2,N})^2 + \ldots + (\bar{y}_{k,N} - \tilde{y}_{k,N})^2.\end{aligned}$$

Hence, Eq. (16) becomes

$$\|\bar{Y}_N - \tilde{Y}_N\|^2_{L^2(0,1)} = \int_0^1 \sum_{i=1}^{k} (\bar{y}_{i,N}(x) - \tilde{y}_{i,N}(x))^2 dx. \tag{17}$$

Considering that

$$\bar{y}_{i,N}(x) = \bar{c}_i^T \psi_i(x),$$
$$\tilde{y}_{i,N}(x) = \tilde{c}_i^T \psi_i(x),$$

and substituting them into (17), we deduce that

$$\|\bar{Y}_N - \tilde{Y}_N\|^2_{L^2(0,1)} = \int_0^1 \sum_{i=1}^{k} \left((\bar{c}_i^T - \tilde{c}_i^T)\psi_i(x)\right)^2 dx.$$

Using the Cauchy inequality and Eq. (2), we arrive at

$$\left\|\bar{Y}_N - \tilde{Y}_N\right\|^2_{L^2(0,1)} \leq \left(\sum_{i=0}^{k} \left|\bar{c}_i^T - \tilde{c}_i^T\right|^2\right)\left(\sum_{i=0}^{k} \int_0^1 \psi_i^2(x)dx\right)$$

$$= \left\|\bar{C} - \tilde{C}\right\|_2^2 \sum_{i=0}^{k} \frac{1}{2i+1}.$$

Consequently,

$$\left\|Y - \tilde{Y}_N\right\|^2_{L^2(0,1)} \leq C^2 N^{-2m} |Y|^2_{H^{m;N}(0,1)} + \left\|\bar{C} - \tilde{C}\right\|_2^2 \sum_{i=0}^{k} \frac{1}{2i+1},$$

as desired. □

The extension of the above result to the Method 2 is stated in the following corollary.

Corollary 4.2. *Let $\tilde{Y}_N^s(x)$ be the approximate solution for (1) in the s^{th} step. Then,*

$$\left\|Y - \tilde{Y}_N\right\|_{L^2(0,T)} \leq CN^{-m}\left(\sum_{s=1}^{M} |Y^s|^2_{H^{m;N}(\xi_s)}\right)^{1/2} + \left(\sum_{s=1}^{M} \left\|\bar{C}_s - \tilde{C}_s\right\|_2^2\right)^{1/2}\left(\sum_{i=0}^{k} \frac{h}{2i+1}\right)^{1/2}.$$

Noting that $\left\|Y - \tilde{Y}_N\right\|^2_{L^2(0,T)} = \sum_{i=1}^{M} \left\|Y^s - \tilde{Y}_N^s\right\|^2_{L^2(\xi_s)}$ and following the same line as in the proof of Theorem 4.1, we can prove the above result.

5. Numerical results

In this section, we consider several numerical examples to demonstrate the accuracy and efficiency of the proposed Legendre collocation methods. For checking the numerical accuracy, we compute the L^∞ errors, i.e., $e_i = \max_{0\leq x\leq T} |y_i(x) - \tilde{y}_i(x)|, \quad i = 1, \ldots, k.$

Example 1.
As the first example consider the following system of Volterra integral equations,

$$\begin{cases} y_1(x) = \cos(x)(2 + \sin(x) - x\cos(x)) + \dfrac{1}{4}(\cos(x-1) - \cos(x+1)) \\ \qquad -\dfrac{1}{2}x\sin(x-1) - 1 + \displaystyle\int_0^x [\sin((x-t)-1)y_1(t) + (1 - t\cos(x))y_2(t)]dt, \\ y_2(x) = \sin(x) - x + \displaystyle\int_o^x [y_1(t) + (x-t)y_2(t)]dt \end{cases}$$

with initial conditions

$$y_1(0) = 1, \quad y_2(0) = 0.$$

The exact solutions to this problem are $y_1(x) = \cos(x)$, $y_2(x) = \sin(x)$. In Tables 1–2 the errors for $y_1(x)$, $y_2(x)$ with $N = 10$ and various values of x are compared with the methods proposed in [21, 43]. The Log-errors for Method 1 and Method 2 with $h = 0.1$ for various N are compared in Fig. 1.

