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gs-Drazin Inverses of Generalized Matrices over Local Rings

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Abstract. An element a in a ring R has a gs-Drazin inverse if there exists $b \in comm^2(a)$ such that $b = b^2a, a - ab \in R^{qnil}$. For any $s \in C(R)$, we completely determine when a generalized matrix $A \in K_s(R)$ over a local ring R has a gs-Drazin inverse.

1. Introduction

Let R be an associative ring with an identity. The commutant of $a \in R$ is defined by $comm(a) = \{x \in R \mid xa = ax\}$. The double commutant of $a \in R$ is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$.

An element a in a ring R has a s-Drazin inverse if there exists $b \in comm^2(a)$ such that $b = b^2a, a - ab \in R$ is nilpotent (see [12]). An element $a \in R$ has a s-Drazin inverse if and only if it is strongly nil-clean, i.e., it is the sum of an idempotent and a nilpotent that commute (see[12, Lemma 2.2]). Strongly nil-clean matrices over local rings were considered by many authors, e.g., [2] and [8].

Following Gurgun, an element a in a ring R has a gs-Drazin inverse if there exists $b \in comm^2(a)$ such that $b = b^2a, a - ab \in R^{qnil}$. Here, $R^{qnil} = \{a \in R \mid 1 + ax \in U(R) \text{ for every } x \in comm(a)\}$. As is well known, an element a in a ring R has a gs-Drazin inverse if and only if there exists $e^2 = e \in comm^2(a)$ such that $a - e \in R^{qnil}$ (see [6, Theorem 3.2]). In [1], Chen and Calci extend Cline's formula and Jacobson's Lemma for gs-Drazin inverses. Various additive properties of gs-Drazin inverses are thereby obtained.

Let R be a ring and $s \in C(R)$. Let $K_s(R) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R\}$, where the operations are defined as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a + a' & b + b' \\ c + c' & d + d' \end{pmatrix},$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + sbc' & ab' + bd' \\ ca' + dc' & scb' + dd' \end{pmatrix}.$$

Then $K_s(R)$ is a ring with the identity $I_2 = \begin{pmatrix} 1_R & 0 \\ 0 & 1_R \end{pmatrix}$. A ring R is local if R has only one maximal right ideal. If $s \in U(R)$, then $K_s(R) \cong M_2(R)$ (see [11, Lemma 1]). Thus, the class of generalized matrices over a

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ring is a generalization of that of matrices. The motivation of this paper is to investigate gs-Drazin inverses of generalized matrices over a local ring.

Let $a \in R$. $l_a : R \to R$ and $r_a : R \to R$ denote, respectively, the abelian group endomorphisms given by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Thus, $l_a - r_b$ is an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$.

Let R be a local ring and $A \in M_2(R)$. In Section 2, we prove that A has a gs-Drazin inverse if and only if $A \in M_2(R)^{qnil}$; or $I_2 - A \in M_2(R)^{qnil}$; or A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in 1 + J(R), \beta \in J(R)$. Further, we characterize matrices having gs-Drazin inverses in terms of quadratic polynomials. These results are also preparations for the general case.

In Section 3, we completely determine when a generalized matrix $A \in K_s(R)$ over a local ring R has a gs-Drazin inverse. Let R be a cobleached local ring and $s \in C(R)$. We prove that $A \in K_s(R)$ has a gs-Drazin inverse if and only if $A \in K_s(R)^{qnil}$; or $I_2 - A \in K_s(R)^{qnil}$; or A is similar to $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$, where $u \in 1+J(R), v \in U(R), w \in J(R)$, $t^2-(vuv^{-1}+w)t+(vuv^{-1}w-sv)$ has a root in $1+J(R)$ and $t^2-(u+w)t+(wu-sv)$ has a root in $J(R)$.

We use $J(R)$, $N(R)$ and $U(R)$ to denote the Jacobson radical of R, the set of nilpotent elements and units in R, respectively. The symbol $C(R)$ stands for the center of a ring R. $GL_2(R)$ denotes the sets of all 2×2 invertible matrices over R.