Table 1: Comparison between errors in approximating $y_1(x)$ for $N = 10$ for Example 1.

x	Method [43]	Method [21]	Method 1	Method 2 (h=0.1)
0	0	0	0	0
0.2	9.27×10^{-4}	1.71×10^{-14}	3.75×10^{-15}	6.11×10^{-31}
0.4	5.45×10^{-3}	1.68×10^{-13}	2.72×10^{-15}	1.43×10^{-30}
0.6	1.39×10^{-2}	1.28×10^{-11}	6.72×10^{-15}	1.57×10^{-30}
0.8	2.56×10^{-2}	2.92×10^{-10}	1.83×10^{-14}	7.81×10^{-31}
1	4.20×10^{-2}	3.20×10^{-9}	2.48×10^{-14}	3.41×10^{-31}
5	–	–	–	2.75×10^{-20}
10	–	–	–	1.12×10^{-17}
15	–	–	–	3.14×10^{-11}

Table 2: Comparison between errors in approximating $y_2(x)$ for $N = 10$ for Example 1.

x	Method [43]	Method [21]	Method 1	Method 2 (h=0.1)
0	0	0	0	0
0.2	3.17×10^{-4}	4.62×10^{-14}	8.41×10^{-15}	1.14×10^{-29}
0.4	4.21×10^{-3}	2.64×10^{-13}	6.93×10^{-15}	2.27×10^{-29}
0.6	6.529×10^{-2}	5.32×10^{-11}	1.51×10^{-14}	3.40×10^{-29}
0.8	1.45×10^{-2}	4.83×10^{-10}	3.36×10^{-14}	4.54×10^{-29}
1	3.75×10^{-2}	7.51×10^{-9}	4.29×10^{-14}	5.69×10^{-29}
5	–	–	–	6.14×10^{-19}
10	–	–	–	3.75×10^{-15}
15	–	–	–	8.63×10^{-9}

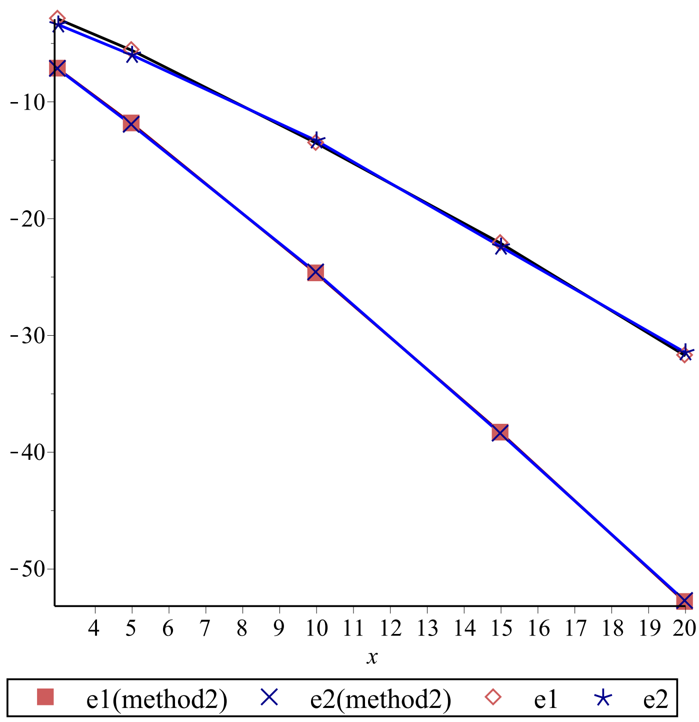

Figure 1: Log-errors using Method 1 and Method 2 for Example 1.