2. gs-Drazin inverses of matrices

This section is devoted to preliminary observations concerning gs-Drazin inverses of a 2×2 matrix over local rings R which will be used in the sequel. Recall that an element a in a ring R is quasipolar if there exists an idempotent $e \in comm^2(a)$ such that $a + e \in U(R)$ and $ae \in R^{qnil}$. As is well known, $a \in R$ is quasipolar if and only if it has a generalized Drazin inverse, i.e., $a - a^2b \in R^{qnil}, b = b^2a$ for some $b \in comm^2(a)$. The following lemma is crucial.

Lemma 2.1. *([4, Theorem 3.4]) Let R be a local ring and $A \in M_2(R)$. Then A is quasipolar if and only if*

(1) *$A \in GL_2(R)$; or*
(2) *$A \in M_2(R)^{qnil}$; or*
(3) *A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in U(R), \beta \in J(R)$.*

Theorem 2.2. *Let R be a local ring and $A \in M_2(R)$. Then A has a gs-Drazin inverse if and only if*

(1) *$A \in M_2(R)^{qnil}$; or*
(2) *$I_2 - A \in M_2(R)^{qnil}$; or*
(3) *A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in 1 + J(R), \beta \in J(R)$.*

Proof. $\Longrightarrow$ In view of [6, Corollary 3.3], A is quasipolar. By virtue of Lemma 2.1, we have three cases.

Case 1. $A \in GL_2(R)$. By virtue of [6, Theorem 3.2], there exists an idempotent $E \in comm^2(A)$ such that $A - E \in M_2(R)^{qnil}$. Hence, $E = A(I_2 - A^{-1}(A - E)) \in GL_2(R)$, and so $E = I_2$. Therefore $I_2 - A \in M_2(R)^{qnil}$.

Case 2. $A \in M_2(R)^{qnil}$.

Case 3. A is similar to $B := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in U(R), \beta \in J(R)$. Then we easily see that $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ has a gs-Drazin inverse. Hence we can find some $E = (e_{ij}) \in comm^2(B)$ such that $B - E \in M_2(R)^{qnil}$. As $EB = BE$, we deduce that $e_{12} = e_{21} = 0$. Hence, $e_{11} \in U(R)$, and so $e_{11} = 1$. Moreover, $e_{22} \in J(R)$, and so $e_{22} = 0$. Therefore $\alpha \in 1 + J(R)$, as desired.

$\Longleftarrow$ We are concern on three cases.

Case 1. $A \in M_2(R)^{qnil}$. Then A has a gs-Drazin inverse.

Case 2. $I_2 - A \in M_2(R)^{qnil}$. Then $A - I_2 \in M_2(R)^{qnil}$, and so A has a gs-Drazin inverse.

Case 3. A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $l_\lambda - r_\mu, l_\mu - r_\lambda$ are injective and $\lambda \in 1 + J(R), \mu \in J(R)$.

Clearly, we have

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \lambda - 1 & 0 \\ 0 & \mu \end{pmatrix},$$

where $\begin{pmatrix} \lambda - 1 & 0 \\ 0 & \mu \end{pmatrix} \in M_2(J(R))$. Let $\begin{pmatrix} x & s \\ t & y \end{pmatrix} \in comm\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. Then

$$\lambda s = s\mu \text{ and } \mu t = t\lambda;$$

hence, $s = t = 0$. This implies that

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Therefore $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in comm^2\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, hence the result. $\square$

A ring R is cobleached provided that for any $a \in J(R), b \in U(R)$, $l_a - r_b$ and $r_b - r_a$ are both injective. For instance, every commutative local ring is cobleached.

Corollary 2.3. *Let R be a local ring and $A \in M_2(R)$. If R is cobleached, then A has a gs-Drazin inverse if and only if*

(1) *$A \in M_2(R)^{qnil}$; or*
(2) *$I_2 - A \in M_2(R)^{qnil}$; or*
(3) *A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in 1 + J(R)and\beta \in J(R)$.*

Proof. This is obvious by Theorem 2.2. $\square$

As an immediate consequence, we can derive the following result.

Corollary 2.4. *Let R be a commutative local ring and $A \in M_2(R)$. Then A has a gs-Drazin inverse if and only if*

(1) *$A = N + W$ with $N^2 = 0, W \in M_2(J(R))$;*
(2) *$A = I_2 + N + W$ with $N^2 = 0, W \in M_2(J(R))$;*
(3) *A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in 1 + J(R), \beta \in J(R)$.*

Proof. Since R is commutative, it is cobleached. In view of [5, Lemma 4.1], $C \in M_2(R)^{qnil}$ if and only if $C^2 \in M_2(J(R))$. It follows by [5, Lemma 3.2] that $C^2 \in M_2(J(R))$ if and only if $C = N + W$, where $N \in N(M_2(R))$ and $W \in M_2(J(R))$. Therefore we complete the proof, by Corollary 2.3. $\square$

Corollary 2.5. *Let D be a division ring. Then $A \in M_2(D)$ has a gs-Drazin inverse if and only if*

(1) *$A^2 = 0$;*
(2) *$(I_2 - A)^2 = 0$;*
(3) *A is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.*

Proof. Since every local ring is a division ring with Jacobson radical 0, we obtain the result by Corollary 2.4. $\square$

Lemma 2.6. *([5, Lemma 3.3]) Let R be a local ring and $A \in M_2(R)$. Then*

(1) *$A \in GL_2(R)$; or*
(2) *$A^2 \in M_2(J(R))$; or*
(3) *A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$.*

We are now ready to prove:

Theorem 2.7. *Let R be a cobleached local ring and $A \in M_2(R)$. Then A has a gs-Drazin inverse if and only if*

(1) *$A \in M_2(R)^{qnil}$; or*
(2) *$I_2 - A \in M_2(R)^{qnil}$; or*
(3) *A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$, the equation $x^2 - \mu x - \lambda = 0$ has a root in $1 + J(R)$ and a root in $J(R)$.*

Proof. $\Longrightarrow$ By virtue of Lemma 2.6, we have three cases.

Case 1. $A \in GL_2(R)$. Then $A - E \in M_2(R)^{qnil}$ for some $E \in comm^2(A)$. Hence $E = I_2$, and so $I_2 - A \in M_2(R)^{qnil}$.

Case 2. $A^2 \in M_2(J(R))$. Hence $A \in M_2(R)^{qnil}$.

Case 3. A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$. It suffices to consider Case 3. In view of Theorem 2.2, there exists $U \in GL_2(R)$ such that

$$U^{-1}\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}U = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

where $\alpha \in U(R), \beta \in J(R)$. Set $U = \begin{pmatrix} x & y \\ s & t \end{pmatrix}$. Then we have

$$\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}\begin{pmatrix} x & y \\ s & t \end{pmatrix} = \begin{pmatrix} x & y \\ s & t \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

This shows that

$$\begin{gathered} \lambda s = x\alpha; \\ \lambda t = y\beta; \\ x + \mu s = s\alpha; \\ y + \mu t = t\beta. \end{gathered}$$

Clearly, $x \in J(R)$. Since $U \in GL_2(R)$, we see that $y, s \in U(R)$, and so $t \in U(R)$. Let $\delta = s\alpha s^{-1}$ and $\gamma = t\beta t^{-1}$. Then $\delta \in U(R), \gamma \in J(R)$. It is easy to verify that

$$\begin{aligned} \delta^2 - \mu\delta &= s\alpha^2 s^{-1} - \mu s\alpha s^{-1} \\ &= (s\alpha - \mu s)(\alpha s^{-1}) \\ &= x\alpha s^{-1} \\ &= \lambda. \end{aligned}$$

Therefore $\delta^2 - \mu\delta - \lambda = 0$. Similarly, $\gamma^2 - \mu\gamma - \lambda = 0$. Consequently, $x^2 - \mu x - \lambda = 0$ has a root $\delta \in U(R)$ and a root $\gamma \in J(R)$, as required.