Example 2.
Consider the Volterra integro-differential system

$$\begin{cases} y_1^{(1)}(x) + y_2(x) = 1 + x + x^2 + \displaystyle\int_0^x [-y_1(t) - y_2(t)]dt, \\ y_2^{(1)}(x) - y_1(x) = -1 - x + \displaystyle\int_0^x [-y_1(t) + y_2(t)]dt, \end{cases}$$

with initial conditions

$$y_1(0) = 1, \quad y_2(0) = -1.$$

The exact solutions to this system are $y_1(x) = x + e^x$ and $y_2(x) = x - e^x$. The errors for $N = 7$ and various values of x are given in Table 3 and Table 4. Fig. 2 compares the log-errors with the those reported in [20]. It is seen that the errors decrease exponentially and the present methods provide more accurate results compared with methods presented in [21, 45].

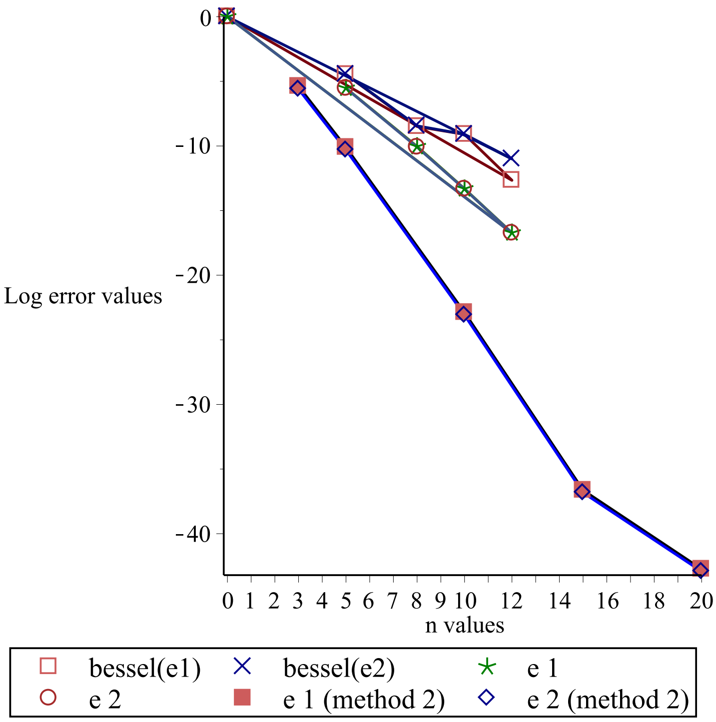

Figure 2: Log-errors for Example 2.

Table 3: Comparison between errors in approximating $y_1(x)$ for $N = 7$ for Example 2.

x	Method [44]	Method [20]	Method 1	Method 2 (h=0.1)
0	0	0	0	0
0.2	3.0×10^{-9}	3.0×10^{-9}	1.4×10^{-10}	8.7×10^{-17}
0.4	3.2×10^{-7}	3.3×10^{-9}	8.7×10^{-10}	2.0×10^{-16}
0.6	5.4×10^{-6}	3.1×10^{-9}	1.3×10^{-9}	3.6×10^{-16}
0.8	3.9×10^{-5}	2.4×10^{-9}	3.2×10^{-10}	5.5×10^{-16}
1	1.8×10^{-4}	8.5×10^{-8}	1.4×10^{-10}	7.8×10^{-16}
5	–	–	–	1.03×10^{-15}
10	–	–	–	4.51×10^{-7}

Table 4: Comparison between errors in approximating $y_2(x)$ for $N = 7$ for Example 2.