$\Longleftarrow$ If $A \in M_2(R)^{qnil}$ or $I_2 - A \in M_2(R)^{qnil}$, then A has a gs-Drazin inverse. Suppose that $x^2 - \mu x - \lambda = 0$ has a root $\alpha \in U(R)$ and a root $\beta \in J(R)$. Then we have

$$\begin{gathered} \alpha^2 - \mu\alpha - \lambda = 0; \\ \beta^2 - \mu\beta - \lambda = 0. \end{gathered}$$

Hence,

$$(\alpha-\mu)\alpha=\lambda;$$
$$(\beta-\mu)\beta=\lambda.$$

Obviously,

$$\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}\begin{pmatrix} \alpha-\mu & \beta-\mu \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} \alpha-\mu & \beta-\mu \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Clearly, we have

$$\begin{pmatrix} \alpha-\mu & \beta-\mu \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 1 & \beta-\mu \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha-\beta & 0 \\ 1 & 1 \end{pmatrix}\in GL_2(R).$$

Therefore $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha\in U(R)$ and a root $\beta\in J(R)$. This completes the proof, by Theorem 2.2. □

Corollary 2.8. *Let R be a commutative local ring and $A\in M_2(R)$. Then A has a gs-Drazin inverse if and only if*

(1) $A=N+W$ *with* $N^2=0, W\in M_2(J(R))$;
(2) $A=I_2+N+W$ *with* $N^2=0, W\in M_2(J(R))$;
(3) $x^2-tr(A)x+det(A)$ *has a root* $\alpha\in 1+J(R)$ *and a root* $\beta\in J(R)$.

Proof. $\Longrightarrow$ In view of Theorem 2.7, we have three cases.

Case 1. $A\in M_2(R)^{qnil}$. In view of [5, Lemma 4.1], $A^2\in M_2(J(R))$. By virtue of [5, Lemma 3.2], we have $A=N+W$ with $N^2=0, W\in M_2(J(R))$.

Case 2. $I_2-A\in M_2(R)^{qnil}$. Similarly, $A-I_2=N+W$ with $N^2=0, W\in M_2(J(R))$, as desired.

Case 3. A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda\in J(R), \mu\in U(R)$, and the equation $x^2-\mu x-\lambda=0$ has a root in $J(R)$ and a root in $1+J(R)$. Hence $\mu=tr(A)$ and $-\lambda=det(A)$. Therefore the equation $x^2-tr(A)x+det(A)=0$ has a root in $J(R)$ and a root in $1+J(R)$.

$\Longleftarrow$ We will suffice to assume that the equation $x^2-tr(A)x+det(A)=0$ has a root in $J(R)$ and a root in $1+J(R)$. By virtue of Lemma 2.6, we may assume that A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda\in J(R), \mu\in U(R)$. Hence $\mu=tr(A)$ and $-\lambda=det(A)$. Thus, the equation $x^2-x\mu-\lambda=0$ has a root in $J(R)$ and a root in $1+J(R)$. Therefore we obtain the result by Theorem 2.7. □

Example 2.9. *Let* $A=\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\in M_2(\mathbb{Z}_3)$. *Then A has a generalized Drazin inverse, but has no gs-Drazin inverse.*

Proof. Clearly, $\mathbb{Z}_3$ is a commutative local ring with $J(\mathbb{Z}_3)=\overline{0}$. Clearly, $A^2,(I_2-A)^2\neq\overline{0}$. Additionally, $tr(A)=\overline{2}$ and $det(A)=\overline{0}$. Taking $p(x)=x(x+1)=x^2+x\in\mathbb{Z}_3[x]$ which has roots $\overline{0}$ and $\overline{2}$. In light of Corollary 2.8, $A\in M_2(\mathbb{Z}_3)$ has no gs-Drazin inverse. As $M_2(\mathbb{Z}_3)$ is a finite ring, we easily see that A has a generalized Drazin inverse, as desired. □

Theorem 2.10. *Let R be a local ring and $A\in M_2(R)$. If R is cobleached, then the following are equivalent:*

(1) *A has a gs-Drazin inverse.*
(2) *There exists* $E^2=E\in comm(A)$ *such that* $A-E\in M_2(R)^{qnil}$.
(3) *There exists* $B\in comm(A)$ *such that* $B=B^2A, A-AB\in M_2(R)^{qnil}$.

Proof. (1) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (2) By hypothesis, there exists $B \in comm(A)$ such that $B = B^2A, A - AB \in M_2(R)^{qnil}$. Set $E = AB$. Then $E \in comm(A)$ and $A - E \in M_2(R)^{qnil}$, as desired.