x	Method [44]	Method [20]	Method 1	Method 2 (h=0.1)
0	0	0	0	0
0.2	2.0×10^{-9}	2.0×10^{-9}	5.5×10^{-10}	6.5×10^{-17}
0.4	3.2×10^{-7}	1.7×10^{-9}	1.0×10^{-9}	1.3×10^{-16}
0.6	5.3×10^{-6}	1.1×10^{-9}	5.0×10^{-10}	2.1×10^{-16}
0.8	3.9×10^{-5}	2.0×10^{-10}	4.2×10^{-10}	3.0×10^{-16}
1	1.8×10^{-4}	8.9×10^{-8}	2.5×10^{-10}	4.2×10^{-16}
5	-	-	-	7.05×10^{-14}
10	-	-	-	4.81×10^{-8}

Example 3.
Consider the following system,

$$\begin{cases} xy_2(x) + y_1(x) = xe^{-x} + e^x(x+1) + \displaystyle\int_0^x -e^{x-t} y_1(t)dt, \\ xy_1(x) + x^2 y_2(x) = 2xe^x + 1 + e^{-x}(x^2 - 1) + \displaystyle\int_o^x -e^{-2t} y_1(t) - e^{x+t} y_2(t)dt, \end{cases}$$

with initial conditions

$$y_1(0) = 1, \quad y_2(0) = 1.$$

The exact solutions are $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. Table 5 gives the maximum errors for different values of N. It is observed that our method, with fewer collocation points, is by far more accurate than the block-pulse method proposed in [45].

Table 5: Maximum absolute error for $y_1(x)$ and $y_2(x)$ for Example 3.

N	15		20		block-pulse [45] ($N = 256$)
	Method 1	Method 2	Method 1	Method 2	
$e1$	1.7×10^{-21}	1.6×10^{-38}	6.2×10^{-30}	3.4×10^{-52}	3.3×10^{-3}
$e2$	2.3×10^{-21}	2.9×10^{-38}	6.3×10^{-30}	3.4×10^{-52}	2.6×10^{-3}

Example 4.
Consider the following system,

$$\begin{cases} t^2 f(t) - \frac{1}{2}t - \frac{1}{2}tf(t) + \frac{1}{16}(f(t) - t^2 g(t)) + \frac{3}{2}tg(t) - \frac{9}{16}g(t) = \int_0^t y_1(x) - y_2(x)dx, \\ \\ -t^2 + t^2 f(t) + t - \frac{1}{2}tf(t) + \frac{1}{16}(f(t) + t^2 g(t)) - \frac{3}{2}tg(t) + \frac{9}{16}g(t) = \int_0^t y_1(x) + y_2(x)dx, \end{cases}$$

with initial conditions $y_1(0) = 0.25,\ \ y_2(0) = 0.75$ and

$$f(x) = \begin{cases} 0 & t < \frac{1}{4} \\ 1 & Otherwise, \end{cases}$$

$$g(x) = \begin{cases} 0 & t < \frac{3}{4} \\ 1 & Otherwise. \end{cases}$$

This problem has the non-smooth exact solutions $y_1(x) = |x - 0.25|$ and $y_2(x) = |x - 0.75|$. Due to piecewise smoothness of this problem, Method 2 is more suited to approximate its solutions. In Table 6, the error values in Method 2 ($h = 0.1, N = 20$) and various values of x are given. It is seen that Method 2 has the advantage of providing high accurate numerical results for non-smooth solutions even in large intervals.