(2) $\Rightarrow$ (1) By hypothesis, there exists $E^2 = E \in comm(A)$ such that $W := A - E \in M_2(R)^{qnil}$. In view of [4, Lemma 2.3], $E = 0$,or $E = I_2$ or E is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Clearly, 0 and $I_2 \in comm^2(A)$. We may assume that

$$U^{-1}EU = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence,

$$U^{-1}AU - U^{-1}EU = U^{-1}WU \in M_2(R)^{qnil}.$$

By hypothesis, $EA = AE$, and so

$$U^{-1}AU \in comm \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Write $U^{-1}AU = \begin{pmatrix} x & y \\ s & t \end{pmatrix}$. It follows from

$$\begin{pmatrix} x & y \\ s & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ s & t \end{pmatrix}$$

that $y = s = 0$.

Moreover, we have

$$\begin{pmatrix} 1+x & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & t \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)^{qnil}.$$

This implies that $1 + x, t \in J(R)$.

For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in comm \begin{pmatrix} x & 0 \\ 0 & t \end{pmatrix}$, we have

$$xb - bt = 0, tc - cx = 0.$$

Since R is cobleached, we see that $b = c = 0$, and so

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in comm \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in comm^2 \begin{pmatrix} x & 0 \\ 0 & t \end{pmatrix},$$

thus, $U^{-1}EU \in comm^2(U^{-1}AU)$. Hence $E \in comm^2(A)$. This completes the proof. □

Corollary 2.11. *Let R be a local ring and $A \in M_2(R)$. If R is cobleached, then the following are equivalent:*

(1) *A has a gs-Drazin inverse.*
(2) *There exists a unique $E^2 = E \in comm(A)$ such that $A - E \in M_2(R)^{qnil}$.*
(3) *There exists a unique $B \in comm(A)$ such that $B = B^2A, A - AB \in M_2(R)^{qnil}$.*

Proof. (1) $\Leftrightarrow$ (2) This is clear, by [6, Theorem 2.7].

(2) $\Rightarrow$ (3) In view of Theorem 2.10, there exists $B \in comm(A)$ such that $B = B^2A, A - AB \in M_2(R)^{qnil}$. Suppose that there exists $C \in comm(A)$ such that $C = C^2A, A - AC \in M_2(R)^{qnil}$. Let $E = AB$ and $F = AC$. Then $E^2 = E, F^2 = F \in comm(A)$ and $A - E, A - F \in M_2(R)^{qnil}$. By the uniqueness, we get $E = F$, and so $B = B(BA) = BE = BF = B(AC) = (BA)C = (CA)C = AC^2 = C$, as desired.

(3) $\Rightarrow$ (1) This is obvious in terms of Theorem 2.10. □

3. Generalized Matrices over local rings

The purpose of this section is to completely characterize gs-Drazin inverses of generalized matrices over a local ring. The following result will play an important role.

Lemma 3.1. *Let R be a local ring and $s \in J(R) \cap C(R)$. Then $A \in K_s(R)$ is quasipolar if and only if*

(1) *$A \in U(K_s(R))$; or*
(2) *$A \in K_s(R)^{nil}$; or*
(3) *A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in U(R), \beta \in J(R)$.*

Proof. $\Longleftarrow$ If $A \in U(K_s(R))$ or $A \in K_s(R)^{nil}$, then $A \in K_s(R)$ is quasipolar. Suppose that A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in U(R), \beta \in J(R)$. Write $U^{-1}AU = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. As R is local, it is quasipolar. Hence, we can find idempotents $e \in comm^2(\alpha), f \in comm^2(\beta)$ such that $\alpha - e, \beta - f \in U(R), \alpha e, \beta f \in J(R)$. Set $E = U\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}U^{-1}$. Then $E^2 = E \in K_s(R)$. We easily check that $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \in comm^2\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Hence $U^{-1}EU \in comm^2(U^{-1}AU)$, and so $E \in comm^2(A)$. Moreover, we see that $A - E \in U\big(K_s(R)\big)$, as desired.

$\Longrightarrow$ Suppose that $A \notin U(K_s(R))$ and $A \notin K_s(R)^{nil}$. Write $A + E = W$ with $E \in comm^2(A), W \in K_s(R)^{qnil}$. Set $E = \begin{pmatrix} c & x \\ y & d \end{pmatrix}$. Let $X \in comm(A)$. Then $EX = XE$, and so $XW = WX$. This shows that $I_2 - WX \in U(K_s(R))$. If $c, d \in J(R)$, then $E \in J(K_s(R))$ by [11, Lemma 2], and so $I_2 - AX = (I_2 - WX) - EX \in U(K_s(R))$. This shows that $A \in K_s(R)^{qnil}$, an absurd. Thus, we see that c or d is not in $J(R)$.