Table 6: Errors in approximating $y_1(x)$ and $y_2(x)$ for $N = 20$ for Example 4.

x	$e1$	$e2$
0	0	0
0.2	2.4×10^{-45}	1.6×10^{-45}
0.4	1.2×10^{-32}	2.6×10^{-33}
0.6	4.0×10^{-28}	4.9×10^{-29}
0.8	2.1×10^{-30}	3.0×10^{-32}
1	1.0×10^{-32}	5.1×10^{-30}
5	2.06×10^{-25}	1.65×10^{-25}
10	1.34×10^{-19}	2.32×10^{-19}
15	2.01×10^{-14}	6.58×10^{-14}
20	5.05×10^{-9}	4.31×10^{-9}

Example 5.
Finally, consider the following second order system

$$\begin{cases} 1 - y_1''(x) + \int_0^x y_2'(x)dt = y_1(x) + y_2(x), \\ \\ -y_2''(x) + \int_0^x y_1'(x)dt = y_1(x) + y_2(x), \end{cases}$$

with initial conditions $y_1(0) = 0,\ y_2(0) = 1,\ y_1'(0) = 1,\ y_2'(0) = 0$ and the exact solutions $y_1(x) = \sin(x),\ \ y_2(x) = \cos(x)$. Table 7 lists the maximum absolute errors in Method 1 and Method 2 with $h = 0.1$ and $N = 15, 20$.

6. Conclusions

Two new collocation methods based on shifted Legendre polynomials have been presented for the numerical solution of systems of Fredholm-Volterra integro-differential equations. In the first method, a

Table 7: Maximum absolute errors of $y_1(x)$ and $y_2(x)$ for Example 5.

N	15		20	
	Method 1	Method 2	Method 1	Method 2
$e1$	8.1×10^{-20}	2.4×10^{-25}	5.1×10^{-29}	2.4×10^{-32}
$e2$	2.0×10^{-19}	5.1×10^{-25}	4.0×10^{-30}	1.6×10^{-35}

one-domain collocation scheme was derived and its numerical accuracy on the interval $[0, 1]$ was assessed. In the second method, a multi-domain version of the first method has been extended. This method is especially suited for periodic solutions or solutions defined on large intervals. The convergence properties of the proposed methods have been investigated, and it was proved that they possess the spectral accuracy. Numerical findings demonstrated the effectiveness and accuracy of the proposed methods even with few collocation points.

References

[1] J. Abdul-Jerri, *Introduction to Integral Equations with Applications*, John Wiley & Sons, New York, NY, USA, 1999.

[2] B. Semper. *Finite-element methods for suspension Bridge models*, Computers. Math. Appl. **5** (1993), 77–91.

[3] M. Tavassoli-Kajani, S. Vahdati, Z. Abbas, M. Maleki. *Application of rational second kind Chebyshev functions for system of integro-differential equations on semi-infinite intervals*, J. Appl. Math. **212** (2012), Article ID 803503, 11 pages.

[4] K. Maleknejad, B. Basirat, E. Hashemizadeh. *A Bernstein operational matrix approach for solving a system of high order linear Volterra-Fredholm-integro-differential equations*, Math. Comput. Modell. **55** (2012), 1363–1372.

[5] M. Ghasemi, M. Tavassoli-Kajani, E. Babolian. *Numerical solutions of the nonlinear integro-differential equations: Wavelet-Galerkin method and homotopy perturbation method*, Appl. Math. Comput. **188(1)** (2007), 450–455.

[6] M. Tavassoli-Kajani, A. Hadi-Vencheh. *Solving linear integro-differential equation with Legendre wavelets* , Int. J. Comput. Math. **81(6)** (2004), 719–726.

[7] S. Abbasbandy, A. Taati. *Numerical solution of the system of nonlinear Volterra-integro-differential equations with nonlinear differential part by the operational Tau method and error estimation*, J. Comput. Appl. Math. **231** (2009), 106–113.

[8] A. Alsaedi, M. Alsulami, H.M. Srivastava, B. Ahmad, S.K. Ntouyas. *Existence Theory for Nonlinear Third-Order Ordinary Differential Equations with Nonlocal Multi-Point and Multi-Strip Boundary Conditions*, Symmetry. **11(2)** (2019), 281–293.

[9] M. Javidi. *Modified homotopy perturbation method for solving system of linear Fredholm integral equations*, Math. Comput. Modell. **50** (2009), 159–165.

[10] B. Asady, M. Tavassoli-Kajani, A. Hadi-Vencheh, A. Heydari. *Direct method for solving integro differential equations using hybrid Fourier and block-pulse functions*, Int. J. Comput. Math. **82(7)** (2005), 889–875.

[11] M. Zarebnia, M.G. Ali-Abadi. *Numerical solutions of system of nonlinear second-order integro-differential equations*, Comput. Math. Appl. **60** (2010), 591–601.

[12] M. Tavassoli-Kajani, M. Maleki. *Numerical approximations for Volterra's population growth model with fractional order via a multi-domain pseudospectral method*, Appl. Math. Modell. **39(19)** (2015), 4300–4308.

[13] M. Izadi, H.M. Srivastava. *Numerical approximations to the nonlinear fractional-order logistic population model with fractional-order Bessel and Legendre bases*, Chaos. Solitons & Fractals. **145** (2021), 110779.

[14] H.M. Srivastava, M. Izadi. *A discretization approach for the nonlinear fractional logistic equation*, Entropy. **22(11)** (2020), 1328.

[15] Q. Zhuang, Q. Ren. *Numerical approximation of a nonlinear fourth-order integro-differential equation by spectral method*, Appl. Math. Comput. **232** (2014), 775–783.

[16] X-J. Yang, F. Gao, H.M. Srivastava, *A new computional approach for solving nonlinear local fractional PDEs*, J. Comput. Appl. Math. **339** (2018), 285-296.

[17] Y. Ordokhani, H. Dehestani. *An application of Bessel function for solving nonlinear Fredholm-Volterra-Hammerstion integro-differential equations*, J. Sci. Kharazmi University. **399** (2018), 285–296.

[18] H.M. Srivastava, R.K. Saxena. *Some Volterra-type fractional integro-differential equations with a multivariable confluent hypergeometric function as their kernel*, J. integral equ. appl. **17(2)** (2005), 199–217.

[19] H. Aminkhah, S. Hoseini. *Numerical solution of nonlinear system of integro-differential equations using Chebyshev wavelets*, J. Appl. Math. Stat. Inform. **11(2)** (2015), 15–34.

[20] S. Yuzbasi. *Numerical solutions of system of linear Fredholm-Volterra integro-differential equations by the Bessel collocation method and error estimation*, Appl. Math. Comput. **250** (2015), 320–338.

[21] M. I. Berenguer, D. Gamez, and A.J. Lopez Linares. *Solution of systems of integro-differential equations using numerical treatment of fixed point*, J. Comput. Appl. Math. **315** (2017), 343–353.

[22] P. Roul, P. Meyer. *Numerical solutions of systems of nonlinear integro-differential equations by Homotopy-perturbation method*, Appl. Math. Modell. **35** (2011), 4234–4242.

[23] Y. Jafarzadeh, B. Keramati. *Numerical method for a system of integro-differential equations and convergence analysis by Taylor collocation*, Ain Shams Eng. J. **9(4)** (2018), 1433–1438.

[24] J. Pour-Mahmoud, M.Y. Rahimi-Ardabili, S. Shahmorad. *Numerical solution of the system of Fredholm integro-differential equations by the Tau method*, Appl. Math. Comput. **168** (2005), 465–478.

[25] F. Gao, X.J. Yang, H.M. Srivastava. *Exact traveling-wave solutions for linear and non-linear heat transfer equations* , Thermal Sci. **21(6)** (2017), 2307–2311.

[26] N. Ebrahimi, J. Rashidinia. *Spline collection for system of Fredholm and Volterra integro-differential equations*, J. Math. Model. **3(2)** (2015), 219–232.

[27] S. Abbasbandy, E. Shivanian. *Application of the variational iteration method for system of nonlinear Volterra integro-differential equations*, Math. Comput. Appl. **14(2)** (2009), 147–158.