Case 1. $c \in U(R)$. Then $\begin{pmatrix} 1 & 0 \\ -yc^{-1} & 1 \end{pmatrix} E \begin{pmatrix} c^{-1} & -c^{-1}x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & d - syc^{-1}x \end{pmatrix}$. This implies that $\begin{pmatrix} 1 & 0 \\ 0 & d - syc^{-1}x \end{pmatrix} \in K_s(R)$ is regular, and then so is $d - syc^{-1}x \in R$. As R is local, we easily check that $d - syc^{-1}x$ is zero or invertible. Hence, we have $P, Q \in U\big(K_s(R)\big)$ such that PEQ is an idempotent diagonal matrix. In light of [11, Lemma 3], E is similar to a diagonal matrix.

Case 2. $d \in U(R)$. Similarly to the discussion in Case 1, we easily verify that E is similar to a diagonal matrix.

Write $P^{-1}EP = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. We may assume that $e = 1$ and $f = 0$. Then $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + P^{-1}UP$ and $P^{-1}AP\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}P^{-1}AP$. This forces that $P^{-1}AP$ is diagonal $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.

Given $\lambda x = x\mu$ with $x \in R$, then

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

Hence, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that $x = 0$. This shows that $l_\lambda - r_\mu$ is injective. Likewise, $l_\mu - r_\lambda$ is injective, as desired. □

Theorem 3.2. *Let R be a local ring and $s \in C(R)$. Then $A \in K_s(R)$ has a gs-Drazin inverse if and only if*

(1) *$A \in K_s(R)^{qnil}$; or*
(2) *$I_2 - A \in K_s(R)^{qnil}$; or*

(3) *A is similar to* $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, *where* $l_\alpha - r_\beta, l_\beta - r_\alpha$ *are injective and* $\alpha \in 1 + J(R), \beta \in J(R)$.

Proof. Since R is local, $s \in U(R)$ or $s \in J(R)$.

Case 1. $s \in U(R)$. Then $K_s(R) \cong M_2(R)$, and so the result follows by Theorem 2.2.

Case 2. $s \in J(R)$.

$\Longrightarrow$ Suppose that $A, I_2 - A \notin K_s(R)^{qnil}$. If $A \in U(K_s(R))$, then $A - E \in K_s(R)^{qnil}$ for some $E^2 = E \in comm^2(A)$. Hence, $E = I_2$, and so $I_2 - A \in K_s(R)^{qnil}$. In view of [6, Corollary 3.3], $A \in K_s(R)$ is quasipolar. It follows by Lemma 3.1 that A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in U(R), \beta \in J(R)$. If $\alpha \in 1 + U(R)$, then $A \in U(K_s(R))$, and so we see that $\alpha \in 1 + J(R)$, as required.

$\Longleftarrow$ If $A \in K_s(R)^{qnil}$ or $I_2 - A \in K_s(R)^{qnil}$, the proof is obvious. Assume that A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $\alpha \in 1 + J(R)$ and $\beta \in J(R)$. Choose $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $A - P \in K_s(R)^{qnil}$ and $P^2 = P$. Let $X \in comm\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. So $X = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$, since $l_\alpha - r_\beta$ and $l_\beta - r_\alpha$ are injective. Hence $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in comm^2\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Hence A has a gs-Drazin inverse, as desired. □

As an immediate consequence of Theorem 3.2, we now derive

Corollary 3.3. *Let R be a cobleached local ring and $s \in C(R)$. Then $A \in K_s(R)$ has a gs-Drazin inverse if and only if*

(1) $A \in K_s(R)^{qnil}$; *or*
(2) $I_2 - A \in K_s(R)^{qnil}$; *or*
(3) *A is similar to* $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, *where* $\alpha \in 1 + J(R), \beta \in J(R)$.

Lemma 3.4. *Let R be a local ring and $s \in C(R)$ and $A \in K_s(R)$. Then*

(1) $A \in U(K_s(R))$; *or*
(2) $I_2 - A \in U(K_s(R))$; *or*
(3) *A or* $I_2 - A$ *is similar to a matrix* $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$, *where* $u \in 1 + J(R), v \in U(R), w \in J(R)$.

Proof. We have two cases to complete to the proof. Assume that $s \in U(R)$. So $K_s(R) \cong M_2(R)$, and the result follows by [13, Lemma 4]. We now assume that $s \in J(R)$. Let $A \in K_s(R)$. In view of [11, Lemma 5], $A \in U(K_s(R))$; or $I_2 - A \in U(K_s(R))$, or A is similar to a matrix $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$, or $\begin{pmatrix} w & 1 \\ v & u \end{pmatrix}$, where $u \in 1 + J(R), v \in U(R), w \in J(R)$. If A is isomorphic to $\begin{pmatrix} w & 1 \\ v & u \end{pmatrix}$, then $I_2 - A$ is isomorphic to $\begin{pmatrix} 1-w & -1 \\ -v & 1-u \end{pmatrix}$. Hence, $I_2 - A$ is isomorphic to $\begin{pmatrix} 1-w & 1 \\ v & 1-u \end{pmatrix}$. This completes the proof. □

We have accumulated all the information necessary to prove the following.

Theorem 3.5. *Let R be a cobleached local ring and $s \in C(R)$. Then $A \in K_s(R)$ has a gs-Drazin inverse if and only if*

(1) $A \in K_s(R)^{qnil}$; *or*
(2) $I_2 - A \in K_s(R)^{qnil}$; *or*
(3) *A is similar to* $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$, *where* $u \in 1 + J(R), v \in U(R), w \in J(R)$, $t^2 - (vuv^{-1} + w)t + (vuv^{-1}w - sv)$ *has a root in* $1 + J(R)$ *and* $t^2 - (u + w)t + (wu - sv)$ *has a root in* $J(R)$.

Proof. $\Longrightarrow$ Write $A = E + W$ with $E^2 = E \in comm^2(A)$ and $W \in K_s(R)^{qnil}$. In view of Lemma 3.4, we have three cases.

Case 1. $A \in U(K_s(R))$. Then $E = I_2$. Hence, $I_2 - A \in K_s(R)^{qnil}$.
Case 2. $I_2 - A \in U(K_s(R))$. Then $E = 0$, and so $A \in K_s(R)^{qnil}$.
Case 3. A or $I_2 - A$ is similar to a matrix $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$, where $u, v \in U(R), w \in J(R)$.

(1) A is similar to a matrix $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$. Then we may assume that there exists $\begin{pmatrix} a & x \\ y & b \end{pmatrix} \in U(K_s(R))$ such that

$$\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}\begin{pmatrix} a & x \\ y & b \end{pmatrix} = \begin{pmatrix} a & x \\ y & b \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Here, $\alpha \in 1 + J(R), \beta \in J(R)$. Thus, we have

$$\begin{array}{rcl} ua + sy & = & a\alpha; \\ va + wy & = & y\alpha; \\ ux + b & = & x\beta; \\ svx + wb & = & b\beta. \end{array}$$

Further, we check that $x, y \in U(R)$. Let $\lambda = y\alpha y^{-1} \in 1 + J(R)$ and $\mu = x\beta x^{-1} \in J(R)$. Then we verify that

$$\begin{array}{rl} & \lambda^2 - (vuv^{-1} + w)\lambda + vuv^{-1}w \\ = & \big((y\alpha)\alpha - (vuv^{-1} + w)y\alpha + vuv^{-1}wy\big)y^{-1} \\ = & \big((va + wy)\alpha - (vuv^{-1} + w)y\alpha + vuv^{-1}wy\big)y^{-1} \\ = & \big(va\alpha - vuv^{-1}(va + wy) + vuv^{-1}wy\big)y^{-1} \\ = & \big(va\alpha - vua\big)y^{-1} \\ = & \big(v(ua + sy) - vua\big)y^{-1} \\ = & sv, \end{array}$$

and

$$\begin{array}{rl} & \mu^2 - (u + w)\mu + wu \\ = & x\beta^2x^{-1} - (u + w)x\beta x^{-1} + wu \\ = & \big((ux + b)\beta - (u + w)x\beta + wux\big)x^{-1} \\ = & (b\beta - wx\beta + wux)x^{-1} \\ = & \big(svx + wb - w(ux + b) + wux\big)x^{-1} \\ = & sv \end{array}$$

as desired.