[28] S. Fazeli, G. Hojjati. *Numerical solution of Volterra Integra-differential equations by superimplicit multistep collocation methods*, Numer. Algorithms. **68(4)** (2015), 741–768.

[29] J. Rong Loh, C. Phang. *A new numerical scheme for solving system of Volterra Integra-differential equation*, Alexandria Eng. J. **57(2)** (2018), 1117–1124.

[30] S. Mahdavi, M. Tavassoli-Kajani. *Nonlinear integro-differential equations*, J. Math. Ext. **4(2)** (2010), 107–117.

[31] A. Arikoglu, I. Ozkol. *Solutins of integral and Integra-differential equation systems by using differential transform method*, Comput. Math. Appl. **56** (2008), 2411–2417 .

[32] S.M. Hosseini, S. Shahmorad. *Tau numerical solution of Fredholm Integra-differential equations with arbitrary polynomial bases*, Appl. Math. Modell. **27** (2003), 145–155.

[33] P.B. Vasconcelos, J. Matos, M.S. Trindade. *Spectral Lanczos' Tau method for systems of nonlinear integro-differential equations*, In: Constanda C., Dalla Riva M., Lamberti P., Musolino P. (eds) Integral Methods in Science and Engineering, Volume 1. Birkhäuser, Cham, 2017.

[34] S. Nemati. *Numerical solution of Volterra-Fredholm integral equations using Legendre collocation method*, Comput. Appl. Math. **315** (2015), 29–36.

[35] A.R. Hadhoud, H.M. Srivastava, A.A.M. Rageh, *Non-polynomial B-spline and shifted Jacobi spectral collocation techniques to solve time-fractional nonlinear coupled Burgers' equations numerically*, Adv. Differ. Equ. **2021** (2021), Article ID: 439.

[36] H. M. Srivastava, A.K.N. Alomari, K.M. Saad, W.M. Hamanah, *Some dynamical models involving fractional-order derivatives with the Mittag-Leffler type kernels and their applications based upon the Legendre spectral collocation method*, Fractal. Frac, **5(3)** (2021), Article ID: 131.

[37] S. Kumar, R.K. Pandey, H.M. Srivastava, G.N. Singh, *A Convergent collocation approach for generalized fractional integro-differential equations using Jacobi poly-fractonomials*, Mathematics. **9(9)** (2021), Article ID: 979.

[38] M. Izadi, H. M. Srivastava. *A novel matrix technique for multi-order pantograph differential equations of fractional order*, Proc. R. Soc. A. **477** (2021), Article ID: 20210321.

[39] H. Singh, D. Baleanu, H.M. Srivastava, H. Dutta, N. Kumar-Jha. *Solution of multi-dimensional Fredholm equations using Legendre scaling functions*, Appl. Numer. Math. **150** (2020), 313–324.

[40] M.R. Ali, A.R. Hadhoud, H.M. Srivastava. *Solution of fractional Volterra-Fredholm integro-differential equations under mixed boundary conditions by using the HOBW method*. Adv. Differ. Equ. **2019** (2019), Article ID: 115.

[41] X-J. Ma, H.M. Srivastava, D. Baleanu, X-J. Yang. *A new Neumann series method for solving a family of local fractional Fredholm and Volterra integral equations*. **2013** (2013), Article ID: 325121.

[42] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang. *Spectral Methods: Fundamentals in Single Domains*, Springer-Verlag, Berlin, 2006.

[43] H.H. Sorkun, S. Yalcinbas. *Approximate solutions of linear Volterra integral equation systems with variable coefficients*. Appl. Math. Modell. **34(11)** (2010), 3451–3464.

[44] J. Biazar. *Solution of systems of integral-differential equations by Adomian decomposition method*. Appl. Math. Comput. **168** (2005), 1232–1238.

[45] V. Balakumar, K. Murugesan. *Numerical solution of Volterra integral-algebraic equations using block-pulse functions*. Appl. Math. Comput. **263** (2015), 165–170.