(2) $I_2 - A$ is similar to a matrix $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$. Clearly, $I_2 - A$ is similar to $\begin{pmatrix} 1-\beta & 0 \\ 0 & 1-\alpha \end{pmatrix}$, and then we are done as in (1).

$\Longleftarrow$ We will suffice to prove $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$ has a gs-Drazin inverse, where $\alpha \in 1 + J(R)$ is the root of $t^2 - (vuv^{-1} + w)t + (vuv^{-1}w - sv)$ and $\beta \in J(R)$ is the root of $t^2 - (u+w)t + (wu - sv)$. Let $P = \begin{pmatrix} v^{-1}(\alpha - w) & 1 \\ 1 & \beta - u \end{pmatrix}$. Then $P \in U(K_s(R))$ by [11, Lemma 2]. It is easy to verify that $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}P = P\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Therefore we complete the proof by Corollary 3.3. $\square$

Let R be a commutative ring and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_s(R)$. Set $tr_s(A) = a + d$ and $det_s(A) = ad - sbc$. We now derive

Corollary 3.6. *Let R be a commutative local ring, $s \in R$ and $A \in K_s(R)$. Then $A \in K_s(R)$ has a gs-Drazin inverse if and only if*

(1) $A^2 \in J(K_s(R))$ *or* $(I_2 - A)^2 \in J(K_s(R))$; *or*
(2) $t^2 - tr_s(A)t + det_s(A) = 0$ *has a root in* $1 + J(R)$ *and a root in* $J(R)$.

Proof. Suppose that $A^2, (I_2 - A)^2 \notin J(K_s(R))$. Then $A, I_2 - A \notin K_s(R)^{qnil}$ by [5, Lemma 4.1]. By virtue of Theorem 3.5, A has a gs-Drazin inverse if and only if A or $I_2 - A$ is similar to $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$, where $u \in 1 + J(R), v \in U(R), w \in J(R)$.

Case 1. A is similar to a matrix $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$. Then $t^2 - tr_s(A)t + det_s(A) = 0$ is solvable if and only if $t^2 - (u + w)t + (uw - sv) = 0$ is solvable, as desired.

Case 2. $I_2 - A$ is similar to a matrix $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$. Then $t^2 - tr_s(A)t + det_s(A) = 0$ is solvable if and only if $x^2 - tr_s(I_2 - A) + det_s(I_2 - A) = 0$ is solvable, if and only if $x^2 - (u + w)x + (uw - sv) = 0$ is solvable, hence the result. $\square$

Example 3.7. *Let* $A = \begin{pmatrix} \overline{1} & \overline{1} \\ \overline{3} & \overline{2} \end{pmatrix} \in K_2(\mathbb{Z}_4)$. *Then A has a gs-Drazin inverse in $K_2(\mathbb{Z}_4)$, but it has no gs-Drazin inverse in $M_2(\mathbb{Z}_4)$.*

Proof. Clearly, $\mathbb{Z}_4$ is a commutative local ring with $J(\mathbb{Z}_4) = \overline{2}\mathbb{Z}_4$. Since $tr_2(A) = \overline{3}$ and $det_2(A) = \overline{0}$, the equation $t^2 - tr_2(A)t + det_2(A) = \overline{0}$ has a root $\overline{3}$ in $1 + J(\mathbb{Z}_4)$ and a root $\overline{0}$ in $J(\mathbb{Z}_4)$. Therefore A has a gs-Drazin inverse in $K_2(\mathbb{Z}_4)$ by Corollary 3.6.

Clearly, $det(A) = -\overline{1}$ and $det(I_2 - A) = \overline{1}$, we see that $A, I_2 - A$ are not nilpotent in $M_2(\mathbb{Z}_4)$. Moreover, the equation $t^2 - tr(A)t + det(A) = \overline{0}$ is not solvable in $\mathbb{Z}_4$. In light of Corollary 2.8, A has no gs-Drazin inverse in $M_2(\mathbb{Z}_4)$, as asserted. $\square$

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